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EACH REGULAR CODE IS INCLUDED IN A MAXIMAL REGULAR CODE (*)

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Abstract. — It is proved that each regular code is included in a maximal regular code. A corollary of this result settles an open question from [R].

Résumé. — On prouve que tout code rationnel est contenu dans un code rationnel maximal. Un corollaire de ce résultat répond à une question ouverte posée dans [R].

INTRODUCTION

A language $C \Sigma^+$ is called a code if $C^*$ is a free submonoid of $\Sigma^*$ with base $C$. The theory of codes initiated by M. Schützenberger [Sch] forms an interesting fragment of formal language theory. A code $C \subseteq \Sigma^+$ is called maximal if, for any $x \in \Sigma^* - C$, $C \cup \{x\}$ is not a code. All codes are subsets of maximal codes and the investigation of maximal codes forms an active research area within the theory of codes (see, e.g., [BPS], [P1], [R] and [SM]). In particular one is often interested in the problem of the following kind: given a code $C$ of type $X$ (e.g. finite or regular) is it possible to find a maximal code $D$ of type $X$ such that $C \subseteq D$?

It was shown in [R] that for finite codes this question gets a negative answer. Since then the following question remained open: is every finite code included in a maximal regular code? Obviously any finite (resp. regular) prefix code is included in a finite (resp. regular) maximal prefix code. Recently it was shown in [P2] that every finite biprefix code is included in a maximal biprefix regular code.

In this paper we provide a positive answer to the above question. As a matter of fact we prove a more general result (theorem 5): each regular code is included in a regular maximal code. We would like to emphasize the

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following: the new result presented in this paper is theorem 5; most of the
other results is in one form or the other (and perhaps in a different terminol-
ygy) retrievable from the literature. However we have decided to make this
paper rather self-contained and to provide all the needed results with their
(sometimes different from the literature) proofs carried out in a "uniform
manner".

We assume the reader to be familiar with basic formal language theory—in
particular with rudimentary theory of regular languages (see, e.g., [S]).

PRELIMINAIRES

We use mostly Standard language theoretic notation and terminology.

For a set $A$, $\#A$ denotes the cardinality of $A$.

For sets $A$, $B$, $A - B$ denotes the set theoretic difference of $A$ and $B$.

For a word $x$, $|x|$ denotes its length and first $(x)$ denotes the first letter
of $x$; if $x = x_1 y x_2$ then $y$ is called a subword of $x$ (also referred to as a
segment or a factor of $x$). The set of all subwords of $x$ is denoted by $\text{sub}(x)$
and for a language $K$, $\text{sub}(K) = \bigcup_{x \in K} \text{sub}(x)$.

A nonempty word $x$ is called bordered if $x = y z y$ for a nonempty word $y$;
otherwise $x$ is called unbordered.

A language $C \subseteq \Sigma^+$ is called a code if every word $y \in C^+$ satisfies the
following condition:

if $y = u_1 \ldots u_n$ and $y = x_1 \ldots x_m$ for $n, m \geq 1$ and $u_1, \ldots, u_n, x_1, \ldots, x_m \in C$
then $n = m$ and $u_i = x_i$ for $1 \leq i \leq n$. (In other words, $y$ has a unique représenta-
tion in $C$; subwords $u_1, \ldots, u_n$ of this representation are referred to as $C$-
blocks of $y$).

A code $C \subseteq \Sigma^+$ is called maximal if, for each $x \in \Sigma^* - C$, $C \cup \{x\}$ is not a
code.

In the sequel of this paper we consider an arbitrary but fixed alphabet $\Sigma
where \sigma = \#\Sigma > 1$; all languages we will consider are over $\Sigma$.

For a language $K$ and a positive integer $n$, $L_n(K) = \{w \in K : |w| = n\}$ and
$\alpha_n(K) = \#L_n(K)$.

We will define now and recall a number of notions concerning lan-
guages— they will be central to our paper.

Let $K \subseteq \Sigma^+$.

(1) $K$ is dense if $x \in \text{sub}(K^*)$ for each $x \in \Sigma^*$.
(2) $K$ is **fast** if there exists a positive integer $n$ such that for each $w \in \text{sub}(K^*)$ there exist $x, y \in \Sigma^*$ such that $|xy| \leq n$ and $xwy \in K^*$.

(3) $K$ is **rich** if there exists a positive integer $e$ such that $\alpha_m(K^*) \geq \sigma^m/e$ for infinitely many positive integers $m$.

**RESULTS**

In this section we investigate the problem how various properties of a code (such as: fast, dense, rich, regular and maximal) influence each other. Once this relationship is explored we can settle the problem of completing a regular code to a regular maximal code.

Our first result is known (see [SM]). However for the sake of completeness we provide its proof (which is different from the proof in [SM]).

**THEOREM 1:** Each maximal code is dense.

**Proof:** First we prove the following result.

**CLAIM 1:** Let $C$ be a code that is not dense. There exists an unbordered word $w_c$ such that $w_c \notin \text{sub}(C^*)$.

**Proof of Claim 1:** Since $C$ is not dense, there exists a word $z \notin \text{sub}(C^*)$. Let $b \in \Sigma$ be such that $b \neq \text{first}(z)$ and let $w_c = zb \mid z\mid$. Clearly $w_c$ is unbordered. Moreover $w_c \notin \text{sub}(C^*)$, because $z \notin \text{sub}(C^*)$.

Thus claim 1 holds. 

Now we prove theorem 1 as follows.

Let $C$ be a maximal code.

Assume to the contrary that $C$ is not dense. Then let $w_c$ be an unbordered word satisfying the statement of claim 1.

Consider $D = C \cup \{w_c\}$. Let $y$ be an arbitrary word in $D^+$. Since $w_c$ is unbordered, $y$ has a unique representation of the form $y = x_0 w_c x_1 w_c \ldots w_c x_m$, where $n \geq 0$ (that is if $y = u_0 w_c u_1 w_c \ldots w_c u_m$ where $m \geq 0$ then $m = n$ and $u_i = x_i$ for $1 \leq i \leq n$). Since $C$ is a code and $w_c \notin \text{sub}(C^*)$, $y$ has a unique representation in $D$. Thus $D$ is a code.

Since $C \subseteq D$ and $w_c \notin \text{sub}(C^*)$ we get a contradiction (to the fact that $C$ is maximal).

Consequently $C$ must be dense and theorem 1 holds. 

**THEOREM 2:** Each rich code is maximal.

**Proof:** Let $C$ be a rich code and let $e$ be a positive integer constant satisfying the definition of richness for $C$. 

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Assume to the contrary that \( C \) is not maximal. Let \( z \) be a word such that \( B = C \cup \{ z \} \) is a code; let \( |z| = t \).

Let \( k \) be a positive integer. Let \( n_1, \ldots, n_k \) be a sequence of positive integers such that:

\[
n_1 < n_2 < \ldots < n_k \quad \text{and} \quad \alpha_{n_i}(C^*) \geq \frac{\sigma^{n_i}}{e}.
\]  

(Since \( C \) is rich and \( e \) satisfies the definition of richness of \( C \), such a sequence exists.)

Consider \( r = n_1 + n_2 + \ldots + n_k + kt \). Clearly:

\[
\alpha_r(B^*) \leq \sigma^r.
\]  

On the other hand let us consider an arbitrary permutation \( i_1, \ldots, i_k \) of the set \( \{1, \ldots, k\} \). Let \( y_{i_1} \in L_{n_{i_1}}(C^*), \ldots, y_{i_k} \in L_{n_{i_k}}(C^*) \) and let

\[
\gamma(i_1, \ldots, i_k) = y_{i_1} z y_{i_2} z \ldots y_{i_k} z.
\]

Since \( B \) is a code, if \((j_1, \ldots, j_k)\) is a permutation of \( \{1, \ldots, k\} \) different from \((i_1, \ldots, i_k)\), then \( \gamma(i_1, \ldots, i_k) \neq \gamma(j_1, \ldots, j_k) \). Consequently from (1) it follows that:

\[
\frac{\sigma^{n_1}}{e} \frac{\sigma^{n_2}}{e} \ldots \frac{\sigma^{n_k}}{e} k! \leq \alpha_r(B^*).
\]

From (2) and (3) it follows that:

\[
k! \leq e^k \sigma^{tk} = (e \sigma^t)^k.
\]

Since \( e \sigma^t \) is a constant (independent of \( k \)), there exists a positive integer \( k_0 \) such that, for all \( s > k_0 \), \( s! > (e \sigma^t)^s \). Consequently (4) yields a contradiction (\( k \) was chosen to be an arbitrary positive integer).

Thus \( C \) must be maximal and theorem 2 holds.

**Theorem 3:** Each regular code is fast.

**Proof:** Obvious. ■

**Theorem 4:** Each dense and fast code is rich.

**Proof:** Let \( C \) be a code that is dense and fast. Then there exists a finite set \( F \) of ordered pairs of words from \( \Sigma^* \) such that for each \( w \in \Sigma^* \) there exists \( (x, y) \in F \) such that \( xwy \in C^* \). Let \( q = \max \{|xy| : (x, y) \in F\} \), \( f = \# F \) and \( d = f \sigma^q \).

**Claim 2:** For each positive integer \( n \) there exists a positive integer \( m \leq n + q \) such that \( \alpha_m(C^*) \geq \sigma^m/d \).
Proof of claim 2: Let for each $w \in \Sigma^*$, pair $(w)$ be a fixed element $(x, y)$ of $F$ such that $xwy \in C^*$.

Let $n$ be a positive integer. Let:

\[ E(n, x, y) = \{ w \in L_n(\Sigma^*): \text{pair}(w) = (x, y) \}. \]

Clearly for some $(x_0, y_0) \in F$, \( \# E(n, x_0, y_0) \geq \sigma^n/f \). Let $p = |x_0 y_0|$. Then $\alpha_{n+p}(C^*) \geq \# E(n, x_0, y_0) \geq \sigma^n/f$.

Hence:

\[ \alpha_{n+p}(C^*) \geq \frac{\sigma^n}{f} = \frac{\sigma^{n+p}}{f\sigma^p} \geq \frac{\sigma^{n+p}}{f\sigma^q} \geq \frac{\sigma^{n+p}}{d}. \]

Thus if we choose $m = n + p$ we get $m \leq n + q$ and claim 2 holds.

Now theorem 4 follows directly from claim 2.

Remark: Theorems 2 and 4 together are more general than theorem 7.4 (due to Schutzenberger) from [E]. However, it is pointed out by D. Perrin in [P3] that a proof of the general case can be retrieved from the proof of theorem 9.3 in [E].

Theorem 5: Let $C$ be a regular code. There exists a code $D$ which is dense, fast, regular and such that $C \subseteq D$.

Proof: Let $C$ be a regular code.

We consider separately two cases.

(i) $C$ is dense. Then the theorem follows from theorem 3 (take $D = C$).

(ii) $C$ is not dense. Then, by claim 1, there exists an unbordered word $w_c$ such that $w_c \notin \text{sub}(C^*)$.

Let:

\[ A = \{ w_c x_1 w_c x_2 \ldots w_c x_n w_c: n \geq 1, x_i \notin C^* \text{ and } w_c \notin \text{sub}(x_i) \} \]

and let $D = C \cup \{ w_c \} \cup A$.

Claim 3: $D$ is a code.

Proof of Claim 3: Let $y \in D^+$. Since $w_c$ is unbordered, $y$ has a unique representation of the form $y = x_1 w_c x_2 w_c \ldots w_c x_n$ (that is we can uniquely distinguish all occurrences of $w_c$ in $y$).

This representation provides the basis for the division of $y$ into $D$-blocks which is obtained as follows:

1. A subword $w_c x_j w_c x_{j+1} \ldots w_c x_{j+l} w_c$ constitutes a $D$-block (corresponding to $A$) if $2 \leq j \leq n-1$, $j+l \leq n-1$, $x_j, \ldots, x_{j+l} \notin C^*$ and $x_{j-1}, x_{j+l+1} \in C^*$; such a $D$-block is referred to as an $A$-block.
(2) All occurrences of \(w_c\) not involved in \(A\)-blocks are also \(D\)-blocks.

(3) All \(x_i\)'s which are not involved in \(A\)-blocks must be in \(C^*\) and so they are uniquely divisible in \(D\)-blocks (really \(C\)-blocks).

The definition of \(A\) and the fact that \(w_c \notin \text{sub}(C^*)\) and \(w_c\) is unbordered guarantee that such a division is unique.

Hence \(D\) is a code and claim 3 holds.

**Claim 4:** \(D\) is dense.

**Proof of claim 4:** Let \(u \in \Sigma^*\).

Consider \(y = w_c u w_c\). Reasoning as in the proof of claim 3 we get a (unique) representation of \(y\) in \(D^+\).

Thus \(D\) is dense and claim 4 holds.

**Claim 5:** \(D\) is regular.

**Proof:** Obvious.

**Claim 6:** \(D\) is fast.

**Proof:** This follows from claim 5 and theorem 3.

Now theorem 5 follows from claims 3 through 5.

Our results yield two interesting corollaries. The first one solves an open problem from the theory of codes (see, e.g., [R] and [P2]). As a matter of fact it provides a more general result: Restivo has asked ([R]) whether an arbitrary finite code can be completed to a maximal regular code—we show that even an arbitrary regular code can be completed to a maximal regular code.

**Corollary 1:** Let \(C\) be a code. If \(C\) is regular, then there exists a code \(D\) such that \(C \subseteq D\), \(D\) is maximal and \(D\) is regular.

**Proof:** Let \(C\) be a regular code.

By theorem 5 there exists a regular code \(D\) such that \(C \subseteq D\), \(D\) is fast and dense.

Thus, by theorem 4, \(D\) is rich and so, by theorem 2, \(D\) is maximal.

Hence corollary 1 holds.

Secondly, we notice that theorems 1 through 4 provide an alternative proof of the theorem by Schützenberger (see [E], p. 94).

**Corollary 2:** Let \(C\) be a regular code. Then \(C\) is maximal if and only if \(C\) is dense.

**Proof:** It follows directly from theorems 1 through 4.
DISCUSSION

We have established a number of relationships between dense, fast, rich, maximal and regular codes. Using these relationships we were able to demonstrate that each regular code is included in a maximal regular code.

In particular we have demonstrated that each rich code is maximal and each maximal code is dense. Hence each rich code is dense. We provide now a "direct" proof of this result—we believe it sheds a different light on this relationship.

COROLLARY 3: Each rich code is dense.

Proof: Let $C$ be a rich code.

Assume that $C$ is not dense. Hence there exists a word $z \notin \text{sub}(C^*)$; let $|z| = t$. Let $n$ be an arbitrary positive integer; $n$ can be represented in the form $n = k_1 t + k_2$ for some $k \geq 0$ and $k_2 < t$. An arbitrary word from $L_n(C^+)$ can be (starting from the left end) divided into $k_1$ consecutive subwords of length $t$ leaving a suffix of length $k_2$. Thus:

$$
\alpha_n(C^+) < (\sigma^t - 1)^{k_1} \sigma^{k_2}.
$$

Consequently:

$$
\frac{\alpha_n(C^+)}{\sigma^n} < \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^n} = \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^{k_2}} = \left(1 - \frac{1}{\sigma^t}\right)^{k_1}.
$$

Hence:

$$
\lim_{n \to \infty} \frac{\alpha_n(C^+)}{\sigma^n} = 0,
$$

which contradicts the fact that $C$ is rich.

Consequently $C$ must be dense and the result holds. ■

To put some of the dependencies we have demonstrated in a better perspective we provide now the following result.

THEOREM 6: There exists a maximal code which is not rich.

Proof: Consider the family of all full binary trees in which leafs are labelled by $a$ and all inner nodes are labelled by $b$. Consider now all postfix notations for these trees—in this way we get the language $P \subseteq \{a, b\}^+$. It is well known that $P$ is a code (every forest of full binary trees has a unique representation in the postfix notation).

Consider an arbitrary word $z \in \{a, b\}^+ - P$. Clearly $a^{\lfloor z \rfloor + 1} z \in P^+$ (we parse $a^{\lfloor z \rfloor + 1} z$ from right to left assigning $+1$ to $a$ and $-1$ to $b$; then each subword yielding by summation weight $+1$ is a tree corresponding to an element of
Hence $P \cup \{z\}$ is not a code, because $a^{z+1}z$ would have two different representations in $P^+$. Thus $P$ is a maximal code.

On the other hand it is known (see, e.g., [F], ch. III, sect. 3) that:

$$\lim_{n \to \infty} \frac{\alpha_n(P^+)}{2^n} = 0.$$ 

(Here one considers random walks on the line of positive integers where $a$ represents a "step up" and $b$ represents a "step down". It turns out that the probability of starting in 0 and not returning to 1 in up to $n$ steps equals 1 in the limit.)

Hence $P$ is not rich and the theorem holds. ■

Perhaps the most significant open question in the area of "extending codes to their maximal counterparts" is (see [P2]): can every biprefix regular code be extended to a maximal biprefix regular code? An answer to this question will certainly make the picture of the whole area clearer.

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REFERENCES