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SUBELLIPTIC ESTIMATES

by J. J. KOHN

Consider the mapping $Q : C_0^\infty(\mathbb{R}^n)^m \times C_0^\infty(\mathbb{R}^n)^m$ given by

$$(1) \quad Q(u, v) = \sum_{i, j=1}^m \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} (a_{\alpha\beta}^{ij} D^\alpha u_i, D^\beta v_j),$$

with $a_{\alpha\beta}^{ij} \in C^\infty(\mathbb{R}^n)$, here $(\cdot, \cdot)$ denotes the L_2 -inner product on $\mathbb{R}^n$. We will assume that

$$(2) \quad Q(u, v) = \overline{Q(v, u)}$$

Definition : Q is subelliptic at $(x_0, \eta_0) \in \mathbb{R}^n \times (\mathbb{R}^n - \{0\})$ if there exist positive constants C, C' and ε and a classical symbol $p(x, \eta)$ of order zero (i.e. $p \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n - \{0\}))$ and $p(x, t\eta) = p(x, \eta)$ for $t > 0$) such that $p(x, \eta) = 1$ in a conic neighborhood of (x_0, η_0) and

$$(3) \quad \|Pu\|^2 \leq CQ(u, u) + C'\|u\|^2$$

for all $u \in C_0^\infty(\mathbb{R}^n)^m$, where P is pseudo-differential operator with symbol $p(x, \eta)$ and $\|f\|_\varepsilon^2 = \sum \|f_j\|_\varepsilon^2$, denotes the Sobolev ε -norms.

It is shown in [1] that subelliptic estimates imply regularity of solutions of the satisfying

$$(4) \quad Q(u, v) = (f, v)$$

for all $v \in C_0^\infty(\mathbb{R}^n)^m$. Here we will outline a microlocal version of the method for obtaining sufficient conditions for subellipticity which is developed in [2]. The advantages of the present treatment is that it can be used to study C-R structures and that it gives results, in at least some cases, when pseudo-convexity fails.

The principal example of the Q comes from the C-R structure described as follows. Let $n = 2k + 1$ and let $L_1, \dots, L_k$ be complex valued vector fields on $\mathbb{R}^n$

such that $[L_i, L_j] = \sum_{s=1}^k b_{ij}^s L_s$ and such that $L_1, \dots, L_k, \bar{L}_1, \dots, \bar{L}_k$ are linearly independent. We define $Q : C_0^\infty(\mathbb{R}^n)^k \times C_0^\infty(\mathbb{R}^n)^k \rightarrow \mathbb{C}$ by

$$(5) \quad Q(u, v) = \sum_{i < j} (\bar{L}_i u_j - \bar{L}_j u_i, \bar{L}_i v_j - \bar{L}_j v_i) + \left(\sum_i L_i u_i, \sum_j L_j v_j \right).$$

This quadratic form controls the regularity of the system $L_i W = f_i$, $i = 1, \dots, k$.

Another example of a Q which can be treated by the methods which we describe below comes from the Hörmander operator $\sum_{j=1}^k X_j^2$, when the X_j are real first order pseudo-differential operators in $\mathbb{R}^n$ and $Q : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ is given by

$$(6) \quad Q(u, v) = \sum_{j=1}^k (X_j u, X_j v).$$

Here subellipticity of Q implies hypoellipticity of the Hörmander operator.

Definition : If Q is given by (1) and if $p(x, \eta)$ is a C^∞ function defined in a conic neighborhood of $(x_0, \eta_0) \in \mathbb{R}^n - \{0\}$ which is homogeneous of zero order in η (i.e. $p(x, \eta) = p(x, t\eta)$ for $t > 0$), we say that p is a subelliptic multiplier for Q at (x_0, η_0) if there exists a pseudo-differential operator P such that the symbol of P equals p in a conic neighborhood of (x_0, η_0) and such that there exist constants C, C' and ε so that (3) is satisfied for all $u \in (C^\infty_0(\mathbb{R}^n))^m$. We say that two subelliptic multipliers are equivalent if they are equal on some conic neighborhood of (x_0, η_0) . We denote the set of equivalence classes of subelliptic multipliers by $\mathcal{P}(Q; (x_0, \eta_0)) = \mathcal{P}$.

Proposition : $\mathcal{P} = \mathcal{P}(Q; (x_0, \eta_0))$ has the following properties

- (a) $\mathcal{P}$ is an ideal in the ring $\mathcal{A}$. Where $\mathcal{A}$ denotes the ring of real-valued C^∞ functions defined in conic neighborhoods of (x_0, η_0) which are homogeneous of order zero.
- (b) $\sqrt[\mathbb{R}]{\mathcal{P}} = \mathcal{P}$. Here $\sqrt[\mathbb{R}]{\mathcal{P}}$ denotes the real radical of $\mathcal{P}$, that is if $g \in \mathcal{A}$ then $g \in \sqrt[\mathbb{R}]{\mathcal{P}}$ if and only if there exists an integer m and $p \in \mathcal{P}$ such that $|g|^m \leq |p|$ in a conic neighborhood of (x_0, η_0) .

Clearly subellipticity of Q at (x_0, η_0) is equivalent to $1 \in \mathcal{P}(Q; (x_0, \eta_0))$. The proposition given below shows how certain types of a priori estimates lead to conditions which imply that $1 \in \mathcal{P}$.

Theorem : Suppose that $A_1, \dots, A_N$ are pseudo-differential operators with symbols $a_1, \dots, a_N \in \mathcal{P}(Q; (x_0, \eta_0))$ such that there exist C and C' so that

$$(7) \quad \sum_1^N \|A_j Pu\|_1^2 \leq CQ(u, u) + C'\|u\|^2$$

for all $u \in (C_0^\infty(\mathbb{R}^n))^m$. Suppose further that, for $i = 1, \dots, M$, $B_i : C_0^\infty(\mathbb{R}^n)^m \rightarrow C_0^\infty(\mathbb{R}^n)$ are first order differential operators such that

$$(8) \quad \|B_i Pu\|^2 \leq CQ(u, u) + C'\|u\|^2$$

for all $u \in C_0^\infty(\mathbb{R}^n)^m$ and

$$(9) \quad \|B_i' Pv\|^2 \leq C \sum_1^N \|A_j v\|_1^2 + C'\|v\|^2$$

for all $v \in C_0^\infty(\mathbb{R}^n)$. Here P denotes a zero order pseudo-differential operator whose symbol equals one in a conic neighborhood of (x_0, η_0) . The operators B_i can be written as

$$(10) \quad B_i u = \sum_{k=1}^m B_i^k u_k$$

and B_i' is then given by

$$(11) \quad B_i' v = ((B_i^1)' v, (B_i^2)' v, \dots, (B_i^m)' v),$$

where $(B_i^k)'$ denotes the formal adjoint of B_i^k .

Suppose that $p_1, \dots, p_m \in \mathcal{P}(Q, (x_0, \eta_0))$ then for each i we have $\det\{p_j, \sigma(B_i^k)\} \in \mathcal{P}(Q, (x_0, \eta_0))$, here $\det$ denotes the determinant of the $m \times m$ matrix, $\{p, q\} = p_x q_\eta - p_\eta q_x$ denotes the Poisson bracket and $\sigma(B_i^k)$ denotes the symbol of B_i^k .

Corollary : Suppose that Q satisfies the hypothesis of the theorem at (x_0, η_0) . Let $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_r \subset \mathcal{P}(Q; (x_0, \eta_0))$ be the ideals defined as follows

$$(12) \quad \mathcal{P}_0 = \sqrt{\mathbb{R}(a_1, \dots, a_N)},$$

where $(a_1, \dots, a_N)$ denotes the ideal in $\mathcal{R}$ generated by the a_j . For $r > 0$ we define

$$(13) \quad \mathcal{P}_r = \sqrt{\mathbb{R}} \left(\mathcal{P}_{r-1}, \{ \det \{ p_j, \sigma(B_i^k) \} \text{ for all } p_1, p_m \in \mathcal{P}_{r-1} \} \right).$$

Then $1 \in \mathcal{P}_r$ implies that Q is subelliptic at (x_0, η_0) .

Returning now to the C-R structures, with Q defined by (5), let γ be a differential form such that in a neighborhood of $x_0 \in \mathbb{R}^n$ we have $\langle \gamma, L_i \rangle = \langle \gamma, \bar{L}_i \rangle = 0$ with $\gamma = -\bar{\gamma}$ and $|\gamma| = 1$. Then γ is determined uniquely up to sign. Let $c_{ij} = \langle \gamma, [L_i, \bar{L}_j] \rangle$ this is the Levi-form and we say that the C-R structure is pseudo-convex at γ if $(c_{ij}) \geq 0$.

Let U be a neighborhood of x_0 and V^+ be a conic neighborhood of $(x_0, [\gamma]_{x_0})$ such that V^+ is also a conic neighborhood of $(x, [\gamma]_x)$ for all $x \in U$.

Let $V^- = \{(x, \eta) \mid (x, -\eta) \in V^+\}$ and let U' be a neighborhood of x_0 with $\bar{U}' \subset U$.

Consider zero order pseudo-differential operators P^0, P^+ whose symbols $p^0(x, \eta)$, $p^+(x, \eta)$ and $p^-(x, \eta)$ are zero for $x \notin U'$ and $p^0(x, \eta) = 0$ if $(x, \eta) \in V^+ \cup V^-$, $p^+(x, \eta) = 0$ if $(x, \eta) \in V^-$ and $p^-(x, \eta) = 0$ if $(x, \eta) \in V^+$. We always have

$$(14) \quad \|P^0 u\|_1^2 \leq CQ(u, u), \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n)^k$$

Furthermore if $(c_{ij}) \geq 0$ on U then

$$(15) \quad \sum_{i,j=1}^k \|\bar{L}_j P^+ u_i\|^2 \leq CQ(u, u)$$

and

$$(16) \quad \sum_{i,j=1}^k \|L_j P^- u_i\|^2 \leq CQ(u, u).$$

To apply our theorem at $(x_0, [\gamma]_{x_0})$ we let

$$(17) \quad A_j = \Lambda^{-1} \bar{L}_j \quad \text{for } j = 1, \dots, k,$$

where Λ denotes the square root of the Laplacian. We define

$$B : C_0^\infty(\mathbb{R}^n)^k \rightarrow C_0^\infty(\mathbb{R}^n)$$

by

$$(18) \quad Bu = \sum_{i=1}^k L_i u_i.$$

The theorem then implies that $\det(C_{ij}(x)) \in \mathcal{C}^{\infty}(Q, (x, [\gamma]_x))$ for $x \in U'$. Applying the corollary we define ideals of germs of C^{∞} functions at x_0 by

$$(19) \quad I_1^+ = \sqrt{\mathbb{R}(\det(C_{ij}))}$$

and inductively

$$(20) \quad I_r^+ = \sqrt{\mathbb{R}(I_{r-1}^+, \det(M_{r-1}^+))},$$

when M_{r-1} runs through all $k \times k$ submatrices of the infinite matrix

$$(21) \quad \begin{pmatrix} C_{11} & \dots & C_{1k} \\ C_{k1} & \dots & C_{kk} \\ L_1(f) & \dots & L_k(f) \\ L_1(g) & \dots & L_k(g) \\ \vdots & & \vdots \end{pmatrix}$$

when $f, g, \dots$ run through all the elements of I_{r-1}^+ .

Hence $1 \in I_r^+$ implies subellipticity at $(x_0, [\gamma]_{x_0})$. Similarly to apply the theorem at $(x_0, -[\gamma]_{x_0})$ we set

$$(22) \quad A_j = \Lambda^{-1} L_j \quad \text{for } j = 1, \dots, k$$

and $B_{ij} : C_0^{\infty}(\mathbb{R}^n)^k \rightarrow C_0^{\infty}(\mathbb{R}^n)$ is defined by

$$(23) \quad B_{ij}u = \tilde{L}_i u_j - \tilde{L}_j u_i \quad \text{for } 1 \leq i < j \leq k.$$

The theorem then applies only when $k \geq 2$ (otherwise there are no B_{ij} and subellipticity does not hold). We then define ideals of germs of C^{∞} functions at x_0 by

$$(24) \quad I_1^- = \sqrt{\mathbb{R} \left(\det \begin{pmatrix} C_{i_1 i_1} & C_{i_1 i_2} \\ C_{i_2 i_1} & C_{i_2 i_2} \end{pmatrix} \right)}$$

and

$$(25) \quad I_r^- = \sqrt{\frac{\mathbb{R}}{(I_{r-1}^-, \det(M_{r-1}^-))}},$$

where the M_r^- run through the 2×2 submatrices of (21) with $f, g, \dots \in I_r^-$. Hence we see that $1 \in I_r^-$ implies subellipticity at $(x_0, -[\gamma]_{x_0})$.

I would conjecture that the conditions $1 \in I_r^+$ and $1 \in I_r^-$, for some r , are also necessary for subellipticity, this is true in the case of real analytic C-R structures.

The method outlined above will also give sufficient conditions in case the Levi form (C_{ij}) is a direct sum in all of U of a non negative semi definite and a non position semi definite form.

In the case of the Hörmander equation, where Q is given by (6). We set $A_j = \Lambda^{-1} X_j$ and $B_j = X_j$ and we obtain the Hörmander condition for subellipticity by applying the theorem.

An example which is related both to the Hörmander equation and to C-R structures is given by a first order pseudo differential operator L on $\mathbb{R}^n$. Here we consider $Q : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ given by

$$(26) \quad Q(u, u) = \|Lu\|^2.$$

The subellipticity of this Q was initiated by Nirenberg and Treves and then taken up by Egorov and Hörmander (see [3]). It is known that a necessary condition for subellipticity is that on the characteristic of L we have

$$(27) \quad \{\sigma(L), \sigma(L^*)\} \geq 0.$$

Furthermore, Egorov has shown that if subellipticity holds at (x_0, η_0) than

$$(28) \quad \|\tilde{L}Pu\| \leq C(\|Lu\| + \|u\|).$$

Hence, if (28) holds problem is reduced to the case of (6) with

$$Q(u, u) = \|X_1 u\|^2 + \|X_2 u\|^2 \quad \text{where } L = X_1 u + \sqrt{-1} X_2 u.$$

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