A NOTE ON UNIVOQUE SELF-STURMIAN NUMBERS∗

JEAN-PAUL ALLOUCHE

Abstract. We compare two sets of (infinite) binary sequences whose suffixes satisfy extremal conditions: one occurs when studying iterations of unimodal continuous maps from the unit interval into itself, but it also characterizes univoque real numbers; the other is a disguised version of the set of characteristic Sturmian sequences. As a corollary to our study we obtain that a real number \( \beta \) in \((1, 2)\) is univoque and self-Sturmian if and only if the \( \beta \)-expansion of 1 is of the form \( 1v \), where \( v \) is a characteristic Sturmian sequence beginning itself in 1.

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1. Introduction

The kneading sequences of a unimodal continuous map \( f \) from \([0, 1]\) into itself, with \( f(1) = 0 \) and \( \text{sup } f = 1 \), are classically studied by first looking at the combinatorial properties of the kneading sequence of 1. Cosnard proved that, using a simple bijection on binary sequences (namely mapping the sequence \((x_n)_{n \geq 0}\) to \((y_n)_{n \geq 0}\), where \( y_n := \sum_{0 \leq j \leq n} x_j \mod 2 \)), the set of kneading sequences of 1 for all maps \( f \) as above, maps to the set \( \Gamma \) defined by

\[
\Gamma := \{ u = (u_n)_{n \geq 0} \in \{0, 1\}^\mathbb{N}, \forall k \geq 0, \overline{u} \leq S^k u \leq u \}
\]

where \( \overline{u} = (\overline{u}_n)_{n \geq 0} \) is the sequence defined by \( \overline{u}_n := 1 - u_n \), where \( S^k \) is the \( k \)th iterate of the shift (i.e., \( S^k((u_n)_{n \geq 0}) := (u_{n+k})_{n \geq 0} \)), and where \( \leq \) is the lexicographical order on sequences induced by \( 0 < 1 \). See [2,10], where the relevant set is actually \( \Gamma \setminus \{(10)^\mathbb{N}\} \). See also [1] for a detailed combinatorial study of the set \( \Gamma \).

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1 CNRS, LRI, UMR 8623, Université Paris Sud, Bâtiment 490, 91405 Orsay Cedex, France; allouche@lri.fr

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A slight modification of the set Γ describes the expansions of 1 in the bases β, where β runs through the univoque numbers belonging to (1, 2). Recall that a number β is called univoque if 1 admits only one expansion in base β. More precisely, the set of expansions of 1 in all the univoque bases β ∈ (1, 2) is the set

\[ \Gamma_1 := \{ u = (u_n)_{n \geq 0} \in \{0,1\}^\mathbb{N}, \forall k \geq 0, \exists \bar{u} < S^k u < u \} \]

(see [11], Rem. 1 p. 379; see also [4] and the bibliography therein).

**Remark 1.1.** Note that a binary sequence belongs to Γ₁ if and only if it belongs to Γ and is not purely periodic.

Other sequences can be defined by extremal properties of their suffixes: characteristic Sturmian sequences and Sturmian sequences. More precisely the following results can be found in several papers (see in particular [7, 8, 14–16]; see also the survey [6] and the discussion therein).

A binary sequence \( u = (u_n)_{n \geq 0} \) is characteristic Sturmian if and only if it is not periodic and belongs to the set \( \Xi \) defined by

\[ \Xi := \{ u = (u_n)_{n \geq 0} \in \{0,1\}^\mathbb{N}, \forall k \geq 0, 0u \leq S^k u \leq 1u \} \].

A binary sequence \( u = (u_n)_{n \geq 0} \) is Sturmian if and only if it is not periodic and there exists a binary sequence \( v = (v_n)_{n \geq 0} \) such that \( u \) belongs to \( \Xi_v \), where

\[ \Xi_v := \{ u = (u_n)_{n \geq 0} \in \{0,1\}^\mathbb{N}, \forall k \geq 0, 0v \leq S^k u \leq 1v \} \].

The sequence \( v \) has the property that \( 1v = \sup_k S^k u \) and \( 0v = \inf_k S^k u \). This is the characteristic Sturmian sequence having the same slope as \( u \).

**Remark 1.2.** The reader can find the essentials on Sturmian sequences in [13] Chapter 2. A hint for the proof of the two assertions above is that a sequence is Sturmian if and only if it is not periodic and for any binary (finite) word \( w \), the words \( 0w0 \) and \( 1w1 \) cannot be simultaneously factors of the sequence. Furthermore a sequence \( u \) is characteristic Sturmian if and only if \( 0u \) and \( 1u \) are both Sturmian.

2. Comparing the sets Γ and Ξ

The analogy between the definitions of Γ and Ξ suggests the natural question whether any sequence can belong to their intersection. The disappointing answer is the following proposition.

**Proposition 2.1.** A sequence \( u \in \{0,1\}^\mathbb{N} \) belongs to \( \Gamma \cap \Xi \) if and only if it is equal to \( 1^\infty \) or there exists \( j \geq 1 \) such that \( u = (1^j0)^\infty \).
Proof. If the sequence \( u \) belongs to \( \Gamma \), we have in particular \( u \geq \overline{\pi} \). Hence \( u = 1w \) for some binary sequence \( w \). If \( u \) is not equal to \( 1^\infty \) (which clearly belongs to \( \Gamma \cap \Xi \)), let us write \( u = 1^j0z \) for some integer \( j \geq 1 \) and some binary sequence \( z \). Since \( u \) belongs to \( \Gamma \) we have \( S^{j+1}u \leq u \), i.e., \( z \leq u \). Now \( u \) belongs to \( \Xi \), thus \( S^ju \geq 0u \), i.e., \( 0z \geq 0u \), hence \( z \geq u \). This gives \( z = u \). Hence \( u = (1^j0)^\infty \), which in turn clearly belongs to \( \Gamma \cap \Xi \). \( \square \)

The next question is whether a Sturmian sequence can belong to \( \Gamma \). The answer is more interesting.

**Proposition 2.2.** A (binary) Sturmian sequence \( u \) belongs to \( \Gamma \) if and only if there exists a characteristic Sturmian sequence \( v \) such that \( v \) begins in 1 and \( u = 1v \).

**Proof.** Let us first suppose that the Sturmian sequence \( u \) belongs to \( \Gamma \). As above, since \( u \) belongs to \( \Gamma \), \( u \) begins in 1. Hence \( u = 1w \) for some binary sequence \( w \). The inequalities \( S^k u \leq u \) for all \( k \geq 0 \) imply that \( \text{sup}_k S^k u = u \) (the inequality \( \geq \) is trivial since \( S^0 u = u \)). This can be written \( \text{sup}_k S^k u = 1w \). On the other hand \( u \) is Sturmian, hence there exists a characteristic Sturmian sequence \( v \) such that \( u \) belongs to \( \Xi \). We also know that \( v \) is such that \( 1v = \text{sup}_{k \geq 0} S^k u \). Hence \( v = w \).

Now \( \inf_k S^k u = 0v = 0w \). But \( S^k u \geq \overline{\pi} \) for all \( k \geq 0 \), since \( u \) belongs to \( \Gamma \). Hence \( 0v = 0w \geq \overline{\pi} = 0\overline{\pi} \), thus \( w \geq \overline{\pi} \), hence \( w \) begins in 1.

If, conversely, \( u = 1v \) where \( v \) is a characteristic Sturmian sequence (which actually implies that \( u \) is Sturmian) beginning in 1, we first note that \( 0v = S^k v \leq 1v \) for all \( k \geq 0 \). Hence, immediately, \( S^k u \leq 1v = u \) for all \( k \geq 0 \) (using that \( S^{k+1} u = S^k v \) and that \( S^0 u = u = 1v \)). And also \( S^k u \geq 0v \geq 0\overline{\pi} = \overline{\pi} \) (using furthermore that \( v \geq \overline{\pi} \) since \( v \) begins in 1, and that \( S^0 u = u \geq \overline{\pi} \) since \( u \) begins in 1). \( \square \)

**Remark 2.3.** We see in particular that a Sturmian sequence belonging to \( \Gamma \) must begin in 11. This is not surprising since the only sequence belonging to \( \Gamma \) that begins in 10 is \( (10)^\infty \). This claim is a particular case of a lemma in [1]: if a sequence \( t \) belonging to \( \Gamma \) begins with \( m\overline{\pi} \), where \( m \) is a (finite) nonempty binary word, then \( t = (m\overline{\pi})^\infty \).

3. Univoque self-Sturmian numbers

Several papers were devoted to univoque numbers having an extra property. For example:

- the smallest univoque number in \((1, 2)\) is determined in [12]; it is related to the celebrated Thue-Morse sequence and was proven transcendental in [3];
- univoque Pisot numbers belonging to \((1, 2)\) are studied in [5].

The notion of self-Sturmian numbers was introduced in [9]. These are the real numbers \( \beta \) such that the greedy \( \beta \)-expansion of 1 is a Sturmian sequence on some two-digit alphabet. It is tempting to ask which univoque numbers are self-Sturmian. We restrict the study to the numbers in \((1, 2)\) for simplicity.
Proposition 3.1. The real self-Sturmian numbers in $(1, 2)$ that are univoque are exactly the real numbers $\beta$ such that $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$, where $u = (u_n)_{n \geq 0}$ is a binary sequence of the form $u = 1v$, with $v$ a characteristic Sturmian sequence beginning in 1.

Proof. This is a rephrasing of Proposition 2.2. □

Remark 3.2.
- The equality $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$, where $u = (u_n)_{n \geq 0}$ is a binary sequence, uniquely determines the real number $\beta$ in $(1, 2)$.
- Self-Sturmian numbers correspond to Sturmian sequences of the form $u = 1v$, where $v$ is any characteristic Sturmian sequence (see [9], Rem. p. 399). All self-Sturmian numbers are transcendental [9].

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References


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