JEAN-JACQUES PANSIOT

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DECIDABILITY OF PERIODICITY
FOR INFINITE WORDS (*)

by Jean-Jacques PANSIOT (1)
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Abstract. — We show that it is decidable whether an infinite word generated by iterated morphism is ultimately periodic or not.

Résumé. — Nous montrons qu'on peut décider si un mot infini engendré par morphisme itéré est ultimement périodique.

1. INTRODUCTION

Let $X$ be a finite alphabet and $g$ a morphism of the free monoid $X^*$, prolongable in $u_0 \in X^+$, i.e. such that $g(u_0) = u_0 u$, $u \in X^+$. Then:

$$g^i(u_0) = g^{i-1}(u_0) g^{i-1}(u)$$

and $g$ defines a unique word, in general infinite, denoted by:

$$g^\omega(u_0) = u_0 u g(u) \ldots g^i(u) \ldots$$

An infinite word $M$ is (ultimately) periodic if $M = vw^\omega = vwww \ldots$ for finite words $v$ and $w$. The question of deciding whether $g^\omega(u_0)$ is periodic or not has been raised recently, in connection with the $\omega$-sequence equivalence problem for DOL systems [1, 2], the adherence equivalence problem for DOL languages [3], and with the subword complexity of infinite words [4].

We give a simple proof of decidability for this question, using the notion of elementary morphism (see [5]). After some preliminaries, we give an algorithm for elementary morphisms in section 2 and for arbitrary morphisms in section 3.

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(1) Université Louis-Pasteur, Centre de Calcul de l’Esplanade, 7, rue René-Descartes, 67084 Strasbourg Cedex.

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A subword $u$ of an infinite word $\mathcal{M}$ is biprolongable if and only if there exist distinct letters $x$ and $y$ such that $ux$ and $uy$ are subwords of $\mathcal{M}$. Let $c(n)$ be the number of distinct subwords of $\mathcal{M}$ of length $n$. Then $c(n) \leq c(n+1)$ and $\mathcal{M}$ is periodic if and only if $c(n)$ is bounded. But if $u$ is a biprolongable subword of $\mathcal{M}$, $|u| = n$, then $c(n+1) \geq c(n) + 1$. Hence the following property.

**Lemma 1:** An infinite word $\mathcal{M}$ is ultimately periodic if and only if the length of its biprolongable subwords is bounded.

Let $g : X^* \to X^*$ be a morphism. It is simplifiable if there exist an alphabet $Y$, $|Y| < |X|$ and two morphisms $f : X^* \to Y^*$, $h : Y^* \to X^*$ such that $g = h \circ f$. A morphism $g$ is elementary if and only if it is not simplifiable. In this case $g$ is injective and the set $\{g(x), x \in X\}$ is a code with bounded delay from left to right ([5], p. 131). In particular if $g(xu)$ is a prefix of $g(yv)$, $x \neq y$ then $g(xu)$ has a bounded length.

Finally a letter $x \in X$ is growing (for $g$) if $|g^n(x)|, n \geq 0$ is unbounded. We denote $C \subset X$ the set of growing letters and $B = X \setminus C$ the set of bounded letters.

2. **THE CASE OF ELEMENTARY MORPHISMS**

**Lemma 2:** The infinite word $\mathcal{M} = g^\omega(u_0)$, with $g$ elementary, is ultimately periodic if and only if $\mathcal{M}$ has no biprolongable subword of the form $xu$, $x \in C$, $u \in B^*$.

**Proof:** Assume $xu_1$ is biprolongable. There exist infinite suffixes $xu_1y_1v_1$ and $xu_1z_1w_1$ of $\mathcal{M}$ for distinct letters $y_1$ and $z_1$. But since $\mathcal{M} = g(\mathcal{M})$, $g(y_1v_1)$ and $g(z_1w_1)$ are also suffixes of $\mathcal{M}$. Because $g$ is elementary, their greatest common prefix, $u_2$, is finite, and $g(xu_1)u_2$ is biprolongable. Similarly there exists $u_3$ such that $g(g(xu_1)u_2)u_3$ is biprolongable, and so on. Thus we can construct an infinite sequence of biprolongable words, with unbounded length since $x$ is growing. Hence $\mathcal{M}$ is not periodic by lemma 1.

Conversely, assume that there is no biprolongable factor of the form $xu$. We consider two cases.

First case: $\mathcal{M}$ contains only a finite number of occurrences of growing letters. Then there is only one such occurrence, and $g(u_0) = u_0u$ with $u \in B^*$. Moreover $|g^i(u)|, i \geq 0$, is bounded, and there is a smallest $n$ such that $g^{n+1}(u) = g^n(u), i \leq n$. But then:

$$\mathcal{M} = u_0ug(u) \ldots g^{i-1}(u)[g^i(u) \ldots g^n(u)]^\omega$$

is ultimately periodic.
Second case: \( M \) contains an infinite number of occurrences of growing letters, \( M = \alpha_0 x_1 \alpha_1 x_2 \alpha_2 \ldots, x_i \in C, \alpha_i \in B^* \). Let \( n \) be the smallest integer such that \( x_{n+1} = x_i, i \leq n \). Since there is no biprolongable word of the form \( xu \), we have \( M = \alpha_0 x_1 \alpha_1 \ldots x_{i-1} \alpha_{i-1} [x_i \alpha_i \ldots x_n \alpha_n]^{\omega} \) and \( M \) is ultimately periodic. \( \blacksquare \)

**Corollary 3:** If \( M = g^\omega(u_0) \), with \( g \) elementary, we can decide if \( M \) is ultimately periodic.

**Proof:** Consider the following procedure:

Compute the subset of growing letters, \( C \).

If \( g(u_0) \) contains only one occurrence of letter from \( C \) then \( M \) is ultimately periodic.

If \( g(u_0) \) contains several occurrences of letters from \( C \) then:

- compute the shortest prefix \( p \) of \( M \) containing two occurrences of the same growing letter \( x_i \);

\[
p = \alpha_0 x_1 \alpha_1 \ldots x_i \alpha_i x_{i+1} \ldots x_n \alpha_n x_i;
\]

- for all \( xu \) prefix of some \( x_j \alpha_j, i \leq j \leq n \), check if \( xu \) is biprolongable;

- \( M \) is ultimately periodic if and only if no \( xu \) is biprolongable.

This procedure gives the right answer by lemma 2. Moreover each step is effectively computable: one can determine if a letter is growing, and one can determine if a given word \( xu \) is biprolongable (this comes from the fact that for a given \( n \) one can compute all subwords of length \( n \) of \( M \), see [5], p. 210-212). \( \blacksquare \)

3. **THE CASE OF ARBITRARY MORPHISMS**

**Theorem 4:** It is decidable whether \( M = g^\omega(u_0) \) is ultimately periodic or not for an arbitrary morphism \( g \).

**Proof:** By induction on the size of the alphabet, \( |X| \).

If \( |X| = 1 \) then \( M \) is always periodic.

Assume the theorem is true for alphabets of size \( < |X| \) and let \( g: X^* \rightarrow X^* \) be an arbitrary morphism. If \( g \) is elementary then we decide if \( M \) is periodic by corollary 3. If \( g \) is not elementary we compute \( Y, f: X^* \rightarrow Y^* \) and \( h: Y^* \rightarrow X^* \) such that \( g = h \circ f \) and \( |Y| < |X| \). Let \( g' = f \circ h \), and \( u'_0 = f(u_0) \).

Then:

\[
g'(u'_0) = g'(f(u_0)) = f(g(u_0)) = f(u_0 u) = u'_0 f(u), \]

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where \( g(u_0) = u_0 u \). So \( g'(u'_0) \) starts with \( u'_0 \) and defines an infinite word \( M' = g^{\omega}(u_0) \). Moreover \( M = h(M') \) and \( M' = f(M) \), and \( M \) is ultimately periodic if and only if \( M' \) is. Therefore, by induction hypothesis we can decide if \( M' \) is periodic or not. Since the construction of \( g' \) from \( g \) is effective (see [5], p. 17), we can decide whether \( M \) is periodic or not. This proves the inductive step. 

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