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ORTHODOX BANDS OF MODULES

par Francis PASTIJN

Summary. - In this paper, we shall consider orthodox bands of commutative groups, together with a ring of endomorphisms. We shall generalize the concept of a left module by introducing orthodox bands of left modules; we shall also deal with linear mappings, the transpose of a linear mapping and with the dual of an orthodox band of left modules.

We shall use the notations and terminology of [1] (chap 2, § 1) and [2].

1. Definition.

Let \((R, +, \cdot)\) be a ring with zero element 0 and identity 1. Let \(S\) be a semigroup and \(R \times S \to S\), \((\alpha, x) \mapsto \alpha x\) a mapping satisfying the following conditions:

(i) \(\alpha(xy) = (\alpha x)(\alpha y)\) for every \(\alpha \in R\) and every \(x, y \in S\),

(ii) \((\alpha + 0)x = (\alpha x) + (0x)\) for every \(\alpha, \beta \in R\) and every \(x \in S\),

(iii) \((\alpha \cdot \beta)x = \alpha(\beta x)\) for every \(\alpha, \beta \in R\) and every \(x \in S\),

(iv) \(1x = x\) for every \(x \in S\).

The so-defined structure will be called an orthodox band of left \(R\)-modules. Next theorem justifies our terminology.

2. Theorem 1. - Let \(R, S\) and mapping \(R \times S \to S\) be as in 1. Then \(S\) is an orthodox band of commutative groups, and the maximal subgroups of \(S\) are left invariant by the elements of \(R\).

Proof. - Let \(x\) be any element of \(S\), and \(\alpha\) any element of \(R\); we then have

\[
(0x)(0x) = (0 + 0)x = 0x,
\]

\[
(\alpha x)(0x) = (\alpha + 0)x = \alpha x = (0 + \alpha)x = (0x)(\alpha x),
\]

\[
(\alpha x)((- \alpha)x) = (\alpha - \alpha)x = 0x = (- \alpha + \alpha)x = ((- \alpha)x)(\alpha x).
\]

This implies that for any \(\alpha \in R\) and any \(x \in S\), \(\alpha x\) belongs to the maximal subgroup of \(S\) with identity \(0x\), the inverse of \(\alpha x\) in this maximal subgroup must be \((- \alpha)x\). More specifically \(1x = x\) belongs to the maximal subgroup of \(S\) with identity \(0x\), and its inverse in this maximal subgroup must be \((- 1)x\). We conclude that \(S\) must be a completely regular semigroup and that all maximal subgroup of \(S\) are left invariant by the elements of \(R\).

For every \(x, y \in S\) we have

\[
(xy)(xy) = (1 + 1)(xy) = ((1 + 1)x)((1 + 1)y) = x^2 y^2.
\]
Let $e, f$ be any idempotents of $S$, then the foregoing implies that

$$ (ef)^2 = e^2 f^2 = ef, \text{ hence } E_S = \{ x \in S ; \ x^2 = x \} $$

must be a subsemigroup of $S$. Let $x$ and $y$ belong to a same maximal subgroup of $S$, then the foregoing implies

$$ xy = ((-1)x)x^2 y^2((-1)x)xyy((-1)y)) = yx, $$

hence $S$ is a union of commutative groups. We conclude that $S$ is an orthodox union of commutative groups [3].

Let $e$ and $f$ be any idempotent of $S$, and $x \in H_e$, $y \in H_f$. We put $(-1)x = x'$ and $(-1)y = y'$, then

$$ ef = (ef)^2 = (1 + 1)(ef) = (1 + 1)(x(x'f)) = x^2(x'f)^2 $$

$$ = x^2 x'fx'f = (xf)(x'f) $$

and analogously

$$ ef = (x'f)(xf). $$

Since $ef$, $x'f$ and $xf$ are elements of rectangular group $D_{ef}$ [3], the foregoing implies that $xf$ and $x'f$ are mutually inverse elements of maximal subgroup $H_{ef}$. Dually, $ey$ and $ey'$ are mutually inverse elements of maximal subgroup $H_{ef}$. Since $(xy)y' = xf$ and $(xf)y = xy$ we have $xy \in xf$, hence $xy \in ef$. Analogously, since $x'(xy) = ey$ and $x(ey) = xy$ we have $xy \in ey$, hence $xy \in ef$. We conclude that $xy \in ef$. Green's relation $\mathcal{H}$ must then be a congruence on $S$. Thus $S$ is an orthodox band of commutative groups [3].

3. Remark.

Let $S$ be an orthodox band of commutative groups. Then, by Yamada's theorem ([3] and [11]), there exists a band $E$, and a semilattice of commutative groups $Q$, both having the same structure semilattice $Y$, such that $S$ is the spined product of $Q$ and $E$ over $Y$: $S = Q \times Y E$. Let $Q = \bigcup_{\kappa \in Y} G_\kappa$ and $E = \bigcup_{\kappa \in Y} E_\kappa$, then $S$ consists of ordered pairs $(x_\kappa, e_\kappa)$, $\kappa \in Y$, $x_\kappa \in G_\kappa$, $e_\kappa \in E_\kappa$; multiplication is defined by

$$ (x_\lambda, e_\lambda)(y_\mu, f_\mu) = (x_\lambda y_\mu, e_\lambda f_\mu), $$

for any $\lambda, \mu \in Y$, $x_\lambda \in G_\lambda$, $y_\mu \in G_\mu$, $e_\lambda \in E_\lambda$, $f_\mu \in E_\mu$. The identity element of $G_\kappa$, $\kappa \in Y$ will be denoted by $1_\kappa$.

The following result will generalize a theorem of [4] about semilattices of left modules. In patching up next theorem and theorem 1, we actually get a characterization for orthodox bands of commutative groups.

4. THEOREM 2. - Let $S$ be any orthodox band of commutative groups, and let $Z$ be the ring of integers. Let $e$ be any idempotent of $S$, and $x$ and $x'$ mutually inverse elements of maximal subgroup $H_e$. Define mapping $Z \times S \rightarrow S$, $(k, x) \rightarrow kx$ by
Then $S$ is an orthodox band of left $\mathbb{Z}$-modules.

Proof. - Conditions (i), (ii), (iii) and (iv) of 1 are checked by some easy calculations.

5. Definitions and remarks.

Let $S$ be an orthodox band of left $R$-modules, and $\tau$ a congruence on semigroup $S$. The natural homomorphism of $S$ onto $S/\tau$, will be denoted by $\tau$. $\tau$ will be called $R$-stable if, and only if, $x \tau y$ implies $(\alpha x) \tau (\alpha y)$ for every $x, y \in S$ and every $\alpha \in R$. We can define a mapping $R \times (S/\tau) \to S/\tau$, $(\alpha, x) \mapsto \alpha x = \bar{\alpha}$. $S/\tau$ will then be an orthodox band of left $R$-modules.

Let $S$ and $T$ be orthodox bands of left $R$-modules. Mapping $\phi : S \to T$ will be called $R$-linear if, and only if,

(i) $\phi(xy) = (\phi(x))(\phi(y))$ for every $x, y \in S$

(ii) $\phi(\alpha x) = \alpha \phi(x)$ for every $x \in S$ and every $\alpha \in R$.

$\bar{\phi}(S)$ will then be an orthodox band of left $R$-modules.

Subset $A$ of $S$ will be called $R$-stable if, and only if, $\alpha x \in A$ for every $x \in A$ and every $\alpha \in R$. If $\phi$ is an $R$-linear mapping of $S$ into $T$, $\phi(S)$ will be an $R$-stable subsemigroup of $T$, and the kernel of $\phi$ will be an $R$-stable subsemigroup of $S$. Any $R$-stable subsemigroup of an orthodox band of left $R$-modules must of course be an orthodox band of left $R$-modules. If $\tau$ is an $R$-stable congruence on $S$, the union of all $\tau$-classes containing an idempotent will be an $R$-stable subsemigroup of $S$.

Mapping $\bar{\phi} : S \to T$ will be $R$-linear if, and only if, $\bar{\phi}^{-1} \phi$ is an $R$-stable congruence on $S$. Equivalence relation $\tau$ on $S$ is an $R$-stable congruence if, and only if, $\bar{\tau}^{-1} \tau$ is an $R$-linear mapping.

Mapping $\bar{\phi} : S \to E_S$, $x \mapsto Ox$ is an $R$-linear mapping of $S$ onto the band consisting of all idempotents of $S$; $\bar{\phi}^{-1} \phi$ is then the $R$-stable congruence $\%$.

Let $S$ be the spined product of semilattice of commutative groups $Q$ and band $E$. We shall use the same notations as in 3. $Q$ is the greatest inverse semigroup homomorphic image of $S$, and the mapping $\Delta : S \to Q$, $(x_\kappa, e_\kappa) \mapsto x_\kappa$ is a homomorphism of $S$ onto $Q$. We shall put $\Delta^{-1} \Delta = \sigma$. This congruence $\sigma$ is the minimal inverse semigroup congruence on $S$, and we will show that $\sigma$ is $R$-stable. Let $G$ be the greatest group homomorphic image of $Q$, and $\Gamma : Q \to G$, $x_\kappa \mapsto \bar{x}_\kappa$ be a homomorphism of $Q$ onto $G$, $\Gamma^{-1} \Gamma$ being the minimal group congruence on $Q$. If $x_\lambda$ and $y_\mu$ are any elements of $Q$, then $x_\lambda \Gamma^{-1} \Gamma y_\mu$, and only if, there exists a $\kappa \in Y$, $\kappa \leq \lambda \wedge \mu$, such that $x_\lambda \Gamma^{-1} \Gamma y_\mu$. We shall
put \((\Gamma\Delta)^{-1}(\Gamma\Delta) = \rho\); this congruence \(\rho\) is the minimal group congruence on \(S\), and we will show that \(\rho\) is \(R\)-stable.

6. **THEOREM 3.** - The minimal inverse semigroup congruence on an orthodox band of left \(R\)-modules is \(R\)-stable.

**Proof.** - Let \(x_\kappa\) be any element of \(Q\), and let us take any two elements \((x_\kappa, e_\kappa)\) and \((x_\kappa, f_\kappa)\) in \(\Delta^{-1}Ax\). Let \(\alpha\) be any element of \(R\). Since \(\kappa\) is an \(R\)-stable congruence on \(S\), \(\alpha(x_\kappa, e_\kappa)\) belongs to the \(\alpha\)-class \(G_\kappa \times e_\kappa\) of \(S\) containing \((x_\kappa, e_\kappa)\), hence,

\[\alpha(x_\kappa, e_\kappa) = (y_\kappa, e_\kappa) \text{ for some } y_\kappa \in G_\kappa.\]

Analogously,

\[\alpha(x_\kappa, f_\kappa) = (z_\kappa, f_\kappa) \text{ for some } z_\kappa \in G_\kappa.\]

Let \((l_\kappa, g_\kappa)\) be \(E\)-related with \((l_\kappa, e_\kappa)\) and \(\alpha\)-related with \((l_\kappa, f_\kappa)\), and let \((l_\kappa, h_\kappa)\) be \(E\)-related with \((l_\kappa, e_\kappa)\) and \(R\)-related with \((l_\kappa, f_\kappa)\).

Since, by the restriction of \(R \times S \to S\) to \(R \times (G_\kappa \times e_\kappa)\), and \(R \times (G_\kappa \times h_\kappa)\) respectively, \(G_\kappa \times e_\kappa\) and \(G_\kappa \times h_\kappa\) become left \(R\)-modules, we must have

\[\alpha(l_\kappa, g_\kappa) = (l_\kappa, g_\kappa) \text{ and } \alpha(l_\kappa, h_\kappa) = (l_\kappa, h_\kappa).\]

Furthermore, we have

\[\begin{aligned}
(z_\kappa, e_\kappa) &= (l_\kappa, h_\kappa)(z_\kappa, f_\kappa)(l_\kappa, e_\kappa) \\
&= (\alpha(l_\kappa, h_\kappa))(\alpha(x_\kappa, f_\kappa))(\alpha(l_\kappa, g_\kappa)) \\
&= \alpha((l_\kappa, h_\kappa)(x_\kappa, f_\kappa)(l_\kappa, g_\kappa)) \\
&= \alpha(x_\kappa, e_\kappa) = (y_\kappa, e_\kappa),
\end{aligned}\]

hence \(z_\kappa = y_\kappa\), and \(\Delta(\alpha(x_\kappa, e_\kappa)) = \Delta(\alpha(x_\kappa, f_\kappa))\).

7. **COROLLARY 1.** - By mapping \(R \times Q \to Q\), \((\alpha, x_\kappa) \to \alpha x_\kappa = \Delta(\alpha \Delta^{-1} x_\kappa)\), \(Q\) becomes a semilattice of left \(R\)-modules, and \(\Delta\) an \(R\)-linear mapping of \(S\) onto \(Q\).

8. **COROLLARY 2.** - Let \(Q\) be any semilattice of left \(R\)-modules, and \(Y\) the structure semilattice of \(Q\), let \(E\) be a band with the same structure semilattice \(Y\), let \(U_{\kappa \in Y} G_\kappa\) and \(U_{\kappa \in Y} E_\kappa\) be the semilattice decompositions of \(Q\) and \(E\) respectively, let \(S\) be the spined product \(Q \times_Y E\) of \(Q\) and \(E\) over \(Y\). By mapping \(R \times S \to S\), \((\alpha, (x_\kappa, e_\kappa)) \to (\alpha x_\kappa, e_\kappa)\) for every \(\alpha \in R\), and every \(\kappa \in Y\), \(x_\kappa \in G_\kappa\), \(e_\kappa \in E_\kappa\), \(S\) become an orthodox band of left \(R\)-modules. Conversely, any orthodox band of left \(R\)-modules can be so constructed.

9. **COROLLARY 3.** - Let \(S\) be an orthodox normal band of left \(R\)-modules, and let \(S = \bigcup_{\kappa \in Y} S_\kappa\) be the semilattice decomposition of \(S\). For any \(\lambda, \mu \in Y\), \(\lambda \geq \mu\), the structure homomorphism \(Y_{\lambda, \mu}\) is an \(R\)-linear mapping of orthodox rectangular band of left \(R\)-modules \(S_\lambda\) into orthodox rectangular band of left \(R\)-modules \(S_\mu\).
Proof. - In a semilattice of left R-modules the structure homomorphisms are R-linear [6]. The theorem now follows from corollary 2 and from a result about normal bands [10].

10. Remark.

Structure theorems for semilattices of left R-modules [6], together with corollary 2 yield structure theorems for orthodox bands of left R-modules.

11. Theorem 4. - The minimal group congruence on an orthodox band of left R-modules is R-stable.

Proof. - Let \( \tilde{x} \) be any element of \( G \), the greatest group homomorphic image of an orthodox band of left R-modules \( S \). Let us take any two elements \( x_\lambda \) and \( y_\mu \) in \( \Gamma^{-1} \tilde{S} \). There exists \( \kappa \in Y \), \( \kappa \leq \lambda \wedge \mu \), such that \( 1_\kappa x_\lambda = 1_\kappa y_\mu \). Let \( \alpha \) be any element of \( R \). From

\[
(\alpha x_\lambda)(1_\kappa) = (\alpha x_\lambda)(1_\kappa) = (1_\mu)(\alpha y_\mu) = (1_\mu)(\alpha y_\mu) = (1_\mu)(y_\mu),
\]

and \( \alpha x_\lambda \in G_\lambda \), \( \alpha y_\mu \in G_\mu \), we conclude that \( \alpha y_\mu \in \Gamma^{-1} \Gamma(\alpha x_\lambda) \), and thus \( \tilde{\alpha x_\lambda} = \tilde{\alpha y_\mu} \).

This implies that the minimal group congruence \( \Gamma^{-1} \Gamma \) on \( G \) must be R-stable. Consequently, the minimal group congruence \( \Gamma^{-1} \Gamma = \rho \) on \( S \) must be R-stable.

12. Corollary 4. - By mapping \( R \times G \to G \), \( (\alpha, \tilde{x}_\mu) \to \tilde{x}_\mu = \tilde{x}_\mu \), \( G \) becomes a left R-module, and the mapping \( \Gamma \Delta \) an R-linear mapping of \( S \) onto \( G \).


An orthodox band of right R-modules \( S \) can be defined in an analogous way as an orthodox band of left R-modules. Condition (iii) of 1 must then be replaced by (iii)'. \( (\alpha \cdot \beta)x = \beta(\alpha x) \) for every \( \alpha, \beta \in R \) and every \( x \in S \). It will be more convenient to denote mapping \( R \times S \to S \), \( (\alpha, x) \to x\alpha \). (iii)', then, becomes

\[
\alpha(x + \beta) = (x\alpha)\beta \quad \text{for every} \quad \alpha, \beta \in R \quad \text{and} \quad x \in S.
\]

If \( S \) is at the same time orthodox band of left R-modules, and orthodox band of right R-modules, then we shall say that \( S \) is an orthodox band of R-bimodules.

Let \( R^\infty = R \cup \{\infty\} \), and define addition in \( R^\infty \) as follows. For any \( \alpha, \beta \in R \), we put \( \alpha + \beta = \gamma \) in \( R^\infty \) if, and only if, \( \alpha + \beta = \gamma \) in \( R \), and

\[
\alpha + \infty = \infty + \alpha = \infty.
\]

\( R^\infty \) will be a group with "zero" \( \infty \). We next define mapping \( R \times R^\infty \to R^\infty \) by

\[
(\alpha, \beta) \to \alpha \beta = \gamma \quad \text{if, and only if,} \quad \alpha \cdot \beta = \nu \quad \text{in} \quad R,
\]

and

\[
(\alpha, \infty) \to \alpha \infty = \infty.
\]

We also define mapping \( R \times R^\infty \to R^\infty \) by
(\alpha, \beta) \rightarrow \beta \alpha = \gamma \text{ if, and only if, } \beta \cdot \alpha = \gamma \text{ in } R,

and

(\alpha, \omega) \rightarrow \omega \alpha = \omega.

By these two mappings \( R^\omega \) becomes a semilattice of \( R \)-bimodules, the structure semilattice being the two element semilattice. We shall use \( R^\omega \) later in this paper.

The next theorem generalizes a result of [9].

14. THEOREM 5. - Let \( S \) be an orthodox band of left \( R \)-modules, and \( T \) an orthodox band of right \( R \)-modules. Let \( S_{ST} \) be the set of all partial mapping of \( S \) into \( T \). Define a multiplication in \( S_{ST} \) as follows: for every \( \psi, \varphi \in S_{ST} \)

\[
\text{dom } \psi \cap \text{dom } \varphi = \text{dom } \psi \cap \text{dom } \varphi,
\]

and for every \( x \in \text{dom } \psi \) we put \( \psi(x) = (\varphi x)(\psi x) \). Define mapping \( R \times S_{ST} \rightarrow S_{ST} \), \( (\alpha, \psi) \rightarrow \alpha \psi \) by \( \text{dom}(\alpha \psi) = \text{dom } \psi \) and

\[
(\alpha \psi)(x) = (\psi x)\alpha,
\]

for every \( x \in \text{dom } \psi \). \( S_{ST} \) will then be an orthodox band of right \( R \)-modules if, and only if, \( T \) is a semilattice of right \( R \)-modules.

Proof. - For any \( \psi, \varphi \in S_{ST} \) and any \( \alpha \in R \) we have

\[
\text{dom}(\psi \varphi) = \text{dom } \psi \cap \text{dom } \varphi = \text{dom } \psi \cap \text{dom } \varphi = \text{dom}(\psi \varphi) \psi \alpha = \text{dom}(\psi \varphi)(\psi \alpha),
\]

and for any \( x \in \text{dom}(\psi \varphi) \) we have

\[
((\psi \varphi)x)\alpha = ((\psi x)(\varphi x))\alpha = ((\psi x)\alpha)((\varphi x)\alpha) = ((\psi x)(\varphi x))x = ((\psi \alpha)(\varphi \alpha))x,
\]

hence \( (\psi \varphi)\alpha = (\psi \alpha)(\varphi \alpha) \). For any \( \psi \in S_{ST} \) and any \( \alpha, \beta \in R \) we have

\[
\text{dom } \psi(\alpha + \beta) = \text{dom } \psi = \text{dom } \psi \alpha \cap \text{dom } \psi \beta = \text{dom}(\psi \alpha)(\psi \beta),
\]

and, for any \( x \in \text{dom } \psi(\alpha + \beta) \) we have

\[
((\psi(\alpha + \beta))x) = (\psi x)(\alpha + \beta) = ((\psi x)\alpha)((\psi x)\beta) = ((\psi x)(\psi \beta)x = (\psi \alpha)(\psi \beta)x,
\]

hence \( \psi(\alpha + \beta) = (\psi \alpha)(\psi \beta) \). Furthermore,

\[
\text{dom } \psi(\alpha \cdot \beta) = \text{dom } \psi = \text{dom } \psi \alpha = \text{dom}(\psi \alpha) \beta,
\]

and for any \( x \in \text{dom } \psi(\alpha \cdot \beta) \) we have

\[
((\psi(\alpha \cdot \beta))x) = (\psi x)(\alpha \cdot \beta) = ((\psi x)\alpha)(\psi \beta x) = ((\psi \alpha)\beta)x,
\]

hence \( \psi(\alpha \cdot \beta) = (\psi \alpha)\beta \). Finally, \( \text{dom } \psi \| = \text{dom } \psi \), and for any \( x \in \text{dom } \psi \| \) we have

\[
(\psi \| x) = (\psi x)\psi = \psi x,
\]

hence \( \psi = \psi \). We conclude that \( S_{ST} \) is an orthodox band of right \( R \)-modules.

From the definition of the multiplication in \( S_{ST} \) follows that \( S_{ST} \) is commutative if, and only if, \( T \) is commutative. From this, follows the last part of the theorem.

15. THEOREM 6. - Let \( S \) be an orthodox band of left \( R \)-modules, \( S' \) the set of \( R \)-linear mappings of \( S \) into \( R \), and \( S^* \) the set of \( R \)-linear mapping of \( S \) into \( R^\omega \). Then \( S^* \) is an \( R \)-stable subsemigroup of \( S_{SR} \) and \( S^* \) is an \( R \)-stable subsemigroup of \( S_{S,R} \).
Proof. - We show that \( S^* \) is an R-stable subsemigroup of \( S_{S,R}^\infty \). The proof of the rest is quite the same. Let \( x^* \) and \( y^* \) be any elements of \( S^* \). Since \( R^\infty \) is a semilattice of commutative groups, \( x^*y^* \) must be a homomorphism of \( S \) into \( R^\infty \). For any \( x \in S \) and any \( x^* \in S^* \) we shall from now put \( x^*(x) = \langle x, x^* \rangle \). For any \( x \in S \), any \( \alpha \in R \) and any \( x^*, y^* \in S^* \) we then have

\[
\langle \alpha x, x^* y^* \rangle = \langle \alpha x, x^* \rangle + \langle \alpha x, y^* \rangle
\]
\[
= \alpha \langle x, x^* \rangle + \alpha \langle x, y^* \rangle
\]
\[
= \alpha \langle x, x^* \rangle + \langle x, y^* \rangle)
\]
\[
= \alpha \langle x, x^* y^* \rangle.
\]

We conclude that for any \( x^*, y^* \in S^* \), \( x^* y^* \) must be an R-linear mapping of \( S \) into \( R^\infty \), hence \( x^* y^* \in S^* \). \( S^* \) is a subsemigroup of \( S_{S,R}^\infty \).

For any \( x, y \in S \), any \( x^* \in S^* \) and any \( \alpha \in R \) we have

\[
\langle xy, x^* \alpha \rangle = \langle xy, x^* \rangle \alpha
\]
\[
= \langle (x, x^*) + \langle y, x^* \rangle \rangle \alpha
\]
\[
= \langle x, x^* \rangle \alpha + \langle y, x^* \rangle \alpha
\]
\[
= \langle x, x^* \rangle \alpha + \langle y, x^* \alpha \rangle,
\]

hence \( x^* \alpha \) must be a homomorphism of \( S \) into \( R^\infty \). For any \( x \in S \), any \( x^* \in S^* \) and any \( \alpha, \beta \in R \) we have

\[
\langle \beta x, x^* \alpha \rangle = \langle \beta x, x^* \rangle \alpha
\]
\[
= \beta \langle x, x^* \rangle \alpha
\]
\[
= \beta \langle x, x^* \alpha \rangle.
\]

We conclude that for any \( x^* \in S^* \) and any \( \alpha \in R \), \( x^* \alpha \) must be an R-linear mapping of \( S \) into \( R^\infty \). Consequently \( S^* \) must be an R-stable subsemigroup of \( S_{S,R}^\infty \).

16. COROLLARY 5. - \( S^* \) is a semilattice of right \( R \)-modules. The structure semilattice of \( S^* \) is isomorphic with the semilattice of prime ideals of \( S \). The mapping \( 1^* : S \rightarrow R^\infty \), \( x \rightarrow 0 \) is the identity of \( S^* \) and the mapping \( 0^* : S \rightarrow R^\infty \), \( x \rightarrow \infty \) is the zero of \( S^* \).

Proof. - \( R^\infty \) is a semilattice of right \( R \)-modules, hence \( S_{S,R}^\infty \) is a semilattice of right \( R \)-modules. Since \( S^* \) is R-stable in \( S_{S,R}^\infty \), \( S^* \) must be a semilattice of right \( R \)-modules too.

Let \( e^* \) be any idempotent of \( S^* \), then

\[
V_{e^*} = \{ x \in S ; \langle x, e^* \rangle = \infty \}
\]
is a prime ideal of \( S \). For any \( x \in S \setminus V_{e^*} \)

\[
\langle x, e^* \rangle \in R \text{ and } \langle x, e^* \rangle = \langle x, e^* \rangle = \langle x, e^* \rangle + \langle x, e^* \rangle,
\]
hence \( \langle x, e^* \rangle = 0 \). Conversely, let \( P \) be any prime ideal of \( S \), then we can define \( e_P^* \in S^* \) by \( \langle x, e_P^* \rangle = \infty \) for all \( x \in P \), and \( \langle x, e_P^* \rangle = 0 \) for all \( x \in S \setminus P \). Furthermore, if \( e^* \) and \( f^* \) are any two idempotents of \( S^* \), we must have
Consequently, the semilattice $E_{S^*}$ consisting of the idempotents of $S^*$ is isomorphic with the $\cup$-semilattice of all prime ideals of $S$. Since $E_{S^*}$ is isomorphic with the structure semilattice of $S^*$, the result stated in the corollary follows.

17. COROLLARY 6. - $S'$ is a right $R$-module which is an $R$-stable subgroup of $S^*$: $S'$ is the maximal submodule of $S^*$ containing the identity $1^*$ of $S^*$.

Proof. - All elements of $S'$ are $R$-linear mappings of $S$ into $R$, hence, they can be considered as $R$-linear mappings of $S$ into $R^\omega$, and consequently $S' \subseteq S^*$. Since $S'$ is $R$-stable in $S$, and since clearly $S,S^*,R$ is $R$-stable in $S,S^*,R^\omega$, $S'$ must be $R$-stable in $S,S^*,R^\omega$; from this we imply that $S'$ is $R$-stable in $S^*$.

It must be evident that $1^* : S \rightarrow R^\omega$, $x \rightarrow 0$ is the identity of $S'$. Let $x^*$ be any element of $S'$, then $x^*(-1) \in S'$, and for any $x \in S$ we have

$$\langle x, x^*(x^*(-1)) \rangle = \langle x, x^* \rangle + \langle x, x^*(-1) \rangle = \langle x, x^* \rangle + \langle x, x^*(-1) \rangle = 0$$

and analogously

$$\langle x, (x^*(-1))x^* \rangle = 0,$$

hence $x^*(x^*(-1)) = (x^*(-1))x^* = 1^*$. This shows that $x^*$ and $x^*(-1)$ are mutually inverse elements of commutative group $H_{1^*}$, the maximal subgroup of $S^*$ containing $1^*$. For any element $y^* \in H_{1^*}$, we must have $V_{y^*} = \emptyset$, hence any element $y^* \in H_{1^*}$ belongs to $S'$. We can conclude that $H_{1^*} = S'$.

18. THEOREM 7. - Let $S$ be an orthodox band of left $R$-modules and $\tau$ any $R$-stable congruence on $S$. The mapping $\bar{\phi} : (S/\tau)^* \rightarrow S^*$, $\bar{x} \rightarrow \bar{x}^*$ defined by

$$\langle x, \bar{\phi}(\bar{x}^*) \rangle = \langle \bar{x}^* x, x^* \rangle \quad \text{for every } x \in S$$

is an $R$-isomorphism of $(S/\tau)^*$ into $S^*$. Whenever $\iota_S \leq \tau \leq \sigma$, $\sigma$ being the minimal inverse semigroup congruence on $S$, this mapping $\bar{\phi}$ is a surjective $R$-isomorphism of $(S/\iota)^*$ onto $S^*$.

Proof. - Let us suppose that $\bar{x}^*$, $\bar{y}^*$ are any elements of $(S/\tau)^*$, and $x$ any element of $S$. We then have

$$\langle x, \bar{\phi}(\bar{x}^*) \rangle = \langle \bar{x}^* x, x^* \rangle$$

$$= \langle \bar{x}^* x, x^* \rangle + \langle \bar{x}^* x, \bar{y}^* \rangle$$

$$= \langle x, \bar{x}^* \rangle + \langle x, \bar{y}^* \rangle$$

$$= \langle x, (\bar{x}^*)\bar{y}^* \rangle,$$

hence $\bar{\phi}(\bar{x}^*) = (\bar{x}^*)\bar{y}^*$. Let us suppose that $\bar{x}^*$ is any element of $(S/\tau)^*$, $\alpha$ any element of $R$ and $x$ any element of $S$, then

$$\langle x, \bar{\phi}(\bar{x}^*) \rangle = \langle \bar{x}^* x, x^* \rangle$$

$$= \langle \bar{x}^* x, x^* \rangle$$

$$= \langle x, \bar{x}^* \rangle = \langle x, (\bar{x}^*) \rangle,$$

hence $\bar{\phi}(\bar{x}^*) = (\bar{x}^*)\alpha$. Since $\tau^\kappa$ is an $R$-linear mapping of $S$ onto $S/\tau$, ...
We conclude that \( \tilde{x} \) is an R-linear mapping of \((S/T)^*\) into \(S^*\). Let us now suppose that \( \tilde{x}\* , \tilde{y} \* \in (S/T)^* \), and \( \tilde{x}\* = \tilde{y}\* \). If for some \( x \in S/T \) \( \langle \tilde{x} , \tilde{x}\* \rangle \neq \langle \tilde{x} , \tilde{y}\* \rangle \), then for any \( x \in (T)^{-1} x \) we should have
\[
\langle x , \tilde{x}\* \rangle = \langle x , \tilde{x}\* \rangle = \langle \tilde{x} , \tilde{x}\* \rangle \neq \langle \tilde{x} , \tilde{y}\* \rangle = \langle x , \tilde{y}\* \rangle ,
\]
and this is impossible. We conclude that \( \tilde{x}\* = \tilde{y}\* \) implies \( \tilde{x}\* = \tilde{y}\* \), hence \( \tilde{x} \) is an isomorphism of \((S/T)^*\) into \(S^*\).

It will be sufficient to show that the mapping \( \phi : (S/\sigma)^* \rightarrow S^* , \ x^* \rightarrow \tilde{x}\* \)
defined by \( \langle x , \tilde{x}\* \rangle = \langle \tilde{x} , \tilde{x}\* \rangle \) for every \( x \in S \), will be an R-isomorphism of \((S/\sigma)^*\) onto \(S^*\). Let \( x^* \) be any element of \( S^* \), and \((x^* , e^* )\) and \((x^* , f^* )\) any two \( \sigma \)-related elements of \( S \). Since \((x^* , e^* )\) and \((x^* , f^* )\) are \( \sigma \)-related in \( S \), they generate a same principal ideal of \( S \), and thus \( \langle (x^* , e^* ) , x^* \rangle = \infty \) if, and only if, \( \langle (x^* , f^* ) , x^* \rangle = \infty \). Let us suppose that \((x^* , e^* )\) and \((x^* , f^* )\) both belong to \( SV_{x^*} \). Let \((i^* , g^* )\) be \( \sigma \)-related with \((x^* , e^* )\) and \( \rho \)-related with \((i^* , f^* )\), and \((i^* , h^* )\) \( \sigma \)-related with \((x^* , e^* )\) and \( \rho \)-related with \((i^* , f^* )\); \((i^* , g^* )\) and \((i^* , h^* )\) are both \( \sigma \)-related with \((x^* , e^* )\), and \((x^* , f^* )\). Hence \((i^* , g^* ) , (i^* , h^* ) \in S \backslash V_{x^*} \), since these two elements are idempotents of \( S \), and since \( x^* \) is an homomorphism of \( S \) into \( R \), we have
\[
\langle (i^* , e^* ) , x^* \rangle = \langle (i^* , h^* ) , x^* \rangle = (i^* , x^* ) = 0 .
\]
From this follows that
\[
\langle (x^* , e^* ) , x^* \rangle = \langle (x^* , f^* ) , x^* \rangle = \langle (i^* , e^* ) , x^* \rangle + \langle (x^* , f^* ) , x^* \rangle = \langle (i^* , h^* ) , x^* \rangle + \langle (i^* , g^* ) , x^* \rangle = \langle (x^* , f^* ) , x^* \rangle .
\]
In any case \( (x^*)^{-1} x^* \equiv \sigma \). Hence the mapping \( \tilde{x}\* \in (S/\sigma)^* \) defined by
\[
\langle \tilde{x}^* x , \tilde{x}\* \rangle = \langle x , x^* \rangle
\]
for all \( x \in S \) is well-defined, and we shall have \( \tilde{x}\* = x^* \). Thus, in this case \( \phi \) must be surjective.

19. COROLLARY 7. - If \( S \) is an orthodox band of left R-modules, and \( Q \) the greatest inverse homomorphic image of \( S \), then \( S^* \) and \( Q^* \) are R-isomorphic.

20. THEOREM 8. - Let \( S \) be an orthodox band of left R-modules and \( \tau \) any R-stable congruence on \( S \). The mapping \( \psi : (S/\tau)^{'} \rightarrow S' , \ x^{'} \rightarrow \psi(x^{'}) \)
defined by
\[
\langle x , \psi(x^{'}) \rangle = \langle \tilde{x}x , \tilde{x}\* \rangle
\]
for any \( x \in S \) is an R-isomorphism of \((S/\tau)^{'}\) into \( S' \). Whenever \( \tau_S \equiv \tau \equiv \rho , \rho \) being the minimal group congruence on \( S \), this mapping \( \psi \) is a surjective R-isomorphism of \((S/\tau)^{'}\) onto \( S' \).

Proof. - It is clear that mapping \( \psi \) must be the restriction of mapping \( \phi \) (of theorem 7) to maximal submodule \((S/\tau)^{'}\) of \((S/\sigma)^*\), hence \( \psi \) is an R-isomorphism of \((S/\tau)^{'}\) into \( S^* \). Since for every \( x \in S \), and every \( \tilde{x}\* \in (S/\tau)^{'} \) we must have \( \langle \tilde{x}x , \tilde{x}\* \rangle \in R \), we conclude \( \tilde{x} \in S^* \) for every \( \tilde{x}\* \in (S/\tau)^{'} \), thus, \( \psi \) is an R-isomorphism of \((S/\tau)^{'}\) into \( S' \).

It will be sufficient to show that the mapping \( \psi : (S/\rho)^{'} \rightarrow S' , \ x^{'} \rightarrow \tilde{x}\* \)
defined by \( \langle x, \overline{x^*} \rangle = \langle \overline{x^*} x, \overline{x^*} \rangle \) for every \( x \in S \) will be an \( R \)-isomorphism of \( (S/\rho) \) onto \( S' \). Let \( x^* \) be any element of \( S' \). Since \( x^* \) must be a homomorphism of \( S \) into the additive group \( R \), we have \( (x^*)^{-1} x^* = \rho \). Hence the mapping \( \overline{x^*} \in (S/\rho) \) defined by \( \langle \rho^* x, \overline{x^*} \rangle = \langle x, x^* \rangle \) for every \( x \in S \) is well-defined, and we shall have \( \overline{x^*} = x^* \). Thus, in this case \( \gamma \) must be surjective.

21. COROLLARY 8. - If \( S \) is an orthodox band of left \( R \)-modules, \( Q \) the greatest inverse homomorphic image of \( S \), and \( G \) the greatest group homomorphic image of \( S \), then \( S' \) and \( Q' \) are both \( R \)-isomorphic with right \( R \)-module \( G' \) which is the dual of left \( R \)-module \( G \).

22. THEOREM 9. - Let \( S \) be an orthodox band of left \( R \)-modules, and \( S = \bigcup_{\kappa \in Y} \bigcup_{\lambda \in Y} S_{\kappa, \lambda} \times E_{\kappa} \) its semilattice decomposition. For any \( \lambda \in Y \), mapping \( 1^*_\lambda : S \to R^{\infty} \) defined by \( \langle x, 1^*_\lambda \rangle = 0 \) if, and only if, \( x \in \bigcup_{\kappa \geq \lambda} S_{\kappa} \), and \( \langle x, 1^*_\lambda \rangle = \infty \) otherwise, is an idempotent of \( S^* \). The maximal submodule \( H_{1^*_\lambda} \) of \( S^* \) containing \( 1^*_\lambda \) is \( R \)-isomorphic with \( (\bigcup_{\kappa \geq \lambda} S_{\kappa})' \) and with right \( R \)-module \( G_{1^*_\lambda} \), the dual of left \( R \)-module \( G_{1^*_\lambda} \).

Proof. - For any \( \lambda \in Y \), \( \bigcup_{\kappa \geq \lambda} S_{\kappa} \) is an \( R \)-stable subsemigroup of \( S \), and \( G_{1^*_\lambda} \) will be the greatest group homomorphic image of \( \bigcup_{\kappa \geq \lambda} S_{\kappa} \). From corollary 8 follows that \( (\bigcup_{\kappa \geq \lambda} S_{\kappa})' \) and \( G_{1^*_\lambda} \) are \( R \)-isomorphic right \( R \)-modules. It is easy to show that \( S \) is a prime ideal of \( S \). From results in the proof of corollary 5 then follows that \( 1^*_\lambda \) must be an idempotent of \( S^* \). We remark that for any \( x^* \in S^* \), \( s^* \in H_{1^*_\lambda} \) if, and only if,

\[ V_{x^*} = \{ x \in S ; \langle x, x^* \rangle = \infty \} = S \setminus \bigcup_{\kappa \geq \lambda} S_{\kappa} \].

Hence the mapping \( H_{1^*_\lambda} \to (\bigcup_{\kappa \geq \lambda} S_{\kappa})' \), \( x^* \to x^* \in \bigcup_{\kappa \geq \lambda} S_{\kappa} \) is an \( R \)-isomorphism of \( H_{1^*_\lambda} \) onto \( (\bigcup_{\kappa \geq \lambda} S_{\kappa})' \).

23. COROLLARY 9. - We use the same notations as in 22. Let \( Q \) be the greatest inverse semigroup homomorphic image of \( S \) and \( Q = \bigcup_{\kappa \in Y} G_{\kappa} \) its semilattice decomposition. For any \( \lambda \), \( \mu \in Y \), \( \lambda \geq \mu \), let \( t_{\lambda, \mu} \) be the structure homomorphism of \( Q \), and \( t_{\lambda, \mu} \) its transpose. Then \( 1^*_\mu > 1^*_\lambda \) in \( S^* \). Let \( t_{\mu, \lambda}^*: H_{1^*_\lambda} \to H_{1^*_\mu} \) be the structure homomorphism of \( S^* \). For any \( \lambda \in Y \) the mapping \( \psi_{\lambda}: H_{1^*_\lambda} \to G_{1^*_\lambda} \), \( x^* \to \psi_{\lambda} x^* \), defined by

\[ \langle (x_{\kappa}, e_{\kappa}), x^* \rangle = \langle t_{\lambda, \mu} x_{\kappa}, x^* \rangle \] for all \( (x_{\kappa}, e_{\kappa}) \in \bigcup_{\kappa \geq \lambda} S_{\kappa} \),

is an \( R \)-isomorphism of \( H_{1^*_\lambda} \) onto \( G_{1^*_\lambda} \), and the following diagram is commutative.
Proof. - The mapping $U : S \rightarrow G_{\lambda}, (x_{\lambda}, e_{\lambda}) \mapsto \phi_{\lambda} \cdot x_{\lambda}$ is an homomorphism of $U_{\lambda \Rightarrow \lambda} S_{\lambda} \rightarrow G_{\lambda}, (x_{\lambda}, e_{\lambda}) \mapsto \phi_{\lambda} \cdot x_{\lambda}$ is an homomorphism of $U_{\lambda \Rightarrow \lambda} S_{\lambda}$ onto its greatest group homomorphic image $G_{\lambda}$. $\psi_{\lambda}$ must then be an $R$-isomorphism of $H_{1,\lambda}^{*}$ onto $G_{\lambda}^{*}$ by theorem 8.

Let $x^{*}$ be any element of $H_{1,\lambda}^{*}$, and $x_{\lambda}$ any element of $G_{\lambda}$. We proceed to show that

$$\langle x_{\lambda}, t_{\lambda} \phi_{\lambda} x^{*} \rangle = \langle x_{\lambda}, \psi_{\lambda} \phi_{\lambda} x^{*} \rangle.$$

Indeed

$$\langle x_{\lambda}, t_{\lambda} \phi_{\lambda} x^{*} \rangle = \langle x_{\lambda}, \psi_{\lambda} \phi_{\lambda} x^{*} \rangle$$

for all $\lambda \geq \mu$, $\phi_{\lambda} x_{\lambda} = x_{\lambda}^{1, \mu}$, $e_{\lambda} \in E_{\lambda}$

$$= \langle x_{\lambda}, e_{\lambda} \rangle, x^{*} \rangle \quad \text{for all } e_{\lambda} \in E_{\lambda}$$

$$= \langle x_{\lambda}, e_{\lambda} \rangle, x^{*} \rangle \quad \text{for all } e_{\lambda} \in E_{\lambda}$$

$$= \langle x_{\lambda}, e_{\lambda} \rangle, \psi_{\lambda} \phi_{\lambda} x^{*} \rangle \quad \text{for all } e_{\lambda} \in E_{\lambda}$$

We conclude that $t_{\lambda} \phi_{\lambda} x^{*} = \psi_{\lambda} \phi_{\lambda} x^{*}$.

24. COROLLARY 10. - We use the same notations as in 22 and 23. Let the structure semilattice of $S$ be a lattice. Consider $V = \bigcup_{\lambda \in \Omega} G_{\lambda}^{*}$, and define multiplication in $V$ by the following. For any $x^{*}, y^{*} \in V$, $x^{*}, y^{*} \in G_{\lambda}^{*}$, put

$$x^{*} y^{*} = (t_{\lambda} \phi_{\lambda} \mu, \lambda x^{*}) \quad \text{in the usual way. Then } V \text{ is a semilattice of right } R\text{-modules, and there exists an } R\text{-isomorphism of } V \text{ into } S^{*}.$$

If $Y$ satisfies the minimal condition, $V$ must be $R$-isomorphic with $S^{*}$.


Corollaries 9 and 10 show that $S^{*}$ could well be named the dual of $S$. If $Y$ is a lattice, the structure semilattice of $V$ is the $V$-semilattice $Y$. Results of [6] make the connections between structure theorems for $S$ and structure theorems for $V$ more explicit.

Theorem 7 is quite analogous with a result in [5] (§ 5) about the character semigroup of a commutative semigroup, and theorem 9, corollary 9 and corollary 10 are in a certain way analogous with results of [7] and [8] (see also [2], chapter 5).

Next theorem generalizes the concept of the transpose of an $R$-linear mapping.

26. THEOREM 10. - Let $S$ and $T$ be orthodox bands of left $R$-modules, and $\Theta : S \rightarrow T$ an $R$-linear mapping. The mapping $T \Theta : T^{*} \rightarrow S^{*}$, $t^{*} \mapsto T \Theta t^{*}$, defined by $\langle x^{*}, T \Theta t^{*} \rangle = \langle \Theta x^{*}, t^{*} \rangle$ for all $x^{*} \in S$, must be an $R$-linear mapping of $T^{*}$ into $S^{*}$, and $T \Theta (T^{*})$ is embeddable in $(\Theta^{-1} @)^{*} \cong (S^{*})^{*}$.

Proof. - It must be clear that for any $t^{*} \in T^{*}$, we must have $T \Theta t^{*} \in S^{*}$, since
is R-linear. It is not difficult either to show that $T\Theta$ is R-linear.

Let $t^*$ and $v^*$ be any elements of $T^*$, then $t^*|\mathcal{G}$ and $v^*|\mathcal{G}$ are both elements of $(\mathcal{G})^*$, since $\mathcal{G}$ is an R-stable subsemigroup of $T$. From the definition of $T\Theta$ we have that $T\Theta t^* = T\Theta v^*$ if, and only if, $v^*|\mathcal{G} = t^*|\mathcal{G}$. This implies that the mapping $T\Theta(T^*) \to (\mathcal{G})^*$, $T\Theta t^* \to t^*|\mathcal{G}$ is an R-isomorphism of $T\Theta(T^*)$ into $(\mathcal{G})^*$.

27. COROLLARY 11. - Let $S$, $T$ and $\mathcal{G}$ be as in theorem 10. The mapping $t\Theta : T' \to S'$, $t^* \to T\Theta t^*$, defined by $\langle x, T\Theta t^* \rangle = \langle \Theta x, t^* \rangle$ for all $x \in S$, must be an R-linear mapping of $T'$ into $S'$, and $\Theta(T')$ is embeddable in $(S/\mathcal{G})^\Theta$. 

28. COROLLARY 12. - We use the same notations as in 26 and 27. Let $\rho_S$ and $\rho_T$ be the minimal group congruences on $S$ and $T$ respectively. Let $\psi_S : (S/\rho_S)' \to S'$, $x^* \to \psi_S x^*$, be the R-automorphism defined by $\langle x, \psi_S x^* \rangle = \langle \psi_S x, x^* \rangle$ for all $x \in S$, and $\psi_T : (T/\rho_T)' \to T'$, $t^* \to \psi_T t^*$, defined by $\langle t, \psi_T t^* \rangle = \langle \rho_T^{-1} \psi_T t^* \rangle$ for all $t \in S$.

Then there exists an R-linear mapping $\Lambda : (S/\rho_S) \to (T/\rho_T)$ such that the following diagrams are commutative:

![Diagram]

Proof. - Since $\rho_T^{\Theta} \Theta$ is an R-linear mapping of $S$ into left R-module $T/\rho_T$, $(\rho_T^{\Theta} \Theta)^{-1} (\rho_T^{\Theta} \Theta)$ must be an R-stable group congruence on $S$, and, since $\rho_S$ is the minimal group congruence on $S$, we must have $\rho_S \subseteq (\rho_T^{\Theta} \Theta)^{-1} (\rho_T^{\Theta} \Theta)$. This implies that $\Lambda$ is a well-defined R-linear mapping of $S/\rho_S$ into $T/\rho_T$. $\Lambda$ is then an R-linear mapping of $(T/\rho_T)'$ into $(S/\rho_S)'$, which is defined by

$\langle \rho_S^h x, \Lambda t^* \rangle = \langle \Lambda \rho_S^h x, t^* \rangle$ for all $x \in S$, and all $t^* \in (T/\rho_T)'$.

But since $\Lambda \rho_S^h = \rho_S^h \Theta$, we then have

$\langle \rho_S^h x, \Lambda t^* \rangle = \langle \rho_T^{\Theta} \Theta x, t^* \rangle$

$= \langle \Theta x, \psi_T t^* \rangle$

$= \langle x, (\psi_T^{\Theta} \Theta) t^* \rangle$

$= \langle \rho_S^h x, (\psi_T^{\Theta} \Theta)^{-1} t^* \rangle$

for all $x \in S$ and all $t^* \in (T/\rho_T)'$, hence $\Lambda = \psi_T^{\Theta} \Theta$. 

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