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REPETITIONS IN THE FIBONACCI INFINITE WORD (*)

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Abstract. — Let $\varphi$ be the golden number; we prove that the Fibonacci infinite word contains no fractional power with exponent greater than $2 + \varphi$ and we prove that for any real number $\varepsilon > 0$ the Fibonacci infinite word contains a fractional power with exponent greater than $2 + \varphi - \varepsilon$.

Résumé. — Soit $\varphi$ le nombre d'or; nous prouvons que le mot infini de Fibonacci ne contient pas la puissance fractionnaire d'exposant supérieur à $2 + \varphi$, et nous prouvons qu'il contient des puissances d'exposant supérieur à $2 + \varphi - \varepsilon$, quel que soit le nombre réel $\varepsilon > 0$.

INTRODUCTION

Many papers are concerned with the existence of integer powers in “long enough” words or in infinite words; a classical combinatorial property is whether a given infinite word is $k$ power-free or not, with $k$ a natural number.

No word on a two letters alphabet can avoid a square but it is well known that the Thue infinite word $t$ on a two letter alphabet does not contain cubes and that the Thue infinite word $m$ on a three letter alphabet does not contain squares (see [9], [10]).

The notion of overlap-free word and more generally the notion of fractional power are considered in many papers (see for instance [4], [7], [9], [10]).

In this paper we prove that the Fibonacci infinite word contains no fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2)$ and that for any real number $\varepsilon > 0$ the Fibonacci infinite word contains a fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2) - \varepsilon$.

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To our knowledge this is the first time that this property for a non rational value is looked for in a given infinite word.

DEFINITIONS AND PRELIMINARY RESULTS

We refer to [6] for the terminology.

Let $A$ be an alphabet. We denote by $A^*$ the free monoid on $A$. The elements of $A^*$ are called words and the elements of $A$ are called letters. We denote by $1$ the empty word which is the identity of $A^*$; we also denote by $|v|$ the length of a word $v$.

A word $v$ is a factor of a word $w$ if there exist $u, u' \in A^*$ such that

$$w = uvu'$$

and we say that $v$ is a left factor of $w$ if $u$ is the empty word.

If a word $w$ is of the form

$$w = v \ldots v = v^k$$

with $u \neq 1$, we say that $w$ is a $k$-power of $v$; $k$ is called the exponent of the power and $v$ is the base of the power.

If a word $w$ is of the form

$$w = v \ldots vu = v^ku$$

with $u \neq 1$, $k \geq 1$ and $u$ left factor of $v$, we say that $w$ is a fractional power of $u$ of exponent $e = |w|/|v|$ and $v$ is the base of the power.

An infinite word $s$ on an alphabet $A$ is a map from the set of positive integers into $A$; we denote by $A^\omega$ the set of all infinite words on the alphabet $A$.

A word $v \in A^*$ is a factor of the infinite word $s$ if there exist $u \in A^*$, $s' \in A^\omega$ such that $s = uvs'$. If $u$ is the empty word then $v$ is a left factor of $s$.

The Fibonacci infinite word $f$ on the alphabet $A = \{a, b\}$ is obtained by iterating the morphism $\psi : \{a, b\} \rightarrow \{a, b\}$ given by

$$\psi(a) = ab, \quad \psi(b) = a$$

starting with the letter $a$ (see [1]). Therefore

$$f = abaababaabaabab...$$
We define the sequence of the finite Fibonacci words by the rule:

\[ f_0 = b, \]
\[ f_{n+1} = \psi(f_n). \]

It is easy to see that \( f_{n+2} = f_{n+1} f_n \) and, consequently, the sequence \( |f_n|, n \in \mathbb{N} \) is the sequence of Fibonacci numbers; moreover for any \( n \geq 1 \), \( f_n \) is a left factor of \( f_{n+1} \) and of \( f \).

For \( n \geq 2 \) we denote by \( g_n \) the word \( f_{n-2} f_{n-1} \). It is easy to see that for each \( n \geq 2 \) there exists a word \( v_n \) such that \( f_n = v_n x y \) and \( g_n = v_n y x \) with \( x, y \in \{ a, b \} \)
and \( x \neq y \) and also that \( f_{n+2} = f_n f_n g_{n-1} \).

The following fact is straightforward

Fact. — If \( u \) is a left factor of \( f \) and also of \( g_{n-1} \) then \( u \) is a left factor of \( v_{n-1} \) and, consequently

\[ |u| \leq |v_{n-1}| = |g_{n-1}| - 2 = |f_{n-1}| - 2. \]

In the sequel we will use the following results.

**Proposition 1** (Karhumäki [4]): *The Fibonacci infinite word \( f \) contains no 4-power.*

**Proposition 2** (Séébold [8]): *Let \( v \neq 1 \); if \( v^2 \) is a factor of the Fibonacci infinite word \( f \) then there exists \( n \) such that \( |v| = |f_n| \); more precisely \( v = wz \) with \( zw = f_n \) for some words \( z \) and \( w \), \( |w| > 0 \), i.e. \( v \) is a conjugate of \( f_n \).

Now let \( u \neq 1 \), \( u \in A^* \) and let \( u = x_1 \ldots x_n \), \( x_i \in A \); we denote by \( \hat{u} \) the mirror image of \( u \), that is \( x_n \ldots x_1 \).

We say that a factor \( u \) of \( f \) is **special** if \( u a \) and \( u b \) are both factors of \( f \).

**Proposition 3** (Berstel [1]): *If \( u \) is a special factor of the Fibonacci infinite word \( f \) then \( \hat{u} \) is a left factor of \( f \).

Since the sequence \( |f_n|, n \in \mathbb{N} \), is the sequence of Fibonacci numbers, we have the following proposition.

**Proposition 4** (Hardy and Wright [5]): *For any \( n > 1 \)

\[
\frac{|f_{n+1}| - 2}{|f_n|} = \frac{|f_n| + |f_{n-1}| - 2}{|f_n|} < \frac{\sqrt{5} + 1}{2}
\]
and

$$\lim_{n \to \infty} \frac{|f_n| + |f_{n-1}| - 2}{|f_n|} = \frac{\sqrt{5} + 1}{2}.$$ 

**Proposition 5 (de Luca [2]):** For each $i$ the word $f_i$ is primitive; therefore for each $i$ the conjugates of $f_i$ are distinct.

**Results and Proofs**

Let us prove the following lemma.

**Lemma:** No fractional power with exponent greater than $1 + (\sqrt{5} + 1)/2$ can be a left factor of the Fibonacci infinite word $f$. More precisely, if $v^u$ is a fractional power which is a left factor of $f$ then $v = f_n$ for some $n$ and $|v^u| \leq |f_n| + |f_n| + |f_{n-1}| - 2$.

**Proof:** Let $v^u$ be a fractional power which is a left factor of $f$.

By using Proposition 2 we have that $|v| = |f_n|$ for some $n$, and, consequently $v^u$ is a fractional power which is a left factor of $f$ with length $2|f_n|$. By inspection one can easily see that $n$ is greater than or equal to 3.

As $f_n$ is a left factor of $f$ we have that $v = f_n$ for some $n \geq 3$. Thus $v^u = f_n f_n u$ and either $u$ is a left factor of $f_n$ or $f_n$ is a left factor of $u$.

But for $n \geq 3 f_{n+2} = f_n f_n g_{n-1}$ is a left factor of $f$.

Hence, since $g_{n-1}$ is not a left factor of $f_n$, we have that $u$ is necessarily a left factor of $g_{n-1}$; by the fact

$$|u| \leq |f_{n-1}| - 2.$$

Thus $|v^u| \leq |f_n| + |f_n| + |f_{n-1}| - 2$ and, by Proposition 4,

$$\frac{|v^u|}{|v|} \leq \frac{|f_n| + |f_n| + |f_{n-1}| - 2}{|f_n|} < 1 + \frac{\sqrt{5} + 1}{2}, \quad \square$$

We are now ready to prove our main result.

**Proposition 6:** The Fibonacci infinite word $f$ contains no fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2)$ and, for any real number $\varepsilon > 0$, it contains a fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2) - \varepsilon.$
Proof: Let $vvvu$ be a fractional power factor of $f$. As in $f$ there are no 4 powers (Proposition 1) one can find in $f$ a factor

$$u' xu'' u' xu'' u' xu'' u' y$$

where $u' xu'' = v$, $u$ is a left factor of $u'$, $u'' \in \{a, b\}^*$ and $x, y \in \{a, b\}$ with $x \neq y$.

It follows that $u' xu'' u' xu'' u'$ is a special factor of $f$. By Proposition 3, $\hat{u} \hat{u}'' xu'' \hat{u}'' xu''$ is a left factor of $f$. From the Lemma

$$\frac{|u' u'' xu'' u' xu''|}{|u' u'' x|} = \frac{|vvvu|}{|v|} < 1 + \frac{\sqrt{5} + 1}{2},$$

and, consequently,

$$\frac{|vvvu|}{|v|} \leq \frac{|vvvu'|}{|v|} < 2 + \frac{\sqrt{5} + 1}{2}.$$

At last, for $n \geq 3$, $f_{n+4} = f_{n+1} f_n f_n f_n g_{n-1} f_{n-1} f_n$.

Hence, for $n \geq 3$, $f_n f_n f_n v_{n-1}$ is always a factor of $f$.

Since

$$\frac{|f_n f_n f_n v_{n-1}|}{|f_n|} = 2 + \frac{|f_n| + |f_n-1| - 2}{|f_n|},$$

the second part of the proposition follows from Proposition 4. □

In the proof of the above proposition we used the fact that for $n \geq 3$, $f_n f_n f_n v_{n-1}$ is a factor of $f$. As a consequence all words of the form $wzwzwz$ with $zw = f_n$ and $|z| \leq |v_{n-1}|$ are factors of $f$; by Proposition 5 all these words are distinct. Since $0 \leq |z| \leq |v_{n-1}|$, the number of these words is $|v_{n-1}| + 1$.

Let us suppose that $vvv$ is a factor of $f$ and that $|v| = |f_n|$ for some $n \geq 3$.

By proposition 2, $v = wz$, $|w| > 0$, and $zw = f_n$.

Suppose that $|z| > |v_{n-1}|$; since $f_n = f_{n-1} f_{n-2} = v_{n-1} y f_{n-2}$ with $x, y \in \{a, b\}$ and $x \neq y$, we can write $f_n = v_{n-1} y u w$ with $z = v_{n-1} y u$ and, consequently, $vvv = w v_{n-1} y u w v_{n-1} y u v_{n-1} y u$.

We know that $f_n f_n f_n g_{n-1} = v_{n-1} y u w v_{n-1} y u w v_{n-1} y u w v_{n-1} x y$ is a factor of $f$; thus $w v_{n-1} y u w v_{n-1} y u w v_{n-1} = w v_{n-1} (y u w v_{n-1})^2$ is a special factor and by Proposition 3 its mirror image must be a prefix of $f$. This is impossible by the Lemma because $|w| > 0$.

Hence we have proved the following proposition.
Proposition 7: For \( n \geq 3 \) the number of distinct factors \( v \) of \( f \) with length \(| f_n | \) such that \( vv v \) is also a factor of \( f \) is exactly \(| v_{n-1} | + 1 \). More precisely they are all the words of the form \( wz \) with \( zw = f_n \) and \(| z | \leq | v_{n-1} | \).

Observation: As \( 2 + ((\sqrt{5} + 1)/2) \) is an irrational number it cannot exist a fractional power with exponent equal to it.

In the Thue infinite word \( t \) on a two letters alphabet \( A \) there are clearly squares but there are no overlaps (that is factors like \( xvxvy \), \( x \in A \), \( v \in A^* \)). On the contrary it is easy to see that, for any \( \varepsilon > 0 \), in the Thue infinite word \( m \) on a three letters alphabet there exists a fractional power with exponent greater than \( 2 - \varepsilon \) but it is a classical result that \( m \) is square free.

Remark: Proposition 6 and 7 were firstly proved by using techniques of Sturmian words. Following the suggestion of P. Séébold we tried to find a simpler proof; actually our proof is simpler than the previous one and use only elementary properties of the Fibonacci infinite word.

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