DEJEAN’S CONJECTURE HOLDS FOR $N \geq 27^*$,**

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Abstract. We show that Dejean’s conjecture holds for $n \geq 27$. This brings the final resolution of the conjecture by the approach of Moulin Ollagnier within range of the computationally feasible.

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Repetitions in words have been studied since the beginning of the previous century [15,16]. Recently, there has been much interest in repetitions with fractional exponent [1,3,6–8,10]. For rational $r$ with $1 < r \leq 2$, a fractional $r$-power is a non-empty word $w = xx'$ such that $x'$ is the prefix of $x$ of length $(r-1)|x|$. For example, 010 is a $3/2$-power. A basic problem is that of identifying the repetitive threshold for each alphabet size $n > 1$:

What is the infimum of $r$ such that an infinite sequence on $n$ letters exists, not containing any factor of exponent greater than $r$?

The infimum is called the repetitive threshold of an $n$-letter alphabet, denoted by $RT(n)$. Dejean’s conjecture [6] is that

$$RT(n) = \begin{cases} 7/4, & n = 3 \\ 7/5, & n = 4 \\ n/(n-1) & n \neq 3,4. \end{cases}$$

Thue[16], Dejean [6] and Pansiot[13], respectively established the values $RT(2)$, $RT(3)$, $RT(4)$. Moulin Ollagnier [12] verified Dejean’s conjecture for $5 \leq n \leq 11$, and Mohammad-Noori and Currie [11] proved the conjecture for $12 \leq n \leq 14$.

Recently, Carpi [3] showed that Dejean’s conjecture holds for $n \geq 33$. Carpi’s result is computation-free, and resolving Dejean’s conjecture is thus reduced to filling a finite gap. Conceptually, one would hope that the gap could now be filled

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from below, using the methods of [11,12]. Since these approaches are computationally intensive, optimizing Carpi’s result is important. The present authors improved part of Carpi’s constructions to show that Dejean’s conjecture holds for \( n \geq 30 \) (see [4]). In the present note we show that in fact Dejean’s conjecture holds for \( n \geq 27 \).

Remark 1. Some months after the first draft of this paper, its goal has been vindicated: the final resolution of the conjecture via methods of Moulin Ollagnier becomes computationally feasible; in a recent paper the present authors proved Dejean’s conjecture by resolving computationally the cases \( n \leq 26 \). Dejean’s conjecture is correct! (See [5] and also [14] for another independent proof.)

The following definitions are from [3]: for any non-negative integer \( r \) let \( A_r = \{1,2,\ldots,r\} \). Fix \( n \geq 27 \). Let \( m = \lfloor (n - 3) / 6 \rfloor \). If \( a \) is a letter, then let \( |v|_a \) denote the number of occurrences of \( a \) in the word \( v \). Let \( \psi = \{ v \in A^*_m \mid \forall a \in A, 4 \text{ divides } |v|_a \} \). (We use this as a definition; it is in fact the assertion of Carpi’s Lem. 9.1.) A word \( v \in A^+_m \) is a \( \psi \)-kernel repetition if it has period \( q \) and a prefix \( v' \) of length \( q \) such that \( v' \in \ker \psi \) and \( (n - 1)(|v| + 1) \geq nq - 3 \).

In [4] we introduced the following definition: if \( v \) has period \( q \) and its prefix \( v' \) of length \( q \) is in \( \ker \psi \), we say that \( q \) is a kernel period of \( v \).

Let \( B = \{0,1\} \) and let \( S_n \) be the permutation group on \( n \) elements. Consider the morphism \( \phi : B^* \to S_n \) generated by

\[
\phi(0) = (1 \ 2 \ 3 \ \cdots \ (n - 1)) \\
\phi(1) = (1 \ 2 \ 3 \ \cdots \ (n - 1) \ n).
\]

This map is due to Pansiot [13]. A word \( u \in B^* \) is a \( k \)-stabilizing word if \( \phi(u) \) fixes \( \{1,2,3,\ldots,k\} \). The set of \( k \)-stabilizing words (for fixed \( n \)) is denoted by \( \text{Stab}_n(k) \). Note that if \( i < j \) then \( \text{Stab}_n(j) \subseteq \text{Stab}_n(i) \).

A map \( \gamma_n : B^* \to A^*_n \) is defined by

\[
\gamma_n(b_1b_2\ldots b_\ell) = a_1a_2\ldots a_\ell
\]

where \( a_i \phi(b_1b_2\ldots b_\ell) = 1 \) for \( 1 \leq i \leq \ell \).

Carpi introduces a morphism \( f : A^*_m \to B^* \) generated by

\[
f(1) = y^p x(101)^{2m} \\
f(a) = y^p x(101)^{2m-2a} 010(101)^{2a-1}
\]

where \( 2 \leq a \leq m, p = \lfloor n/2 \rfloor, y \) is the suffix of \( (01)^n \) of length \( n - 1 \) and \( x \) is the suffix of \( y \) of length \( |y| - 6m \).

The concepts of so-called short repetitions and kernel repetitions were introduced by Moulin Ollagnier [12]. His work is complicated by the fact that his short repetitions are words over \( A_n \), while his kernel repetitions are words over \( B \) (although they code words over \( A_n \) via Pansiot’s map). Without going into the details, we recall that he reduced the construction of an infinite word over \( n \) letters
attaining threshold \( n/(n-1) \) to avoiding both short repetitions and kernel repetitions. Moulin Ollagnier’s binary words were fixed points of morphisms. In [11], a technique was introduced for dealing separately with short repetitions and kernel repetitions; the binary words given there can be viewed as being produced by HD0L’s: they have the form \( g(h^\gamma(0)) \) where all words coded by \( g(B^*) \) avoid short repetitions, and each \( h \) is chosen to eliminate kernel repetitions.

Carpi’s work follows essentially this strategy. The lemmas of his paper show that \( f(B^*) \) avoids short repetitions if \( n \geq 30 \). For \( m = 5 \) (corresponding to \( n \geq 33 \)) he produces an infinite word \( w_5 \) over \( A_m \) such that \( f(w_5) \) avoids kernel repetitions. The exact statement of this division of work into short vs. kernel repetitions is the following:

**Proposition 2** ([3], Prop. 3.2). Let \( v \in B^* \). If a factor of \( \gamma_n(v) \) has exponent larger than \( n/(n-1) \), then \( v \) has a factor \( u \) satisfying one of the following conditions:

(i) \( u \in \text{Stab}_n(k) \) and \( 0 < |u| < k(n-1) \) for some \( k \leq n-1 \);

(ii) \( u \) is a kernel repetition of order \( n \).

In our previous note, we improved only the second part of Carpi’s construction; he had shown that for \( n \geq 30 \), no factor \( u \) of \( f(A^n_m) \) satisfied condition (i) above. As Carpi therefore states at the beginning of Section 9 of [3]:

By the results of the previous sections, at least in the case \( n \geq 30 \), in order to construct an infinite word on \( n \) letters avoiding factors of any exponent larger than \( n/(n-1) \), it is sufficient to find an infinite word \( w \) on the alphabet \( A_m \) avoiding \( \psi \)-kernel repetitions.

For \( m = 5 \), Carpi was able to produce such an infinite word, based on a paper-folding construction. He thus established Dejean’s conjecture for \( n \geq 33 \). The present authors refined this by constructing an infinite word \( w_4 \) on the alphabet \( A_4 \) avoiding \( \psi \)-kernel repetitions. This established Dejean’s conjecture for \( n \geq 30 \). We remark that for \( 30 \leq n \leq 32 \) the word on \( A_n \) verifying Dejean’s conjecture for \( n \) is \( \gamma_n(v) \), where \( v = f(w_4) \).

In the present note, we improve on the first aspect of Carpi’s attack, by showing that for \( 27 \leq n \leq 29 \), no factor \( u \) of \( v = f(w_4) \) satisfies (i) above. This implies that Dejean’s conjecture holds for \( n \geq 27 \). Since \( f \) is \( r \)-uniform where \( r = (p + 1)(n-1) \), to show that (i) holds for \( v \) it suffices to check that no factor \( u \) of \( f(B^3) \) satisfies (i). In principle, this involves considering all factors of \( f(B^3) \) of length less than \( (n-1)^2 \). However, we shorten this computation considerably by combining several of Carpi’s lemmas.

**Lemma 3.** Suppose \( n \geq 18 \). Suppose that \( u \in f(A^n_m) \cap \text{Stab}_n(k) \) and \( |u| < k(n-1) \) for some \( k \in \{1, 2, \ldots, n-1\} \). Then \( |u| = r(n-1) \) for some \( r, p+1 \leq r < k \leq 16 \).

**Proof.** Propositions and lemmas referenced in this proof are in [3]. By Proposition 5.1, \( k \geq 4 \) so that \( u \in \text{Stab}_n(4) \). It then follows from Proposition 6.3 that \( |u| \geq (p + 1)(n-1) \). Since \( |u| < k(n-1) \), we deduce that \( k > p + 1 \). From \( n \geq 18 \) this means that \( k > 10 \), so that surely \( u \in \text{Stab}_n(7) \). Applying Lemma 7.1, we see
that $|u|$ is divisible by $n-1$. We may thus write $|u| = r(n-1)$, $p+1 \leq r < k$. By the contrapositive of Proposition 7.2, $u \notin \text{Stab}_n(17)$. It follows that $k \leq 16$. □

We verify that Dejean’s conjecture holds for $n = 27, 28, 29$ by exhaustively examining factors $u$ of $f(B^3)$ of length $r(n-1)$ for $p+1 \leq r \leq 15$, and verifying that such $u$ are not in $\text{Stab}_n(k)$ for any $k, r < k \leq 16$. For $n = 28, 29$, the check only involves $r = 15, k = 16$. For $n = 27$, we also must consider $r = 14$. Code written in SAGE running on a PC performed the necessary verifications in about half an hour. The code is available at www.uwinnipeg.ca/~currie/kstab.sage.

References


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