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The Cauchy problem and Hadamard’s example


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Let $1 > 0$ and $m > 0$ be integers. Let $P(D)$ be a linear operator in $\mathbb{R}^n$. Let $P_m$ be its principal part. We say that the Cauchy problem

$$(1) \quad P(D)u = f, \ u - g = 0(x_1^1)$$

is uniquely solvable in the class of analytic functions if to each $f$ analytic in $\mathbb{R}^n$ and each $g$ analytic in a neighbourhood of $x_1 = 0$ there is an unique function $u$ analytic in $\mathbb{R}^n$ such that (1) is true. We show the following theorem [5].

**Theorem 1.** The problem (1) is uniquely solvable in the class of analytic functions if and only if $m = 1$ and $P_m$ is hyperbolic in the $(1, 0, \ldots, 0)$ direction.

In the proof we use

**Theorem 2.** Let $P(D)$ be a linear operator with constant coefficients such that $P_m$ is not hyperbolic in the $(1, 0, \ldots, 0)$ direction. Then there is a $v$ such that $v$ is analytic in $x_1 > 0$, $P(D)v = 0$ in $x_1 > 0$ and $v$ is not bounded near $x = 0$.

The proof of Theorem 2 makes use of

**Theorem 3.** Let $P(D)$ be a linear operator in $\mathbb{C}^n$ of the form

$$P(D) = D_1^{m-1} + \sum_{|\alpha| = m} a_{\alpha} D^\alpha + \sum_{|\alpha| < m} a_{\alpha} D^\alpha$$

with $0 \leq 1 < m$.

Then there is a function $v$ holomorphic when $z_1 \in (0, 0]$ such that

$$P(D)v = 0, \ v(z_1, 0) = z_1^{-1}, \ z_1 \in (-\infty, 0].$$

Hadamard's example with $u = n^{-1}\sin nx_2 \sinh nx_1$ shows that the Cauchy problem for the Laplace equation is not uniquely solvable in $\mathbb{C}^n$. The function $u = (1 - x_1 + ix_2)^{-1}$ shows that this is
also the case in the smaller class of analytic functions.

Theorem 2 is a generalization of this example to general operators.

We like to remark that the "if" part of Theorem 1 is due to J.-M. Bony and P. Schapira [1].

As another application of Theorem 2 we prove

**Theorem 4.** Let \( P(D) \) be an operator with constant coefficients in \( \mathbb{R}^n \). Let \( \omega \) and \( \Omega \) be open convex sets in \( \mathbb{R}^n \) such that \( \omega \subset \Omega \). Then the following two conditions are equivalent.

a) Let \( u \) be analytic in \( \omega \) and assume that \( P(D)u \) can be continued analytically to \( \Omega \). Then \( u \) can be continued to a function analytic in \( \Omega \).

b) Every hyperplane intersecting \( \Omega \) but not \( \omega \) has a normal hyperbolic with respect to \( P \).

**Proof.** It follows from [1, Théorème 4.2, p. 88-89] that b) implies a). Here we notice that the set of hyperbolic directions is open when the coefficients are constant. See [3, Lemma 5.5.1, p. 133].

Assume that there is a hyperplane \( H \) with non-hyperbolic normal with respect to \( P_m \) such that \( H \cap \Omega \neq \emptyset \) and \( H \cap \omega = \emptyset \). We rotate and translate the coordinate system such that \( H = \{ x; x_1 = 0 \} \), \( \omega \subset \{ x; x_1 > 0 \} \), \( 0 \in \Omega \). Then we choose \( u \) from Theorem 2 and get a \( u \) analytic in \( \omega \) and fulfilling \( P(D)u = 0 \) there. But \( u \) cannot be continued analytically to \( \Omega \). The theorem is proved.

A local version of Theorem 3 for operators with holomorphic coefficients in \( \mathbb{C}^n \) can be found in [4, Theorem 4.1]. We may also notice that a refinement of the technique in [4] has been used to
prove an existence theorem for the non-characteristic Cauchy problem when data are singular. See J. Persson [6]. A similar but much more complicated technique has been used on the same problem by Y. Hamada, J. Leray and C. Wagschal [2].

References.


