RECURSIVE COALGEBRAS OF FINITARY FUNCTORS

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Abstract. For finitary set functors preserving inverse images, recursive coalgebras \(A\) of Paul Taylor are proved to be precisely those for which the system described by \(A\) always halts in finitely many steps.

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1. Introduction

For finitary endofunctors \(H\) of the category of sets we study recursive coalgebras. A coalgebra for \(H\) is recursive if it admits a unique homomorphism into every algebra for \(H\). This concept stems from the work of Osius [11] (see also Montague [10]) on coalgebras for the power-set functor. For an arbitrary endofunctor the notion of a recursive coalgebra appears in the monograph of Taylor [13] under the name “coalgebra obeying the recursion scheme”, and the name recursive coalgebra stems from a recent paper of Capretta et al. [6]. It was proved by Taylor that whenever a set functor \(H\) preserves inverse images, then recursive coalgebras are precisely the well-founded ones. In the present paper we prove that if \(H\) is, moreover, finitary, then recursive coalgebras are precisely those having the halting property which means that the corresponding systems halt in finitely many steps no matter what the initial state is and what input is processed.

Recall that a coalgebra is a set \(A\) of states together with a function \(\alpha : A \rightarrow HA\) assigning to every state \(a\) the collection \(\alpha(a)\) of all observations about \(a\). For example, if \(H = H\Sigma\) is the polynomial functor of a signature \(\Sigma\), then a coalgebra can be understood as a deterministic system given by a set \(A\) of states...
and by a dynamics

$$\alpha : A \longrightarrow H \Sigma A = \bigoplus_{n \in N} \bigoplus_{\sigma \in \Sigma_n} A^n$$

assigning to every state an expression of the form $\sigma(a_0, \ldots, a_{n-1})$ for some $n$-ary symbol $\sigma$. The states with $n = 0$ are the halting states of the system, the states with $n > 0$ react to an $n$-ary input, they have the output $\sigma$, and $a_0, \ldots, a_{n-1}$ are the successor states. The initial algebra $I_\Sigma$ can be described as the algebra of all finite $\Sigma$-trees (i.e., trees labeled by $\Sigma$ so that an $n$-ary label implies that the node has $n$ children). The systems with a homomorphism into $I_\Sigma$ are precisely those which always halt in finitely many steps; this is called the halting property of the system. Thus, recursive coalgebras are precisely the systems having the halting property.

P. Taylor also mentioned in [13] that every recursive coalgebra $\alpha : A \longrightarrow HA$ satisfies an inductive principle called parametric recursivity in [6] which states that for the endofunctor $H(-) \times A$ the coalgebra $\langle \alpha, id_A \rangle : A \longrightarrow HA \times A$ is recursive. Explicitly: for every morphism $e : HX \times A \longrightarrow X$ there exists a unique morphism $e^\dagger : A \longrightarrow X$ such that the square

$$
\begin{array}{ccc}
A & \xrightarrow{(\alpha, id_A)} & HA \times A \\
\downarrow e^\dagger & & \downarrow He^\dagger \times id_A \\
X & \xleftarrow{e} & HX \times A
\end{array}
$$

commutes. This is the dual concept of the concept of a completely iterative algebra of [9].

We believe that in addition to their theoretical importance our results have many interesting applications which we illustrate with several examples. In particular, in functional programming one often uses the universal property of an initial algebra to provide a semantics of a recursive program. Recursive coalgebras extend that universal property beyond the initial algebra (considered as a coalgebra). So this provides a larger set of tools for semantics of functional programs. For example, divide-and-conquer algorithms like Quicksort can easily be formulated using recursive coalgebras. Furthermore, our characterization of recursive coalgebras gives a necessary and sufficient condition which is easy to check in order to establish recursivity in concrete examples. Finally, parametric recursivity yields an extended universal property of recursive coalgebras that is useful for the semantics of programs where the calling parameter is used not only in the base case of the recursion. This happens frequently, for example in primitive recursion.
The above results hold for every finitary endofunctor $H$ which preserves inverse images or satisfies $H\emptyset = \emptyset$. In case $H$ is a nontrivial, connected functor, we prove that, conversely, if for every coalgebra the equivalence

\[ \text{a homomorphism into the initial algebra exists} \iff \text{recursive} \]

is valid, it follows that $H$ preserves inverse images or satisfies $H\emptyset = \emptyset$.

Preservation of inverse images is a relatively weak assumption on $H$: it is weaker than the (often used, see e.g. [12]) assumption that $H$ preserves weak pullbacks. We provide a complete description of finitary functors preserving inverse images in Section 2 based on the concept of regular equations well known in universal algebra; our characterization appears to be new. We also present simple functors which fail to preserve inverse images but have the above equivalence property. The results of this article were announced at the workshop CALCO-jnr 2005, see [4].

\section{Preservation of inverse images}

\textbf{Assumption 2.1.} Throughout this section $H$ denotes a finitary endofunctor of $\text{Set}$.

\textbf{Notation 2.2.} For every signature $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ we define the polynomial endofunctor $H_\Sigma : X \mapsto \coprod_{n \in \mathbb{N}} \prod_{\sigma \in \Sigma_n} X^n$.

\textbf{Remark 2.3.} Recall that an endofunctor $H$ of $\text{Set}$ is finitary, if it fulfils one of the equivalent conditions:

(i) $H$ preserves directed colimits;

(ii) every element of $HX$, where $X$ is an arbitrary set, lies in the image of $Hm$ for some finite subset $m : M \rightarrow X$;

(iii) $H$ is a quotient of some polynomial functor.

See [3]. For example, the passage (ii) $\implies$ (iii) is provided by the Yoneda Lemma: given $H$ satisfying (ii), let $\Sigma$ be the signature with $\Sigma_n = H(n)$ for all $n \in \mathbb{N}$. By the Yoneda Lemma each element of $\Sigma_n$ corresponds to precisely one natural transformation $(-)^n \rightarrow H$. These natural transformations for every $n$ and every element of $\Sigma_n$ give a natural transformation $\epsilon : H_\Sigma \rightarrow H$, and it is easy to see that each component $\epsilon_X$ is surjective.

\textbf{Definition 2.4.} We call a functor $F$ a \textit{quotient} of a functor $H$, if there is a natural transformation $\epsilon : H \rightarrow F$ with surjective components. In case $H = H_\Sigma$, we call $(\Sigma, \epsilon)$, a \textit{presentation} of $F$.

\textbf{Example 2.5.} The finite-power-set functor $\mathcal{P}_n : X \mapsto \{A \subseteq X \mid A \text{ finite}\}$ is finitary. It has a presentation with $\Sigma$ having a unique $n$-ary symbol $\sigma_n$ for every $n \in \mathbb{N}$, and $\epsilon_X(\sigma_n(x_0, \ldots, x_{n-1})) = \{x_0, \ldots, x_{n-1}\}$.

\textbf{Remark 2.6.} Every finitary functor has a presentation $(\Sigma, \epsilon)$. And $\epsilon$ is completely described by the kernel pairs of each component $\epsilon_X$, where $X$ is a finite set, which are written in the form of equations

\[ \sigma(x_0, \ldots, x_{n-1}) = \varphi(y_0, \ldots, y_{k-1}), \quad (2.1) \]
where \( \sigma \) and \( \varrho \) are operation symbols from \( \Sigma \) and where \( x_0, \ldots, x_{n-1} \) and \( y_0, \ldots, y_{k-1} \) are variables from \( X \), see [3], III.3.3. The above equations are called \( \varepsilon \)-equations. Notice that for every \( \varepsilon \)-equation the function \( \varepsilon_X : H\Sigma X \rightarrow HX \) merges both sides (which are elements of \( H\Sigma X \)).

**Definition 2.7.** A presentation is called *regular* provided that every \( \varepsilon \)-equation has the same set of variables on both sides; more precisely: \( \{x_0, \ldots, x_{n-1}\} = \{y_0, \ldots, y_{k-1}\} \) in the equations (2.1) above.

**Remark 2.8.** Recall that an inverse image of a subobject \( m : B_0 \hookrightarrow B \) under a morphism \( f : A \rightarrow B \) is simply a pullback of \( f \) along \( m \).

A functor preserving such pullbacks is said to preserve inverse images.

Polynomial functors \( H\Sigma \) and the functor \( P_{\text{fin}} \) preserve inverse images. Moreover, products, coproducts and composites of functors preserving inverse images also preserve them.

**Examples 2.9.**

(i) The functor \( (\cdot)^3 \), which is the subfunctor of \( X \rightarrow X \times X \times X \) given by all triples \( (x_1, x_2, x_3) \) which do not have pairwise distinct components, does not preserve weak pullbacks, see [1], but it of course preserves inverse images.

(ii) Let \( R \) be the functor defined on objects by \( RX = \{(x, y) \in X \times X \mid x \neq y\} + \{d\} \) and on morphisms \( f : X \rightarrow X' \) by

\[
Rf(d) = d \quad \text{and} \quad Rf(x, y) = \begin{cases} d & \text{if } f(x) = f(y) \\ (f(x), f(y)) & \text{else.} \end{cases}
\]

This functor does not preserve inverse images, consider e.g.

\[
\begin{array}{ccc}
\emptyset & \rightarrow & \{0\} \\
\downarrow & & \downarrow \\
\{0, 1\} & \xrightarrow{\text{const}} & \{0, 1\}
\end{array}
\]

(for the elements \( (0, 1) \in R\{0, 1\} \) and \( d \in R\emptyset \) there is no suitable element of \( R\emptyset \)).

**Theorem 2.10.** A finitary endofunctor \( H \) of Set preserves inverse images iff it has a regular presentation.
Proof. (1) Let $H$ preserve inverse images. Recall from [3], VII.2.5, that a presentation $\varepsilon : H_\Sigma \to H$ is minimal provided that no $n$-ary operation of $\Sigma$ can be substituted by an operation of arity $k < n$. More precisely, that means that for every $n$-ary $\sigma \in \Sigma$ the element

$$\hat{\sigma} = \varepsilon_n(\sigma(0, 1, \ldots, n - 1)) \in H_n \quad \text{(where } n = \{0, 1, \ldots, n - 1\})$$

does not lie in the image of $H_r$ for any function $r : k \to n$ with $k < n$. Every finitary functor obviously has a minimal presentation: every operation $\sigma$ with $\hat{\sigma} \in H_r([H_k])$ can be substituted by a $k$-ary operation.

We prove that every minimal presentation of $H$ is regular. In fact, let

$$\sigma(x_0, \ldots, x_{n-1}) = \varrho(y_0, \ldots, y_{k-1})$$

be an $\varepsilon$-equation. We derive a contradiction from the assumption, that

$$x_{i_0} \not\in \{y_0, \ldots, y_{k-1}\}$$

for some $i_0$. By symmetry, this proves the regularity. Let

$$B = \{ i \in n \mid x_i \not\in \{y_0, \ldots, y_{k-1}\} \} \neq \emptyset.$$

Consider the $n$-tuple $(x_0, \ldots, x_{n-1})$ as a function $x : n \to X$, and denote by $\bar{x} : n - B \to \bar{X}$ its domain-codomain restriction, where $\bar{X} = X - \{x_i \mid i \in B\}$. For the inclusion map $v : \bar{X} \to X$, form the inverse image

The element $\sigma(0, 1, \ldots, n - 1)$ of $H_\Sigma(n)$ is mapped by $\varepsilon_n$ to $\hat{\sigma}$ and the element $\varrho(y_0, \ldots, y_{k-1})$ of $H_\Sigma \bar{X}$ is mapped by $\varepsilon_{\bar{X}}$ to

$$\varepsilon_{\bar{X}}(\varrho(y_0, \ldots, y_{k-1})) = \varepsilon_{\bar{X}}(\sigma(x_0, \ldots, x_{n-1})) \in H_{\bar{X}}.$$

Thus in the pullback

the elements $\hat{\sigma}$ and $\varepsilon_{\bar{X}}(\varrho(y_0, \ldots, y_{k-1}))$ are mapped by $Hx$ and $ Hv$, respectively, to the same element of $H_{\bar{X}}$. This implies that $\hat{\sigma}$ lies in the image of $Hw$, in contradiction to the minimality of the presentation $\varepsilon$. 

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(2) Let \( H \) have a regular presentation. Suppose we have an inverse image
\[
\begin{array}{c}
X_0 \\ f_0 \downarrow \\
\downarrow w \\
Y_0 \\
\end{array}
\]
where \( v \) and \( w \) are inclusion maps, and two elements
\[
a = \varepsilon_X(\sigma(x_0, \ldots, x_{n-1})) \in HX
\]
\[
b = \varepsilon_0(\rho(y_0, \ldots, y_{k-1})) \in HY_0
\]
with \( Hf(a) = Hv(b) \). Then
\[
\sigma(f(x_0), \ldots, f(x_{n-1})) = \rho(y_0, \ldots, y_{k-1})
\]
is an \( \varepsilon \)-equation because
\[
\varepsilon_Y(\sigma(f(x_0), \ldots, f(x_{n-1}))) = \\
\varepsilon_Y \cdot Hf(\sigma(x_0, \ldots, x_{n-1})) = \\
Hf(\varepsilon_X(\sigma(x_0, \ldots, x_{n-1}))) = \\
Hf(a) = Hv(b) = \\
Hv(\varepsilon_0(\rho(y_0, \ldots, y_{k-1}))) = \\
\varepsilon_Y(\rho(y_0, \ldots, y_{k-1})).
\]
Consequently, \( \{f(x_i) \mid 0 \leq i \leq n - 1\} = \{y_j \mid 0 \leq j \leq k - 1\} \subseteq Y_0 \). Therefore, the subset \( X_0 = f^{-1}(Y_0) \) contains all the variables of \( \sigma(x_0, \ldots, x_{n-1}) \). Thus the element \( a_0 = \varepsilon_X(\sigma(x_0, \ldots, x_{n-1})) \) of \( HX_0 \) fulfils \( Hw(a_0) = a \) and \( Hf_0(a_0) = b \).

3. Recursive coalgebras

**Notation 3.1.** Throughout this section \( H \) denotes a finitary endofunctor of \( \text{Set} \). Recall from [5] that \( H \) has a terminal coalgebra
\[
\tau : T \longrightarrow HT
\]
and an initial algebra
\[
\varphi : HI \longrightarrow I.
\]
We consider \( I \) as a coalgebra via \( \varphi^{-1} \). (Recall that \( \varphi \) is invertible due to Lambek’s Lemma, see [8].) We denote by \( u : I \longrightarrow T \) the unique coalgebra homomorphism.
Example 3.2. For a polynomial functor $H_\Sigma$ we can describe a terminal coalgebra $T_\Sigma$ as the coalgebra of all $\Sigma$-trees and an initial algebra $I_\Sigma$ as the algebra of all finite $\Sigma$-trees. A coalgebra $\alpha : A \rightarrow H_\Sigma A$ yields the unique homomorphism $h : A \rightarrow T_\Sigma$ assigning to every state the tree unfolding.

Definition 3.3. We say that a $H_\Sigma$-coalgebra $A$ has the halting property, if every tree in the image of the unique homomorphism $h : A \rightarrow T_\Sigma$ is finite.

Example 3.4 (Ex. 3.2 continued). If a system $A$ has the halting property, then it halts after finitely many steps (no matter what the initial state is and what input string comes), and vice versa. This property becomes trivial if $\Sigma$ has no constant symbols: then $I = \emptyset$ and only the empty coalgebra has the halting property.

Definition 3.5 (see [6, 13]). A coalgebra $(A, \alpha)$ is called recursive if for every algebra $(X, e)$ there exists a unique coalgebra-to-algebra morphism $e^\dagger : A \rightarrow X$:

$\begin{align*}
A & \xrightarrow{\alpha} HA \\
X & \xleftarrow{e} HX \\
\downarrow e^\dagger & \downarrow He^\dagger \\
\end{align*}$

A coalgebra $(A, \alpha)$ is called parametrically recursive if for every morphism $e : HX \times A \rightarrow X$ there exists a unique morphism $e^\dagger : X \rightarrow A$ such that the diagram (1.1) commutes.

Remarks 3.6.
(i) It is obvious that the implications

parametrically recursive $\Rightarrow$ recursive $\Rightarrow$ has a homomorphism into $I$

hold for all endofunctors $H$: for the first one, turn every algebra $e : HX \rightarrow X$ into a morphism

$\begin{align*}
HX \times A & \xrightarrow{\text{out}} HX \\
\downarrow e & \downarrow X.
\end{align*}$

(ii) The converse implications need not hold. In fact, for the functor $R$ of 2.9(ii) both fail. Observe that here $I = T = 1$, thus every coalgebra has a homomorphism into $I$. However, the coalgebra $A = \{0, 1\}$ with

$\alpha(0) = (0, 1) \quad \text{and} \quad \alpha(1) = d$

is not recursive. In fact, let $e : RX \rightarrow X$ be any algebra which contains an element $x \in X$ such that $e(x, y) = e(y, x) = x$ for $x \neq y = e(d)$. Then any candidate of $e^\dagger : A \rightarrow X$ must satisfy $e^\dagger(1) = y$. But, there are two possible choices $e^\dagger(0) = y$ and $e^\dagger(0) = x$.

And the recursive coalgebra $B = \{0, 1\}$ with

$\beta(0) = \beta(1) = (0, 1)$
is not parametrically recursive. In fact, recursivity is easily seen: for every algebra $e : RX \to X$ the only candidate of $e^! : B \to X$ sends both 0 and 1 to $y = e(d)$. But consider any morphism $e : RX \times \{0, 1\} \to X$ such that $RX$ contains more than one pair $(x_0, x_1), x_0 \neq x_1$, with $e((x_0, x_1), i) = x_i$ for $i = 0, 1$. Each such pair yields $e^! : B \to X$ by $e^!(i) = x_i$. Thus, $B$ is not parametrically recursive.

Remark 3.7. In the definition of recursive coalgebras the uniqueness of the morphism cannot be lifted. In fact, a coalgebra with a (not necessarily unique) homomorphism into every algebra is precisely a coalgebra with a homomorphism into $I$. So the non-recursive coalgebra $A$ of Remark 3.6(ii) has, for every algebra $e : RX \to X$, a coalgebra-to-algebra morphism, e.g., the constant function with value $e(d)$. Notice that our result of Theorem 3.17 shows that the uniqueness can be lifted whenever $H$ preserves inverse images.

Remark 3.8. The equivalence of the conditions (i), (iii) and (iv) in the following theorem can be deduced from results of Taylor [13], see Proposition 6.3.9, Theorem 6.3.13, Corollary 6.3.6 and Exercise 6.24. We present a (short) full proof for the sake of completeness:

**Theorem 3.9.** For every $\Sigma$-coalgebra $A$ the following conditions are equivalent:

(i) $A$ is recursive;

(ii) $A$ has the halting property;

(iii) a coalgebra homomorphism from $A$ to $I_\Sigma$ exists; and

(iv) $A$ is parametrically recursive.

**Proof.** The equivalence of (iii) and (ii) is obvious from the fact that the unique coalgebra homomorphism $A \to T_\Sigma$ assigns to every state the tree-unfolding. And $A$ has the halting property if and only if the unique homomorphism from $A$ to $T_\Sigma$ factors through $u : I_\Sigma \to T_\Sigma$.

It remains to prove (iii) $\Rightarrow$ (iv). Given $e : H_\Sigma X \times A \to X$, we are to prove that there exists precisely one $e^! : A \to X$ equal to $e \cdot (H_\Sigma e^! \times id_A) \cdot (\alpha, id_A)$. We start with a homomorphism

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & H_\Sigma A \\
\downarrow{h} & & \downarrow{H_\Sigma h} \\
I_\Sigma & \xrightarrow{\varphi_\Sigma^{-1}} & H_\Sigma I_\Sigma
\end{array}
$$

Then $A = \bigcup_{i \in \mathbb{N}} A_i$ where $A_0$ are the halting states,

$$
A_0 = \{ a \in A \mid \alpha(a) \in \Sigma_0 \}
$$

and given $A_i$ then

$$
A_{i+1} = A_i \cup \{ a \in A \mid \alpha(a) \in H_\Sigma A_i \}.
$$
In fact, since $h$ is a homomorphism, it is easy to prove by induction on $i$ that $A_i$ is the inverse image of the set of all $\Sigma$-trees of depth $\leq i$ under $h$, therefore, every element of $A$ lies in some $A_i$.

The morphism $e^\dagger$ is uniquely determined

(a) on $A_0$, since if $\alpha(a) = \sigma \in \Sigma_0$, then $e^\dagger(a) = e(H \Sigma e^\dagger(\sigma), a) = e(\sigma, a);$

(b) on $A_{i+1}$ whenever it is uniquely determined on $A_i$ since if $\alpha(a) = \sigma(a_0, \ldots, a_{n-1})$ for some $\sigma \in \Sigma_n$ and $a_t \in A_i$ with $0 \leq t < n$, then

$$e^\dagger(a) = e(H \Sigma e^\dagger(\sigma(a_0, \ldots, a_{n-1})), a) = e(\sigma(e^\dagger(a_0), \ldots, e^\dagger(a_{n-1})), a).$$

Therefore, $A$ is parametrically recursive.

□

**Example 3.10.** The functor

$$HX = X + 1$$

has unary algebras with a constant as $H$-algebras, and partial unary algebras as $H$-coalgebras. The coalgebra $\mathbb{N}$ of natural numbers with the partial operation $n \mapsto n - 1$ (defined iff $n > 0$) obviously has the halting property. Consequently, it is parametrically recursive. Thus every function

$$e = [u, v] : HX \times \mathbb{N} = X \times \mathbb{N} + \mathbb{N} \longrightarrow X$$

(with $u : X \times \mathbb{N} \longrightarrow X$ and $v : \mathbb{N} \longrightarrow X$) defines a unique sequence

$$e^\dagger : \mathbb{N} \longrightarrow X, \quad e^\dagger(n) = x_n$$

in $X$ such that the diagram (1.1) commutes, which means that $x_0 = v(0)$ and $x_{n+1} = u(x_n, n+1)$. For example, the factorial function is then given by the choice $X = \mathbb{N}$; $u(n, m) = n \cdot m$ and $v(0) = 1$.

**Example 3.11.** For the functor $H$ given by

$$HX = X \times X + 1$$

$H$-algebras are the algebras on one binary operation and one constant. Coalgebras are deterministic systems with a binary input and with halting states (expressed by the inverse image of the right hand summand 1 under the dynamics $\alpha : A \longrightarrow A \times A + 1$).

The coalgebra $\mathbb{N}$ of natural numbers with halting states 0 and 1 and dynamics $\alpha : n \mapsto (n-1, n-2)$ for $n \geq 2$ obviously has the halting property. Consequently, $\mathbb{N}$ is parametrically recursive.

To define the Fibonacci sequence, consider the morphism

$$e : H\mathbb{N} \times \mathbb{N} = \mathbb{N}^3 + \mathbb{N} \longrightarrow \mathbb{N}$$

given by

$$(i, j, k) \mapsto i + j \quad \text{and} \quad n \longmapsto \begin{cases} a_0 & n = 0 \\ a_1 & n = 1 \\ 0 & n \geq 2. \end{cases}$$
We know that there is a unique sequence $e^i$ such that the diagram (1.1) commutes, which means $x_0 = a_0$, $x_1 = a_1$ and $x_{n+2} = x_{n+1} + x_n$.

**Example 3.12** (Quicksort, see [6]). Let $A$ be any linearly ordered set (of data elements). Then Quicksort is usually given in terms of the following recursive definition

\[ q\text{sort} : A^* \longrightarrow A^* \]

\[ \varepsilon \longmapsto \varepsilon \]

\[ a \cdot w \longmapsto q\text{sort}(w_{\leq a}) \ast (a \cdot q\text{sort}(w_{> a})) , \]

where $A^*$ is the set of all lists on $A$, $\varepsilon$ is the empty list, $\ast$ is the concatenation of lists and $w_{\leq a}$ and $w_{> a}$ denote the lists of those elements of $w$ which are less than or equal, or greater than $a$, respectively. Now consider the functor $HX = A \times X \times X + 1$, where $1 = \{\bullet\}$, and form the coalgebra

\[ q\text{split} : A^* \longrightarrow A \times A^* \times A^* + 1 \]

\[ \varepsilon \longmapsto \bullet \]

\[ a \cdot w \longmapsto (a, w_{\leq a}, w_{> a}) . \]

This coalgebra obviously has the halting property. Thus, for the $H$-algebra

\[ q\text{merge} : A \times A^* \times A^* + 1 \longrightarrow A^* \]

\[ \bullet \longmapsto \varepsilon \]

\[ (a, w, v) \longmapsto w \ast (a v) \]

there exists a unique function $q\text{sort}$ on $A^*$ such that

\[ q\text{sort} = q\text{merge} \cdot H(q\text{sort}) \cdot q\text{split}. \]

Notice that the last equation reflects the idea that Quicksort is a “divide-and-conquer”-algorithm. The coalgebra structure $q\text{split}$ divides a list into two parts $w_{\leq a}$ and $w_{> a}$, then $H(q\text{sort})$ sorts these two smaller lists, and finally in the “combine”-step (or “conquer”-step) the algebra structure $q\text{merge}$ merges the two sorted parts to obtain the desired whole sorted list.

Similarly, functions defined by parametrical recursivity, see Diagram (1.1), can be understood as “divide-and-conquer”-algorithms, where the “combine”-step is allowed to access the original parameter additionally. For instance, in our current example the “divide”-step $(q\text{split}, id_A)$ produces the pair consisting of $(a, w_{\leq a}, w_{> a})$ and the original parameter $a \cdot w$, and the “combine”-step which is given by an algebra $HX \times A^* \longrightarrow X$ will by the commutativity of (1.1) get $a \cdot w$ as its right-hand input.

**Definition 3.13.** Let $\varepsilon : H_S \longrightarrow H$ be a presentation of a set functor $H$. A coalgebra $\alpha : A \longrightarrow HA$ for $H$ is said to be presented by a $H_S$-coalgebra $\tilde{\alpha} : A \longrightarrow H_S A$ if $\alpha = \varepsilon_A \cdot \tilde{\alpha}$. If some presentation of $A$ has the halting property (w.r.t. $H_S$), we say that the $H$-coalgebra $A$ has the halting property.

**Observation 3.14.** Let $\varepsilon : H_S \longrightarrow H$ be a presentation, and $\alpha : A \longrightarrow HA$ be a coalgebra. Choose any $m : HA \longrightarrow H_S A$ with $\varepsilon_A \cdot m = id_{HA}$ and consider
A as a Σ-coalgebra via \( \bar{\alpha} = m \cdot \alpha \). Clearly, this is a presentation of \( A \), and \( A \) is a (parametrically) recursive coalgebra for \( H \) if it is (parametrically) recursive for \( H_\Sigma \). In fact, given \( e : HX \times A \to X \), then morphisms \( f = e^\dagger \) for \( H \) are precisely the morphisms \( f = \bar{e}^\dagger \) for \( H_\Sigma \), where \( \bar{e} = e \cdot (\varepsilon_X \times id_A) \):

\[
\begin{array}{ccc}
A & \xrightarrow{(\alpha, id_A)} & H_\Sigma A \times A \\
f & \downarrow & \downarrow H_\Sigma f \times id_A \\
X & \xrightarrow{e} & H_\Sigma X \times A \\
& \varepsilon_A \times id_A & H_\Sigma f \times id_A
\end{array}
\]

In fact, the outer square of this diagram commutes iff the left-hand inner square does since all other parts trivially commute.

**Remarks 3.15.**

(i) For every presentation \( \varepsilon : H_\Sigma \to H \) we have the initial \( H \)-algebra \( I \) as a quotient of the initial Σ-algebra \( I_\Sigma \) via the unique Σ-algebra homomorphism

\[
i : I_\Sigma \to I,
\]

where \( I \) is considered as the Σ-algebra

\[
H_\Sigma I \xrightarrow{\varepsilon} HI \xrightarrow{\varphi} I.
\]

In fact, \( I \) can be considered as the quotient of the Σ-algebra \( I_\Sigma \) modulo the congruence generated by \( \varepsilon \)-equations, see Remark 2.6.

(ii) Let \( \alpha : A \to H A \) be a coalgebra with a presentation \( \bar{\alpha} : A \to H_\Sigma A \). Every homomorphism \( f : A \to I_\Sigma \) of \( H_\Sigma \)-coalgebras defines a homomorphism \( i \cdot f : A \to I \) of \( H \)-coalgebras. In fact, from the equations \( i \cdot \varphi = (\varphi \cdot \varepsilon_I) \cdot H_\Sigma i \) and \( \varphi^{-1} \cdot f = H f \cdot \bar{\alpha} \) we easily derive \( \varphi^{-1} \cdot f = H f \cdot \alpha \).

(iii) We also have the terminal \( H \)-coalgebra \( T \) as a quotient of the terminal Σ-coalgebra \( T_\Sigma \) via the unique \( H \)-coalgebra homomorphism

\[
j : T_\Sigma \to T,
\]

where \( T_\Sigma \) is considered as the \( H \)-coalgebra

\[
T_\Sigma \xrightarrow{\tau_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon T_\Sigma} HT_\Sigma.
\]

In fact, as proved in [2], \( T \) can be considered as the quotient of the Σ-coalgebra \( T_\Sigma \) modulo infinite application of \( \varepsilon \)-equations.

Finally, for every functor \( H \) we have the unique coalgebra homomorphism

\[
u : I \to T.
\]
In case \( H = H_\Sigma \) we denote it by \( u_\Sigma : I_\Sigma \longrightarrow T_\Sigma \); this is the inclusion map (of all finite \( \Sigma \)-trees into all \( \Sigma \)-trees).

**Lemma 3.16.** If \( H \) is a finitary functor preserving inverse images, then a regular presentation leads to a pullback

\[
\begin{array}{ccc}
I_\Sigma & \xrightarrow{u_\Sigma} & T_\Sigma \\
\downarrow{i} & & \downarrow{j} \\
I & \xrightarrow{u} & T
\end{array}
\]

**Proof.** It is quite easy to show that \( j \cdot u_\Sigma \) and \( u \cdot i \) are both \( H \)-coalgebra homomorphisms, and since \( T \) is the terminal \( H \)-coalgebra, we obtain that they are equal. Given \( \Sigma \)-trees \( s \in I_\Sigma \) and \( t \in T_\Sigma \) with \( u(i(s)) = j(t) \), it is our task to show that \( t \in I_\Sigma \)—it then follows that the above square is a weak pullback, and since \( u_\Sigma \) is a monomorphism, it is a pullback. The proof is an easy induction on the depth \( n \) of the finite tree \( s \): we prove that \( t \) and \( s \) have the same depth. The equality \( u(i(s)) = j(t) \) implies, due to the regularity of the presentation, that we can obtain \( t \) from \( s \) by applying \( \varepsilon \)-equations on (subtrees of) nodes of \( s \). Since \( s \) is finite, it is sufficient to consider one \( \varepsilon \)-equation applied to one node of \( s \).

Case \( n = 0 \): the regularity of the presentation tells us that since \( s \) is a nullary symbol, every \( \varepsilon \)-equation with \( s \) on one side has a constant symbol on the other side. Thus, \( t \) is a nullary symbol.

Induction step: if the node of \( s \) to which the given \( \varepsilon \)-equation is applied is not the root, use the induction hypothesis. And if it is the root, then we consider the form

\[
\sigma(x_0, \ldots, x_{m-1}) = g(y_0, \ldots, y_{k-1})
\]

of the \( \varepsilon \)-equation used, see Remark 2.6: it follows that

\[
s = \sigma(s_0, \ldots, s_{m-1})
\]

for trees \( s_0, \ldots, s_{m-1} \), and since the variables \( y_0, \ldots, y_{k-1} \) form the same set as \( x_0, \ldots, x_{m-1} \), we conclude that \( t \) has the root labeled by \( g \) and has the same set of children as \( s \), thus, \( t \) has the same depth as \( s \). \( \square \)

**Theorem 3.17.** Let \( H \) be a finitary endofunctor of \( \text{Set} \) preserving inverse images. Then for every \( H \)-coalgebra \( A \) the following conditions are equivalent:

(i) \( A \) is recursive;
(ii) \( A \) has the halting property;
(iii) a coalgebra homomorphism from \( A \) to \( I \) exists; and
(iv) \( A \) is parametrically recursive.
Proof. (i) ⇒ (iii) is trivial.

(iii) ⇒ (ii) Let $h : A \rightarrow I$ be a homomorphism. For the coalgebra $\alpha : A \rightarrow HA$ choose a presentation by putting $\bar{\alpha} = m \cdot \alpha : A \rightarrow H\Sigma A$ where $m : HA \rightarrow H\Sigma A$ is a morphism with $\varepsilon_A \cdot m = id_{HA}$. Let $k : A \rightarrow T\Sigma$ be the unique $\Sigma$-coalgebra homomorphism from $(A, m \cdot \alpha)$ to $(T\Sigma, \tau\Sigma)$. Then $u \cdot h$ and $j \cdot k$ are both $H$-coalgebra homomorphisms from $(A, \alpha)$ to $(T, \tau)$, in fact, for $j \cdot k$ consider the commutative diagram

\[
\begin{array}{cccccccccc}
A & \xrightarrow{\alpha} & HA & \xrightarrow{m} & H\Sigma A & \xrightarrow{\varepsilon_A} & HA \\
& \downarrow{k} & \downarrow{Hk} & & \downarrow{Hk} & & \\
T\Sigma & \xrightarrow{\tau\Sigma} & H\Sigma T\Sigma & \xrightarrow{\varepsilon\tau\Sigma} & H\Sigma T\Sigma \\
& \downarrow{j} & \downarrow{Hj} & & \downarrow{Hj} & & \\
T & \xrightarrow{\tau} & HT & \xrightarrow{\varepsilon\tau} & HT \\
\end{array}
\]

Due to the pullback in Lemma 3.16 we obtain the unique morphism

$$l : A \rightarrow I\Sigma \quad \text{with} \quad h = i \cdot l \quad \text{and} \quad k = u\Sigma \cdot l.$$ 

Then $l$ is a $\Sigma$-coalgebra homomorphism because $H\Sigma u\Sigma$ is a monomorphism and in the diagram

\[
\begin{array}{cccccccccc}
A & \xrightarrow{\alpha} & HA & \xrightarrow{m} & H\Sigma A & \xrightarrow{H\Sigma l} & H\Sigma T\Sigma \\
& \downarrow{i} & \downarrow{\varphi^{-1}\Sigma} & \downarrow{H\Sigma u\Sigma} & \downarrow{H\Sigma k} & \downarrow{H\Sigma u\Sigma} & \downarrow{H\Sigma k} \\
T\Sigma & \xrightarrow{\tau\Sigma} & H\Sigma T\Sigma & \xrightarrow{\varepsilon\tau\Sigma} & H\Sigma T\Sigma & \xrightarrow{\varepsilon\tau\Sigma} & H\Sigma T\Sigma \\
\end{array}
\]

the outside square and all inner parts except the upper one commute. Thus, the $H\Sigma$-coalgebra $A$ has the halting property by Theorem 3.9.

(ii) ⇒ (iv) Let $\alpha : A \rightarrow HA$ have the halting property, and choose some presentation $\bar{\alpha} : A \rightarrow H\Sigma A$ having the halting property, too (see Def. 3.13). By Theorem 3.9, $A$ is parametrically recursive for $H\Sigma$. Finally, the same argument as in Observation 3.14 shows that $A$ is parametrically recursive for $H$.

(iv) ⇒ (i) is trivial. \qed

Example 3.18. A $\mathcal{P}_{\text{fin}}$-coalgebra is a finitely branching graph $A$: the structure map $\alpha : A \rightarrow \mathcal{P}_{\text{fin}} A$ assigns to every node the set of all neighbor nodes. Such a graph is recursive iff it has no infinite paths.
Example 3.19. Finitely branching labelled transition systems are coalgebras for the functor $P_{\text{fin}}(\Sigma \times -)$, where $\Sigma$ is the set of all actions. Recursivity means that every development ends in finitely many steps in a state without transitions.

Remark 3.20. Recall from [14] that a set functor $H$ is connected (i.e., is not a coproduct of proper subfunctors) iff $H1 \cong 1$. We call $H$ trivial if $HA \cong 1$ for all sets $A \neq \emptyset$.

Theorem 3.21. For a nontrivial, connected endofunctor $H$ the following conditions are equivalent:

(i) every coalgebra, for which a homomorphism into $I$ exists, is recursive;
(ii) $H\emptyset = \emptyset$.

Proof. It is obvious that (ii) $\Rightarrow$ (i) since $I = \emptyset$, thus, only the empty coalgebra has a homomorphism into $I$. Conversely, suppose $H\emptyset \neq \emptyset$, then we construct a non-recursive coalgebra. This is sufficient because every coalgebra has a homomorphism into $I$: since $H$ is connected, $T = 1$, and since $H\emptyset \neq \emptyset$, we have $I \neq \emptyset$. However, there always exists a monomorphism $u : I \hookrightarrow T$, thus,

\[ I \cong T. \]

By Lemma 4.3 in [7], since $H$ is nontrivial, there exists a set $A$ such that

\[ \text{card } HA \geq \text{card } A > 1. \]

Choose $e : HA \rightarrow A$ and $\alpha : A \rightarrow HA$ with $e \cdot \alpha = \text{id}_A$. Then the coalgebra $(A, \alpha)$ is not recursive: for the algebra $(A, e)$ one candidate of $e^!$ is $\text{id}_A$:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & HA \\
\downarrow{\text{id}_A} & & \downarrow{H\text{id}_A} \\
A & \leftarrow & HA
\end{array}
\]

Another candidate is obtained by choosing an element $d \in H\emptyset$: for every set $X$ the empty map $r_X : \emptyset \rightarrow X$ yields an element $d_X = Hr_X(d)$ such that

\[ Hf(d_X) = d_Y \quad \text{for all functions} \quad f : X \rightarrow Y. \]

Consequently, the constant function $c : A \rightarrow A$ of value $e(d_A)$ also makes the square commute: in fact, since $c$ factorizes through $A \rightarrow 1$, it follows that $Hc$ factorizes through $H(A \rightarrow 1)$, thus, since $H$ is connected, $Hc$ is the constant function of value $d_A$. And $c \neq \text{id}_A$ because $\text{card } A > 1$. □
Example 3.22. There exists a functor not preserving inverse images, but having the property that for all coalgebras the equivalences

\[
\text{a homomorphism into } I \text{ exists } \iff \text{recursive } \iff \text{parametrically recursive}
\]

hold. Change the value of \( R \), see Example 2.9(ii), in the empty set to the value \( \emptyset \). The only coalgebra having a homomorphism into \( I = \emptyset \) is the empty one.

4. Conclusions

We study coalgebras for finitary set functors \( H \), making use of the presentation of such functors as (precisely all) quotients of polynomial functors \( H_\Sigma \) modulo \( \varepsilon \)-equations. We proved that the condition of \( H \) preserving inverse images, useful in various parts of coalgebra theory, is equivalent to the fact that \( \varepsilon \)-equations have the same set of variables on both sides.

Our main result is a characterization of recursive \( H \)-coalgebras as studied by Taylor [13] and recently by Capretta et al. [6]; those are coalgebras with a unique morphism into every algebra. We prove that recursive coalgebras are precisely those describing systems with the “halting property”, i.e., such that when started in any fixed state, the system halts in finitely many steps. This holds for finitary set functors preserving inverse images.

References


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