

SIGNED CHIP FIRING GAMES AND SYMMETRIC SANDPILE MODELS ON THE CYCLES*

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Abstract. We investigate the Sandpile Model and Chip Firing Game and an extension of these models on cycle graphs. The extended model consists of allowing a negative number of chips at each vertex. We give the characterization of reachable configurations and of fixed points of each model. At the end, we give explicit formula for the number of their fixed points.

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1. INTRODUCTION

The Chip Firing Game (CFG) was introduced by Björner, Lovász and Shor in 1991 [2]. A CFG is defined on an underlying directed graph $G = (V, E)$ and described by its *configurations* and the *firing rule*. Each configuration is a distribution of chips on V . The firing rule is that at each step, a vertex containing as many chips as its out-going degree gives one chip along each of its out-going edges. CFG is then showed having numerous applications in various fields, such as theoretical computer sciences, combinatorics, mathematics [2, 3, 5, 7, 14].

The Sandpile Model (SPM) is introduced independently by Bak, Tang and Wiesenfeld [1] to study a popular phenomena in physics called Self-Organized

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Criticality (SOC). It obeys the evolution rule that sandpiles with high energy collapse to sandpiles with lower energy to become more stable. The system then has been developed and broadened deeply in many different directions such as avalanche, cellular automata, discrete dynamical system [5, 6, 15]. Although these two systems (SPM and CFG) are introduced for different purposes at the beginning, they have actually a very close relation. Particularly, it is possible to code a SPM by a CFG on a linear graph [10]. So investigating CFG contributes to investigating SPM. Conversely, investigating SPM gives as well the results which are very specific and difficult to be done in CFG in general. For instance, it is proved that the system CFG on graphs with no closed component converges to a unique fixed point (*i.e.*, configuration on which no rule can be applied) [14], but while the formula of fixed point of SPM was given explicitly [7], this is not yet the case for CFG.

One important research exploring the systems is to study all their configurations. More details, it comes up with the qualitative research on the convergence, on the characterizations of reachable configurations, of fixed point, and the various structures, such as order, lattice and algebra, of the configuration spaces, *etc.* Simultaneously, it also suggests the quantitative research on the complexity, the convergent time of the system. These problems are solved partly for the systems CFG and SPM on some particular graphs such as the line, the rectangular grids or even on general graphs.

In this paper, we study these problems for the systems SPM and CFG on cycle graphs. These problems also have a strong relation to the class of problems on cycles such as games of cards [4, 8, 12]. Furthermore, we are also interested in the signed versions of these systems, *i.e.*, we allow the vertices to contain negative numbers of chips for CFG and the sandpiles to have negative heights for SPM. This also reflects deeply some natural phenomena: between sandpiles there may be holes (of negative heights), and besides the delivering chips from vertices containing many chips, it is dually possible receiving chips from vertices lacking (negative enough) chips [13].

The paper is organized as follows. In Section 2, we present some preliminary definitions and notations of SPMs and CFGs (signed and non-signed versions) on cycles and we then give the characterizations for their configurations. In Section 3, we point out the characterizations for their stable configurations and count them by giving explicit formulas.

2. SPM, CFG, SSPM AND SCFG ON CYCLES: DEFINITIONS AND NOTATIONS

In this section, we present four models on cycles defined in term of discrete dynamical systems: Sandpile Model (SPM), Chip Firing Game (CFG) and their two generalizations called Symmetric Sandpile Model (SSPM) and signed chip firing game (SCFG). Then we show the close relationship between these systems.

Let C_n be a cycle graph of n vertices $\{1, 2, \dots, n\}$ ($n \geq 3$). Each sequence of integers (a_1, a_2, \dots, a_n) on vertices of C_n is called *circular distribution* and we say that vertex i contains a_i chips (note that a_i may be negative). We identify two circular distributions if they differ by a rotation of the cycle.

Definition 2.1. Let k be a non-negative integer. *The Sandpile Model on C_n of weight k* , denoted by $\text{SPM}(C_n, k)$, is described as follows:

- (i) The initial configuration is $(k, 0, 0, \dots, 0)$.
- (ii) The evolution rule is the *right rule* as follows: a vertex gives one chip to its right neighbor vertex if it is at least 2 higher than this neighbor.

Definition 2.2. Let k be a non-negative integer. *The Symmetric Sandpile Model on C_n of weight k* , denoted by $\text{SSPM}(C_n, k)$, is described as follows:

- (i) The initial configuration is $(k, 0, \dots, 0)$.
- (ii) The evolution rule: addition to the right rule in SPM, there is also the *left rule*, that means a vertex gives one chip to its left neighbor vertex if it is at least 2 higher than this left neighbor.

Definition 2.3. Let k be a non-negative integer. *The Chip Firing Game on C_n of weight k* , denoted by $\text{CFG}(C_n, k)$, is described as follows:

- (i) The initial configuration is $(k, 0, 0, \dots, 0, -k)$.
- (ii) The evolution rule is the *positive rule* as follows: a vertex containing at least 2 gives one chip to each of its two neighbors.

Definition 2.4. Let k be a non-negative integer. *The Signed Chip Firing Game on C_n of weight k* , denoted by $\text{SCFG}(C_n, k)$, is described as follows:

- (i) The initial configuration is $(k, 0, \dots, 0, -k)$.
- (ii) The evolution rule: addition to the positive rule in CFG, there is also the *negative rule*, that means a vertex containing at most -2 receives one chip from each of its neighbors.

Notations.

- We also denote by $\text{SPM}(C_n, k)$, $\text{SSPM}(C_n, k)$, $\text{CFG}(C_n, k)$, $\text{SCFG}(C_n, k)$ the set of all reachable configurations, called *configuration spaces*, of the Sandpile Model, the Symmetric Sandpile Model, the Chip Firing Game and the Signed Chip Firing Game on C_n of weight k respectively.
- We then denote by $\text{SPM}(C_n)$ the disjoint union of $\text{SPM}(C_n, k)$ for all $k \geq 0$, and similarly for $\text{SSPM}(C_n)$, $\text{CFG}(C_n)$, $\text{SCFG}(C_n)$.
- Let a and b be two circular distributions on C_n , we write $a \xrightarrow{(i,r)} b$ (resp. $a \xrightarrow{(i,l)} b$) if b is obtained from a by applying the right (resp. left) rule at vertex i ; and $a \xrightarrow{(i,+)} b$ (resp. $a \xrightarrow{(i,-)} b$) if b is obtained from a by applying the positive rule (resp. negative rule) at vertex i .

Figure 1 illustrates the configuration space of $\text{SSPM}(C_4, 4)$ as an example.

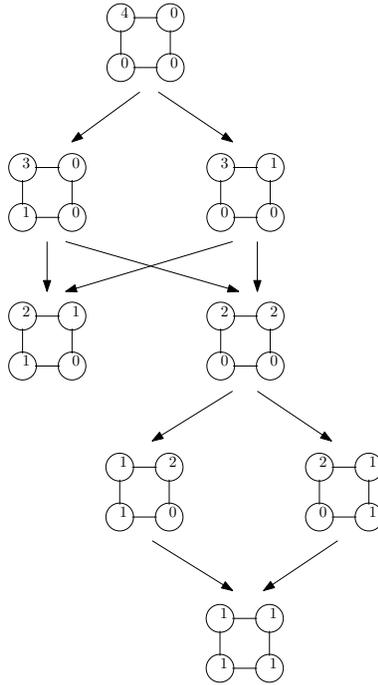


FIGURE 1. The space of configurations of $SSPM(C_4, 4)$.

Remark 2.5. It is straightforward from the definitions that

- (i) The configurations of $SPM(C_n)$ and $SSPM(C_n)$ are circular distributions of non-negative integers whereas the ones of $CFG(C_n)$ and $SCFG(C_n)$ are circular distributions of integers (may be negative).
- (ii) We have the two following inclusions

$$SPM(C_n, k) \subseteq SSPM(C_n, k) \text{ and } CFG(C_n, k) \subseteq SCFG(C_n, k).$$

As mentioned above, on the line each SPM can be coded by a CFG. Studying the model SPM can give results on CFG and *vice versa*. However, we also perceive that the configurations of the $SPM(C_n)$ (resp. $SSPM(C_n)$) and $CFG(C_n)$ (resp. $SCFG(C_n)$) on the cycle are very different. For instance, given a circular distribution we can calculate the weight in the model SPM (which is equal to the sum of all its parts) whereas we even do not know exactly this quantity in the model $CFG(C_n)$. Nevertheless, next we prove that for each given weight they are in fact isomorphic.

Definition 2.6. Two discrete dynamical systems are called *isomorphic* if there exists a bijection between their configuration spaces and this bijection preserves their evolution rule.

Let $a = (a_1, \dots, a_n)$ be a circular distribution on C_n . We define

$$d(a) = (a_1 - a_2, \dots, a_{n-1} - a_n, a_n - a_1).$$

It is straightforward that d is a well-defined map from the set of circular distribution on C_n to itself. Furthermore, we have the following

Proposition 2.7. *Under the map d two systems $SPM(C_n, k)$ and $CFG(C_n, k)$ are isomorphic; and two systems $SSPM(C_n, k)$ and $SCFG(C_n, k)$ are isomorphic.*

Proof. By the definition, $d(k, 0, \dots, 0) = (k, 0, \dots, -k)$ and so d maps the initial configuration of $SPM(C_n, k)$ (resp. $SSPM(C_n, k)$) to the initial configuration of $CFG(C_n, k)$ (resp. $SCFG(C_n, k)$).

We prove that d preserves the evolution rule between corresponding systems by showing that:

- (i) $a \xrightarrow{(i,r)} b$ if and only if $d(a) \xrightarrow{(i,+)} d(b)$ (so that d also preserves the rule of $SPM(C_n, k)$ and $CFG(C_n, k)$).
- (ii) $a \xrightarrow{(i,l)} b$ if and only if $d(a) \xrightarrow{(i,-)} d(b)$.

Thus, $a \xrightarrow{(i,r)} b$, then $a_i - a_{i+1} \geq 2$ and $d(a)_i \geq 2$. Hence, it is possible to apply the positive rule at vertex i on $d(a)$ and obtain the configuration

$$(d(a)_1, \dots, d(a)_{i-1} + 1, d(a)_i - 2, d(a)_{i+1} + 1, \dots, d(a)_n).$$

On the other hand,

$$b = (a_1, \dots, a_i - 1, a_{i+1} + 1, \dots, a_n)$$

and

$$d(b) = (d(a)_1, \dots, d(a)_{i-1} + 1, d(a)_i - 2, d(a)_{i+1} + 1, \dots, d(a)_n).$$

Hence, $d(a) \xrightarrow{(i,+)} d(b)$. Similarly for (ii) and so that d preserves the evolution rule. So d is isomorphic. Furthermore,

$$d^{-1}(u) = (\alpha, \alpha - u_1, \alpha - u_1 - u_2, \dots, \alpha - u_1 - \dots - u_{n-1}),$$

where $u = (u_1, \dots, u_n) \in CFG(C_n, k)$ (resp. $SCFG(C_n, k)$) and $\alpha = \frac{k + \sum_{i=1}^{n-1} (n-i)u_i}{n}$. □

It is remarkable that although d is bijective from $SSPM(C_n, k)$ (resp. $SPM(C_n, k)$) to $SCFG(C_n, k)$ (resp. $CFG(C_n, k)$), it is not bijective from $SSPM(C_n)$ (resp. $SPM(C_n)$) to $SCFG(C_n)$ (resp. $CFG(C_n)$). Moreover, while $SSPM(C_n, k)$ and $SPM(C_n, k)$ are absolutely disjoint for different values k , $SCFG(C_n, k)$ and $CFG(C_n, k)$ may overlap each other, especially for values k differing a multiple of n . So that a configuration of $SCFG(C_n)$ may correspond to many configurations of $SSPM(C_n)$ whose weights differ a multiple of n .

On the other hand, for different values k , the systems evolve differently. The larger k is, the more complicated the systems are. However, as far as we have known in many systems, their configurations are not so complicated. Particularly, we prove in Proposition 3.1 that for large enough values k in the same residue class modulo n , the set of stable configurations of $\text{SCFG}(C_n, k)$ (resp. $\text{CFG}(C_n, k)$) coincide.

Next, we study a characterization for the configurations of the four systems. To do this we represent the characterization for the configurations of SPM on the line given by Goles and Kiwi [7]. Recall that the Sandpile Model on the line, denoted by $\text{SPM}(k)$, is defined by its initial configuration (k) and the right rule. We also denote by

$$\text{SPM} = \sqcup_{k=0}^{\infty} \text{SPM}(k)$$

the configuration space of the SPM on the line.

Let $a = (a_1, a_2, \dots)$ be a sequence of positive integers. A pair (a_i, a_{i+1}) is called a *cliff* (resp. *plateau*) of a at position i if $a_i - a_{i+1} \geq 2$ (resp. $a_i - a_{i+1} = 0$).

Lemma 2.8 ([7]). *A non-increasing sequence (a_1, a_2, \dots, a_n) of positive integers is a configuration of SPM on the line if between two consecutive plateaus there exists at least one cliff.*

Theorem 2.9. *Let a be a circular distribution on C_n . Then a is a configuration of $\text{SPM}(C_n, k)$ if and only if there is a rotation vertices of C_n such that a (in the sequence form) is a configuration of $\text{SPM}(k)$ with the length at most n .*

Proof. Let $a \in \text{SPM}(C_n, k)$. Without loss of the generality, we assume that a is reachable from $(k, 0, \dots, 0)$ where k is placed at the first vertex of C_n . Since only the right rule is applied, it creates the intermediate non-increasing sequences in the evolution to reach a . So vertex n of C_n never gives back to vertex 1 during the evolution. So a is also a configuration of $\text{SPM}(k)$. The converse is straightforward. \square

The following corollary is direct from Proposition 2.7 and Theorem 2.9.

Corollary 2.10. *Let $a = (a_1, a_2, \dots, a_n)$ be a circular distribution. Then a is a configuration of $\text{CFG}(C_n, k)$ if and only if $d^{-1}(a)$ is a configuration of $\text{SPM}(C_n, k)$.*

We recall that given a system we can define a 2-ary relation \preceq on its configuration space such that $a \preceq b$ if a is reachable from b by applying several steps of the evolution rule. This relation very depends on the evolution rule of the system and the distinct systems give the distinct relations. Furthermore, the relation \preceq is in general not an ordered relation since a configuration can be reachable from itself and creating a cyclic in the orbit graph of its configuration space. However, in many investigated systems, the relation is an ordered relation. Particularly, the systems SPM and SSPM on the line investigated in [6, 11, 15] together with the relation \preceq on those are partially ordered sets (poset) and moreover the poset SPM

forms a lattice in which any two its elements have a unique supremum and a unique infimum. Next we prove that $SPM(C_n, k)$, and so $CFG(C_n, k)$, also inherits the lattice structure of $SPM(k)$.

Proposition 2.11. *The poset $(SPM(C_n, k), \preceq)$ (resp. $(CFG(C_n, k), \preceq)$) is a sub-lattice of the lattice $(SPM(k), \preceq)$ (resp. $(CFG(k), \preceq)$).*

Proof. By Proposition 2.7, it is sufficient to prove the statement for $SPM(C_n, k)$. Let a and b be configurations of $SPM(C_n, k)$. By Theorem 2.9, we assume that $a, b \in SPM(k)$ and $l(a), l(b) \leq n$. We prove that $c = \inf(a, b)$ and $d = \sup(a, b)$, where the supremum and infimum are taken in the lattice $SPM(k)$, are of the length at most n . We recall that if $u \preceq v$ in $SPM(k)$, i.e. u is reachable from v in $SPM(k)$, then $l(v) \leq l(u)$. So $l(d) \leq l(a) \leq n$ and $d \in SPM(C_n, k)$. The rest is implied from the explicit formulae of $\inf(a, b)$ in $SPM(k)$ ([9]). More precisely, we have $c = (c_1, c_2, \dots, c_l)$, where c_i is defined recursively as follows $c_1 = \min\{a_1, b_1\}$ and $c_i = \min\{\sum_{j=1}^i a_j, \sum_{j=1}^i b_j\} - \sum_{j=1}^{i-1} c_j$. So that $l(c) \leq \max\{l(a), l(b)\} \leq n$ and $c \in SPM(C_n, k)$. \square

Figure 2 illustrates the lattice of $SPM(10)$ containing $SPM(C_3, 10)$ as a sub-lattice. The fixed point of $SPM(10)$ is $(4, 3, 2, 1)$ whereas the fixed point of $SPM(C_3, 10)$ is $(4, 3, 3)$.

By some simple calculations, we get

Corollary 2.12. *The unique fixed point of $SPM(C_n, k)$ is of the form*

(i) $(p, p - 1, \dots, q, q, q - 1, \dots, 1, 0, \dots, 0)$ if $k \leq \frac{n(n-1)}{2}$, where

$$p = \left\lceil \frac{3 + \sqrt{9 + 8k}}{2} \right\rceil \quad \text{and} \quad q = k - \frac{p(p + 1)}{2}.$$

(ii) $(p, p - 1, \dots, q + 1, q, q, q - 1, \dots, p - n + 3, p - n + 2)$ if $k \geq \frac{n(n-1)}{2} + 1$, where

$$p = \left\lceil \frac{2k + n(n - 2)}{2n} + 1 \right\rceil \quad \text{and} \quad q = k - \frac{(2p - n + 2)(n - 1)}{2}.$$

Here $\lceil x \rceil$ denotes the largest integer no greater than x .

Next, we give a characterization for the configurations of $SSPM(C_n)$ as well as $SCFG(C_n)$. To do this we first present the concept of 2-decomposition of the circular distributions which is more general than the concept of LR-decomposition (left-right decomposition) on the line [6, 15].

Definition 2.13. Let $a = (a_1, a_2, \dots, a_n)$ be a circular distribution. Then a is called 2-decomposable at (i, j) (with $1 \leq i \leq j \leq n$) if $(a_{i-1}, a_{i-2}, \dots, a_1, a_n, \dots, a_{j+1})$ and $(a_i, a_{i+1}, \dots, a_j)$ are SPM configurations. Furthermore, a is called 2-decomposable if there exist i, j such that a is 2-decomposable at (i, j) .

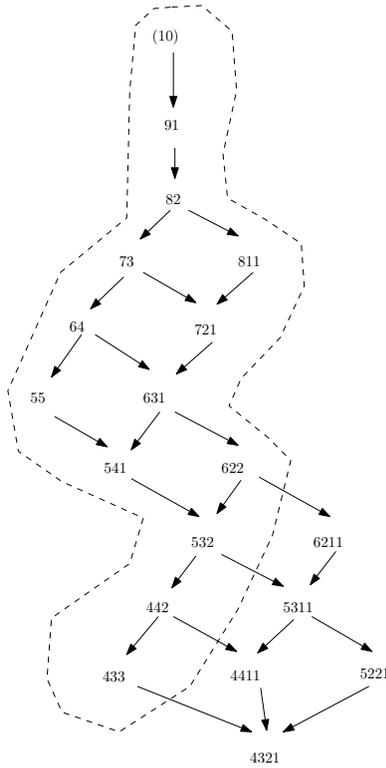


FIGURE 2. The sub-lattice $SPM(C_3, 10)$ in the lattice $SPM(10)$.

Remark 2.14. A 2-decomposable configuration a may be 2-decomposable at many (i, j) . For instance, $(2, 5, 5, 4, 1, 1)$ is 2-decomposable at $(1, 2)$ and $(2, 5)$ but not 2-decomposable at $(5, 5)$. Furthermore, $(1, 2, 2, 3, 3, 7, 4, 4, 1)$ is not 2-decomposable.

Theorem 2.15. *Let a be a circular distribution on C_n . Then a is a configuration of $SSPM(C_n)$ if and only if a is 2-decomposable.*

Proof. To prove the inference indicator, by recurrence, we show that if a is 2-decomposable at (i, j) $(1 \leq i \leq j \leq n)$ and a evolves to b by applying one step of the evolution rule then b also is 2-decomposable. We only consider the following cases:

- (i) $a \xrightarrow{(j,r)} b$;
- (ii) $a \xrightarrow{(i,l)} b$;
- (iii) $a \xrightarrow{(j+1,l)} b$;
- (iv) $a \xrightarrow{(i-1,r)} b$.

The others are deduced from the result in [7] saying that applying one step (or several steps) of the right rule on a SPM configuration (on the line) gives also a SPM configuration. So that if a is 2-decomposable at (i, j) then b is also 2-decomposable at (i, j) .

We now prove the statement for four cases above. By Lemma 2.8, adding cliffs to or removing parts from a SPM configuration gives a SPM configuration. So if $a \xrightarrow{(j,r)} b$, then $a_j - a_{j+1} \geq 2$ and a is also 2-decomposable at $(i, j + 1)$. According to this new decomposition of a then b is obtained from a not by one of four above cases. Therefore, b is 2-decomposable at $(i, j + 1)$.

Similarly, b is 2-decomposable at $(i + 1, j)$ if (ii); b is 2-decomposable at $(i, j - 1)$ if (iii) and b is 2-decomposable at $(i - 1, j)$ if (iv).

Conversely, assume that a is 2-decomposable at (i, j) and that a is of weight k . We need to show that a is reachable from (k) in $SSPM(C_n, k)$. Put

$$k_1 = \sum_{t=i}^j a_t \quad \text{and} \quad k_2 = \sum_{t=j+1}^n a_t + \sum_{t=1}^{i-1} a_t,$$

and so $k_1 + k_2 = k$.

Since (a_i, \dots, a_j) is a SPM configuration of weight k_1 , so it is reachable from (k_1) in SPM. Hence, a is reachable from $(a_1, \dots, a_{i-1}, k_1, 0, \dots, 0, a_{j+1}, \dots, a_n)$ by a sequence of applications the right rule. Similarly, $(a_{i-1}, \dots, a_1, a_n, \dots, a_{j+1})$ is reachable from (k_2) in SPM and, equivalently, $(a_{j+1}, a_{j+2}, \dots, a_n, a_1, \dots, a_{i-1})$ is reachable from (k_2) by a sequence of applications the left rule. Therefore, a is reachable from $(0, \dots, 0, k_2, k_1, 0, \dots, 0)$ by a sequence of applications the rule of $SSPM(C_n)$. And the latter configuration is reachable from $(0, \dots, 0, k, 0, \dots, 0)$ by a sequence of applications the rule of $SSPM(C_n, k)$, where the column of height k is at the position i if $k_1 \geq k_2$ and at the position $(i - 1)$ if $k_1 < k_2$. \square

The following is direct from Proposition 2.7 and Theorem 2.15.

Corollary 2.16. *Let $u = (u_1, \dots, u_n)$ be a circular distribution. Then u is a configuration of $SCFG(C_n, k)$ if and only if $d^{-1}(u)$ is 2-decomposable.*

3. FIXED POINTS OF $CFG(C_n)$ AND $SCFG(C_n)$

Although Corollary 2.16 gives a criteria for the configurations of $SCFG(C_n)$, it requires us to calculate their inverse images by d and then to check their 2-decomposability in $SSPM(C_n)$. In this section, we present a simple and direct characterization for the fixed points (not all their configurations) of $SCFG(C_n)$. Based on this characterization, we give an enumeration for these fixed points. We first classify the configurations of $CFG(C_n)$ and those of $SCFG(C_n)$.

Proposition 3.1. *Let k, l be non-negative integers. Then,*

(i) *If $k \not\equiv l \pmod n$ then*

$$CFG(C_n, k) \cap CFG(C_n, l) = \emptyset$$

and

$$SCFG(C_n, k) \cap SCFG(C_n, l) = \emptyset.$$

Consequently, the intersection of the set of fixed points of $SCFG(C_n, k)$ and those of $SCFG(C_n, l)$ is empty.

(ii) *If $k \equiv l \pmod n$ and $k, l \geq \lceil \frac{n+1}{2} \rceil^2$ then the set of fixed points of $CFG(C_n, k)$ (resp. $SCFG(C_n, k)$) is equal to those of $CFG(C_n, l)$ (resp. $SCFG(C_n, l)$).*

Proof.

(i) We prove that if $u = (u_1, u_2, \dots, u_n) \in SCFG(C_n, k)$, then

$$\sum_{i=1}^{n-1} iu_i \equiv k \pmod n$$

by showing that if $u \xrightarrow{(i,+)} v$ (similarly, $u \xrightarrow{(i,-)} v$) then

$$\sum_{t=1}^{n-1} (n-t)u_t \equiv \sum_{t=1}^{n-1} (n-t)v_t \pmod n. \tag{*}$$

Recall that

$$v = (u_1, \dots, u_{i-1} + 1, u_i - 2, u_{i+1} + 1, \dots, u_n).$$

By simple calculations, the expression (*) is deduced easily from the fact that

$$\sum_{t=1}^{n-1} tu_t = \sum_{t=1}^{n-1} tv_t \text{ for } i = 1, 2, \dots, n-2,$$

and

$$\sum_{t=1}^{n-1} tu_t = \sum_{t=1}^{n-1} tv_t + n \text{ for } i = n-1$$

and

$$\sum_{t=1}^{n-1} tu_t = \sum_{t=1}^{n-1} tv_t - n \text{ for } i = n.$$

(ii) Let u be a fixed point of $SCFG(C_n, k)$. By Proposition 2.7 and Theorem 2.15, $d^{-1}(u)$ is a fixed point of $SSPM(C_n, k)$ and 2-decomposable at (i, j) ($1 \leq i \leq j \leq n$). Therefore, the configuration $d^{-1}(u) + 1$, which is obtained from $d^{-1}(u)$ by adding 1 to each its part, is also 2-decomposable at (i, j) and on which neither the right nor the left rule can be applied. So $d^{-1}(u) + 1$ is a fixed point of $SSPM(C_n, n+k)$. Furthermore, $d(d^{-1}(u) + 1) = u$. Hence, u is a fixed point of $SCFG(C_n, n+k)$. Conversely, let u be a fixed point of $SCFG(C_n, k+n)$. Similarly, $d^{-1}(u) - 1$ contains the non-negative parts (since $k \geq \lceil \frac{n+1}{2} \rceil^2$) and is a fixed point of $SSPM(C_n, k)$. Hence, $d(d^{-1}(u) - 1) = u$ is also a fixed point of $SCFG(C_n, k)$. \square

As we remarked in the previous section that for large enough values of k in a residue class modulo n , although the set of fixed points of $\text{SSPM}(C_n, k)$ (resp. $\text{SPM}(C_n, k)$) are disjointed, the heights of their columns differ up-to a constant. In the other word, if (a_1, \dots, a_n) is a fixed point of $\text{SSPM}(C_n, k)$ (resp. $\text{SPM}(C_n, k)$), then $(a_1 + 1, \dots, a_n + 1)$ is a fixed point of $\text{SSPM}(C_n, k + n)$ (resp. $\text{SPM}(C_n, k + n)$) (Prop. 3.1). And so that their images by d in $\text{SCFG}(C_n, k)$ (resp. $\text{CFG}(C_n, k)$) and in $\text{SCFG}(C_n, k + n)$ (resp. $\text{CFG}(C_n, k + n)$) coincide.

By Corollary 2.12, $\text{CFG}(C_n, k)$ has a unique fixed point whereas $\text{SCFG}(C_n, k)$ may have many fixed points. By Proposition 3.1(ii), the set of fixed points of $\text{SCFG}(C_n)$ includes the fixed points of $\text{SCFG}(C_n, k)$ for small values of k and the n distinct residue classes of fixed points of $\text{SCFG}(C_n, k)$ for large values of k . For the small k , their fixed points can be found directly by taking the inverse images of d of 2-decomposable fixed points. We next characterize and enumerate the fixed points of $\text{SCFG}(C_n)$ for all $k \geq \lceil \frac{n+1}{2} \rceil^2$.

For convenience, we denote by $FP(\text{SCFG}(C_n, k))$ the set of fixed points of $\text{SCFG}(C_n, k)$ and

$$FP(\text{SCFG}(C_n)) = \bigcup_{k \geq \lceil \frac{n+1}{2} \rceil^2} FP(\text{SCFG}(C_n, k)).$$

Recall that each fixed point of $\text{SCFG}(C_n)$ is a circular distribution on C_n and its chips at vertices are $0, 1, -1$. By a rotation, next we can consider $FP(\text{SCFG}(C_n))$ as words on the alphabet $\{0, 1, \bar{1}\}$ where the letter $\bar{1}$ is understood as -1 .

Theorem 3.2. *The set $FP(\text{SCFG}(C_n))$ is determined as follows*

- (1) $FP(\text{SCFG}(C_3)) = \{(000); (10\bar{1}); (1\bar{1}0)\}$.
- (2) $FP(\text{SCFG}(C_4)) = \{(0000); (1\bar{1}00); (10\bar{1}0); (100\bar{1}); (11\bar{1}\bar{1})\}$.
- (3) $FP(\text{SCFG}(C_n))$, with $n \geq 5$, includes the words w on the alphabet $\{0, 1, \bar{1}\}$ satisfying the following properties:
 - (i) w starts from 1;
 - (ii) in w , the number of occurrences of 1 is equal to that of $\bar{1}$;
 - (iii) w avoids the subsequences: $\bar{1}1, 1001, \bar{1}00\bar{1}$ and 00000 ;
 - (iv) If w has 4 occurrences of 0 then it must end by 0 and does not contain the subword $1\bar{1}$.

Proof.

- (1) It is straightforward from the fact that all 2-decomposable fixed points of $\text{SSPM}(C_3)$ are of the form (a, a, a) ; $(a + 1, a, a)$ and $(a + 1, a, a)$.
- (2) It is straightforward from the fact that all 2-decomposable fixed points of $\text{SSPM}(C_4)$ are (a, a, a, a) ; $(a + 1, a, a + 1, a + 1)$; $(a + 1, a, a, a + 1)$; $(a + 1, a, a, a)$ and $(a + 1, a, a - 1, a)$.
- (3) Let $w \in FP(\text{SCFG}(C_n))$. Since the sum of the parts of w is equal to 0, the statement (ii) is obvious. By a rotation, we assume that w avoids the subsequence $\bar{1}1$. By Theorem 2.15, $d^{-1}(w)$, taken in $\text{SSPM}(C_n, k)$ for $k \geq \lceil \frac{n+1}{2} \rceil^2$,

is 2-decomposable. So each its decomposed part does not contain more than one plateau corresponding to one occurrence of 0 and the subsequences 1001 and $\bar{1}00\bar{1}$ are forbidden in w . Furthermore, we may allow to have at most two plateaus at two splitting positions. Therefore, we have at most 4 plateaus in $d^{-1}(w)$ corresponding 4 occurrences of 0s and so the subsequence 00000 is forbidden in w . Moreover, for $n \geq 5$, w contains at least 1 occurrence of 1 and we can assume w starts from 1, avoids the subsequence $\bar{1}1$. The statements (i) and (iii) are so satisfied. To prove (iv) we remark that if w contains exactly 4 occurrences of 0s then $d^{-1}(w)$ must have two plateaus at two splitting positions. Rotating w such that (i) and (iii) are satisfied says that the two plateaus at two splitting positions will give one occurrence of 0 at the end and one occurrence of 0 between the last 1 and the first $\bar{1}$ of w . So (iv) is satisfied. \square

We now give an explicit formula for the number of fixed points of $SCFG(C_n)$ by counting words in the above theorem.

Theorem 3.3. *The cardinality of $FP(SCFG(C_n))$ is*

- (i) 3 if $n = 3$;
- (ii) 5 if $n = 4$;
- (iii) $\frac{(n-1)^2}{2}$ if n is odd and $n \geq 5$;
- (iv) $\frac{n(n-2)}{2}$ if n is even and $n \geq 6$.

Proof. It is sufficient to prove (iii) and (iv). To do this, by Theorem 3.2 we count the number of ways to insert some 0s into a sequence including the 1s before the $\bar{1}$ s such that the conditions of Theorem 3.2(3.2) are satisfied.

(iii) $n = 2l + 1$: Let $w \in \mathcal{L}(SCFG)$ then w has either one or three occurrences of 0 (by 3.2(ii) of Theorem 3.2).

- (a) w has one occurrence of 0. Then this 0 may appear at any position except for the first position (since w starts from 1). So we have $n - 1$ such words w .
- (b) w has 3 occurrences of 0s. Then w has $(l - 1)$ occurrences of 1 and so that of $\bar{1}$. Denote by A the set of words of $(l - 1)$ occurrences of 1s; of $(l - 1)$ occurrences of $\bar{1}$ s and 3 occurrences of 0s satisfying the conditions 3.2(i), 3.2(ii) in Theorem 3.2 and avoiding the subsequence $\bar{1}1$. Denote by B the set of words in A not satisfying all conditions of Theorem 3.2(3.2). Then $|A|$ is equal to the number of ways to choose 3 positions for 0s from $n - 1$ positions except for the first position. Hence, $|A| = C_{n-1}^3$.

On the other hand, the words of B must contain the subsequence either 1001 or $\bar{1}00\bar{1}$. The number of words of B containing the subsequence 1001 (resp. $\bar{1}00\bar{1}$) and not containing the subsequence 10001 (resp. $\bar{1}000\bar{1}$) is $(n-l-1)C_{l-1}^2$. Here, we have C_{l-1}^2 ways to choose 2 positions of 0 from $(l - 1)$ its possible positions such that 1001 (resp. $\bar{1}00\bar{1}$) is its the subsequence; and $(n - l - 1)$ ways to choose the rest 0. Similarly, the number of words of B containing the

subsequence 10001 (resp. $\bar{1}000\bar{1}$) is C_l^3 . Hence,

$$|B| = 2(n - l - 1)C_{l-1}^2 + 2C_l^3.$$

Therefore,

$$|FP(SCFG(C_n))| = (n - 1) + (|A| - |B|) = \frac{(n - 1)^2}{2}.$$

(iv) $n = 2l$: Let $w \in \mathcal{L}(SCFG(C_n))$. w has no or 2 or 4 occurrences of 0.

- (a) if w has no occurrence of 0 then $w = 1 \dots 1\bar{1} \dots \bar{1}$ and we have a unique w .
- (b) if w has exactly 2 occurrences of 0s then it has $(l - 1)$ occurrences of 1 and also $(l - 1)$ occurrences of $\bar{1}$ s. So that the number of such words w is $C_{n-1}^2 - 2C_{l-1}^2$. Here, C_{n-1}^2 is the number of words of 2 occurrences of 0s satisfying the conditions 3.2(i) and 3.2(ii) of Theorem 3.2 and avoiding the subsequence $\bar{1}1$; and C_{l-1}^2 is the number of words satisfying the conditions 3.2(i) and 3.2(ii) of Theorem 3.2, avoiding the subsequence $\bar{1}1$ but containing the subsequence 1001 (resp. $\bar{1}00\bar{1}$).

(c) if w has 4 occurrences of 0s then it has $(l - 2)$ occurrences of 1 and also $(l - 2)$ occurrences of $\bar{1}$ s. By 3.2(iv) of Theorem 3.2, w ends by 0 and has at least 1 occurrence of 0 between the last 1 and the first $\bar{1}$.

- w is of the form $(1 \dots 0 \dots 10\bar{1} \dots \bar{1}0)$ that means the first 0 is between two 1s. We have $(l - 3)$ ways to choose the first 0; 1 way for the second 0 (as just after the last 1); and $(l - 1)$ ways for the third 0 at any positions after the second 0; and 1 way for the last 0 at the end of w . So we have $(l - 1)(l - 3)$ such words w .
- w is of the form $(1 \dots 100\bar{1} \dots \bar{1}0)$ that means the first two 0s are between the first 1 and the last $\bar{1}$. So we have $(l - 1)$ ways to choose the third 0 at any positions after the first two 0s and we have $(l - 1)$ such words w .
- w is of the form $(1 \dots 10\bar{1} \dots 0 \dots \bar{1}00)$ that means the first 0 is at the between the last 1 and the first $\bar{1}$, the second 0 is between two $\bar{1}$ s. So we have $(l - 3)$ ways to choose the second 0 and hence we have $(l - 3)$ such words w .
- w if of the form $(1 \dots 10\bar{1} \dots \bar{1}000)$ that means the first 0 is between the last 1 and the first $\bar{1}$ and the three last 0s are at the end. So we have a unique such word w .

Taking the sum of these values we obtain $l(l - 2)$ words of w in this case. Hence,

$$|FP(SCFG(C_n))| = 1 + C_{n-1}^2 - 2C_{l-1}^2 + l(l - 2) = \frac{n(n - 2)}{2}. \quad \square$$

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