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Extendability of C. R. functions : a microlocal version of Bochner’s tube theorem


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EXTENDABILITY OF C. R. FUNCTIONS :
A MICROLOCAL VERSION OF BOCHNER'S TUBE THEOREM

by M. S. BAOUENDI

We present some recent results obtained jointly with F. Treves. Details and complete proofs can be found in [1].

Let m and n be two positive integers, we shall denote by $t = (t_1, \ldots, t_m)$ the variable in $\mathbb{R}^m$ and by $x = (x_1, \ldots, x_n)$ the variable in $\mathbb{R}^n$. Let $U$ be an open connected set in $\mathbb{R}^m$ and $\phi = (\phi_1, \ldots, \phi_n)$ a Lipschitz continuous mapping $U \to \mathbb{R}^n$. We consider the associated complex vector fields in $U \times \mathbb{R}^n$

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{n} \frac{\partial \phi_k(t)}{\partial t_j} \frac{\partial}{\partial x_k}, \quad j = 1, \ldots, m.$$  \hfill (1)

We have

$$\begin{cases}
L_j z_k = 0 & 1 \leq j \leq m, \quad 1 \leq k \leq n \\
z_k(t,x) = x_k + i \phi_k(t).
\end{cases} \hfill (2)$$

We denote by $z = z(t,x)$ the mapping $U \times \mathbb{R}^n \to \mathbb{C}^n$ defined by $z = (z_1, \ldots, z_n)$.

**Definition 1**: Assume $\phi$ to be real analytic and let $t^0 \in U$ and $x^0 \in \mathbb{R}^n$. The system $\mathfrak{L} = (L_1, \ldots, L_m)$ defined by (1) is said to be analytic hypoelliptic at $(t^0, x^0)$ if and only if any distribution $u$ in some open neighborhood $\omega$ of $(t^0, x^0)$, such that $L_j u$ is analytic for $j = 1, \ldots, m$, is itself analytic in a possibly smaller open neighborhood $\omega'$ of $(t^0, x^0)$.

Before giving a necessary and sufficient condition for the system $\mathfrak{L}$ to be analytic hypoelliptic at $(t^0, x^0)$ we state some simple reductions and remarks.

**Remarks**

1. In order to prove the analytic hypoellipticity of $\mathfrak{L}$ it suffices to prove the analyticity of the solutions of the homogeneous equations

$$L_j h = 0 \quad 1 \leq j \leq m.$$  \hfill (3)

Indeed if $L_j u = f_j$ is analytic, we can solve $L_j v = f_j$ with an analytic solution $v$. 

Since $L (u-v) = 0$ it suffices to show the analyticity of $u - v$.

2. We can restrict ourselves to the study of the $C^1$ solutions of (3). Indeed it can be easily proved [2] that any distribution solution of (3) near $(t^0, x^0)$ is of the form

$$h = \Delta_{x}^{q} h'$$

where $h'$ is of class $C^1$ and also solution of (3).

3. In order to prove the analytic hypoellipticity of $L$ at $(t^0, x^0)$ it suffices to show that if $h$ is a $C^1$ solution of (3) near $(t^0, x^0)$ then the function

$$h_o(x) = h(t^0, x)$$

is analytic at $x^0$. This can be easily seen using Remarks 1, 2 and the fact that the local Cauchy problem $L_j v = 0$, $1 \leq j \leq m$, with Cauchy datum at $t = t^0$, has a solution in the class of analytic functions and uniqueness holds in the class $C^1$ functions.

C.R. Functions

Let $V$ be an open set of $\mathbb{R}^n$. We denote

$$\Omega = U \times V.$$ 

We consider the "tuboid" of $\mathbb{C}^n$

$$z(\Omega) = V + i\phi(U).$$

Définition 2 : A function $u$ defined on the set $z(\Omega)$ is said to be Lipschitz continuous if its pull-back via $z$, $\bar{u} = u \circ z$ is Lipschitz continuous on $\Omega = U \times V$.

Moreover $u$ is said to be a C.R. function if $\bar{u}$ satisfies (3) in $U \times V$.

Observe that the push via $z$ of $L_j$, $1 \leq j \leq m$ is given by

$$\sum_{k=1}^{n} (L_j z_k) \frac{\partial}{\partial z_k} + (L_j \bar{z}_k) \frac{\partial}{\partial \bar{z}_k} = -2i \sum_{k=1}^{n} \frac{\partial \phi_k}{\partial t_j} \frac{\partial}{\partial \bar{z}_k} \cdot$$

Therefore if $\phi(U)$ is an immersed submanifold of $\mathbb{R}^n$, a function $u$ is a C.R. function according to Definition 2 if and only if it satisfies the usual induced Cauchy-Riemann equations on $z(\Omega)$. 

If \( f \) is a holomorphic function in an open neighborhood of \( z(\Omega) \) in \( \mathbb{C}^n \), clearly its restriction to \( z(\Omega) \) is a C.R. function. We are interested here in the following local extendability question: Let \( (t^0, x^0) \in \Omega \) and \( u \) a C.R. function on \( z(\Omega) \) when does \( u \) extend holomorphically to a neighborhood of \( z(t^0, x^0) \)?

We have the following:

**Proposition 1**: Let \( u \) be a C.R. function defined on \( z(\Omega) \) and \( (t^0, x^0) \in \Omega \). The function \( u \) extends holomorphically to a neighborhood of \( z(t^0, x^0) \) if and only if the function

\[
x \mapsto \tilde{u}(t^0, x) = u(z(t^0, x))
\]

is analytic at \( x^0 \).

When \( \phi \) is analytic the analytic hypoellipticity of the system \( L \) defined by (1) and the local holomorphic extendability are therefore equivalent (Prop. 1 and Remark 3).

**Theorem 1**: Assume \( \phi \) to be analytic. The following conditions are equivalent:

(i) The system \( L = (L_1, \ldots, L_m) \) defined by (1) is analytic hypoelliptic at \( (t^0, x^0) \).

(ii) Any C.R. function defined on a neighborhood of \( z(t^0, x^0) \) in \( z(\Omega) \) extends holomorphically to a full neighborhood of \( z(t^0, x^0) \) in \( \mathbb{C}^n \).

(iii) For every \( \xi \in \mathbb{R}^n \setminus 0 \), \( t^0 \) is not a local extremum of the function \( t \mapsto \phi(t) \cdot \xi \).

Theorem 1 follows essentially from the following microlocal result.

**Theorem 2**: Assume \( \phi \) to be analytic and let \( \xi^0 \in \mathbb{R}^n \setminus 0 \). The following conditions are equivalent:

(a) For every distribution \( h \) defined in some neighborhood of \( (t^0, x^0) \) and satisfying (3) \( (x^0, \xi^0) \) is not in the analytic wave-front set of \( h_0 \) (defined by (4)).

(b) \( t^0 \) is not a local minimum of the function \( t \mapsto \phi(t) \cdot \xi^0 \).

We can assume that \( (t^0, x^0) \) is the origin of \( \mathbb{R}^m \times \mathbb{R}^n \) and that \( \phi(0) = 0 \). In order to prove that (a) implies (b) it suffices to observe that if \( \phi(t) \cdot \xi^0 > 0 \) for all \( t \in U \), the function

\[
h(t, x) = (x \cdot \xi^0 + i \cdot \phi(t) \cdot \xi^0)^{3/2},
\]

with the principal determination of \( ^{3/2} \) for \( \zeta \in \mathbb{C} \) \( \text{Im } \zeta > 0 \), satisfies (3) and
(0, \xi^0) is in the analytic wave-front set of \( h_0(x) = (x.\xi^0)^{3/2} \).

The proof of (b) \( \Rightarrow \) (a) is an easy corollary of the following more general result:

**Theorem 3**: Assume \( \phi \) to be Lipschitz continuous in \( U(0 \in U) \) and let \( V \) be the open ball of \( \mathbb{R}^n \) centered at the origin of radius \( r > 0 \). Let \( \xi^0 \in \mathbb{R}^n \setminus 0 \) and assume there are \( t \in U \setminus 0 \) and a Lipschitz curve \( \gamma \) in \( U \) with \( 0 \) and \( t^* \) as its end-points satisfying:

\[
\begin{align*}
5) & \ - \phi(t^*).\xi^0 > 0, \\
6) & \ \sup_{t \in \gamma} |\phi(t)| < r, \\
7) & \ \left| \phi(t^*) \right|^2 \sup_{t \in \gamma} \phi(t).\xi^0 < \left[ r^2 - \sup_{t \in \gamma} \phi(t) \right]^2 \left[ - \phi(t^*).\xi^0 \right].
\end{align*}
\]

Then if \( h \) is any Lipschitz continuous solution of (3) in \( \Omega = U \times V \), \( (0,\xi^0) \) is not in the analytic wave-front set of \( h_0(x) = h(0,x) \).

**Idea of the proof of Theorem 3**

Let \( \epsilon > 0 \) and \( K > 0 \) be determined later. Let \( g \in C^\infty_0(V) \), \( g(x) = 1 \) for \( |x| \leq (1 - \epsilon)r \). Consider the integral

\[
I(x,\xi) = \int_{\mathbb{R}^n} \int_{\gamma} e^{i(x-y-\phi(t^*).\xi)} K(x-y-\phi(t)) \xi^0 L[g(y)h(t,y)] dt dy.
\]

We have used the notation \( z^2 = \sum_{j=1}^n z_j^2 \), and

\[
Lf(t,y) dt = \sum_{j=1}^m L_j f(t,y) dt_j
\]

which is a one form on \( U \) depending on \( y \).

Integrating (8) by parts with respect to \( t \) and \( y \) and using (2) we obtain

\[
I(x,\xi) = I_*(x,\xi) - I_0(x,\xi)
\]

with

\[

I_*(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-y-\phi(t^*).\xi)} K(x-y-\phi(t^*)) \xi^0 L[g(y)h(t^*,y)] dy
\]

\[
I_0(x,\xi) = \int_{\mathbb{R}^n} e^{i(x-y).\xi} K(x-y)^2 \xi^0 L[g(y)h_0(y)] dy.
\]
In order to show that \((0,\xi^0)\) is not in the analytic wave front set of \(h_0\), it suffices to show that the estimate

\[
|I_0(x,\xi)| \leq C e^{-|\xi|/C}
\]

with \(C > 0\), holds for \((x,\xi)\) in a conic neighborhood of \((0,\xi^0)\) (see Sjöstrand [3]). Assumptions (5), (6), (7) and (3) allow us to find \(\varepsilon > 0\) and \(K > 0\) so that estimates of the form (10) hold for \(I(x,\xi)\) and \(I_\ast(x,\xi)\); thus the desired estimate (10) follows from (9).

Other remarks

4. The microlocal results of this paper can yield holomorphic extendability of C.R. functions not only to full neighborhood of a point in \(z(\Omega)\) in \(\mathbb{C}^n\), but also to open sets of \(\mathbb{C}^n\) whose boundary contains part of \(z(\Omega)\).

5. It should be mentioned that other extendability results generalizing Bochner's tube theorem appeared in the literature: H. Lewy, Hörmander, Komatsu, Hill, Kazlow (see [1] for references).

REFERENCES

