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ANDERS MELIN

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ON SOME LOCALIZED ESTIMATES FOR
PSEUDO-DIFFERENTIAL OPERATORS

by A. MELIN

Let $P = p^w(x, D)$ (Weyl convention) be a classical ps.d.op. in $\mathbb{R}^n$ with principal symbol p_m positively homogeneous of degree m . Let ρ_0 , say $\rho_0 = (0, \xi_0)$, be a point in the cotangent space of $\mathbb{R}^n$ and consider problems of the following types :

- (A) Determine those μ for which there exist a constant C and a ps.d.op. R of order strictly less than μ near ρ_0 such that
- $$\|u\|_{(\mu)}^2 \leq C \|Pu\|^2 + \|Ru\|^2 ; u \in C_0^\infty(\mathbb{R}^n),$$
- or in case p is real, $p_m \geq 0$.
- (B) Determine those μ for which there is a lower bound
- $$(Pu, u) \geq C \|u\|_{(\mu)}^2 + \|Ru\|^2 .$$

Sometimes when μ is kept fixed we also look for the possible constants C that can occur.

In case $p_m(\rho_0) \neq 0$ the standard calculus for pseudo-differential operators gives us a simple answer. In the other cases one has to localize the estimates near the characteristic variety Σ and get corresponding problems for operators with polynomial coefficients obtained from the Taylor series of $(x, \xi) \rightarrow p(\rho + \lambda(x, \xi))$ when $\rho \in \Sigma$, and λ is a small parameter. Thus for example to have (B) with $\mu = (m-1)/2$ (the sharp Gårding inequality) and a constant C implies lower bounds for the eigenvalues of the harmonic oscillator type operators which are obtained from a Taylor expansion up to the second order along Σ . Sharper results in these direction are obtained by Hörmander [2].

In Egorov [1] it is shown that his theorem about the validity of (A) with $\mu = m - \delta$ under the condition (ψ) when not all the commutators p_I of length $|I|$ of $\text{Re } p$ and $\text{Im } p$ vanish when $|I|(1-\delta) \leq 1$, essentially relies upon estimates of the following form :

$$(1) \quad M \|u\| + \|u'_t\| \leq C_0 \|u'_t - Q(t)u\| .$$

Here Q is either multiplication by a polynomial $q(t)$ or an operator $v(y) \rightarrow [F(t, y) + G(t) D_y] v(y)$ with polynomial coefficients acting on $L^2(\mathbb{R}_y)$. The condition (ψ) implies that $(Q(t)v, v)$ can only change sign

from - to + for fixed v and M is related to $\sum_{|I|(1-\delta) \leq 1} |P_I|^{1/|I|}$. Thus

for example $M = \sum |q^{(j)}(0)|^{1/(j+1)}$ in the first case. In this case a simple proof for (1) is given if one observes that there is a constant C_A only depending on A so that :

$$(2) \quad \|u'_t\| + \|u\| \leq C_A \|u' - hu\|$$

if h can only change sign from - to + and in addition satisfies the following :

$$(3) \quad \text{measure } \{t; |h(t)| < A^{-1}\} < 1 ,$$

$$(4) \quad \int_{\mathbf{R}} \max(0, -h'(t) - |h(t)|) dt \leq A.$$

One then obtains (1) when $Q(t) = q(t)$ by a symplectic dilation.

References

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