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NON EMBEDDABLE CR-STRUCTURES

by François TREVES

A CR-structure on a smooth manifold $\Omega$ is the datum of a closed (see [5], Ch. 1, Def. 1.1) vector subbundle $T'$ of the complex cotangent bundle $\mathbb{C}T^*\Omega$, such that

\[ \mathbb{C}T^*\Omega = T' + \overline{T'} . \]

We shall call $m$ the fiber dimension of $T'$. Note that, by (1), $\dim \Omega \leq 2m$. (If $\dim \Omega = 2m$ the structure is a complex one, a case in which we are not interested here). The structure $T'$ is said to be locally integrable or, equivalently, the CR manifold $(\Omega, T')$ is said to be locally embeddable if every point of $\Omega$ has an open neighborhood over which $T'$ is generated by $m$ closed (or exact) one-forms. A function, or a distribution, $f$, such that $df$ is a section of $T'$ is said to be a CR function, or distribution. It ought perhaps to be said that CR stands for Cauchy-Riemann.

H. Lewy [3] (1956) has raised the question as to whether a strongly pseudoconvex CR structure, on a $(2m-1)$-dimensional manifold $S^{2m}$, is always locally embeddable. Pseudoconvexity is defined by means of the Levi form (see below, (8)). That the answer is no was shown by L. Nirenberg [4] (1972) when $\dim \Omega = 3$, in which case the Levi form is a scalar (and $m = 2$). Here we show that the CR-structures that have non degenerate Levi forms, with one eigenvalue of one sign and all others of the opposite sign, and which are not locally embeddable, are dense (in a sense made precise below: see Theorem and remarks that follow).

Our viewpoint will be strictly local. We shall henceforth suppose that $\Omega$ is an open neighborhood of the origin in an Euclidean space, specifically $\mathbb{R}^{2n+1}$. We shall limit ourselves to the case where

\[ n = m - 1 . \]

Thus the fiber dimension of $T' \cap \overline{T'}$ is one. We shall begin by assuming that there are $m$ $C^\infty$ functions $z^1, \ldots, z^m$ in $\Omega$, complex valued, such that $dz^1, \ldots, dz^m$ span $T'$ at

(*) The present work is a generalization of some recent joint work, [2], with H. Jacobowitz (Rutgers University).

(•) For a positive answer to the global embeddability question, when $\Omega$ is compact and has dimension $\geq 5$, see Boutet de Monvel [1].
each point of \( \Omega \). After a contraction of \( \Omega \) about the origin, possibly a modification of the coordinates in \( \mathbb{R}^{2n+1} \), which we denote by \( x^1, \ldots, x^m, y^1, \ldots, y^n \), and a \( \mathcal{C} \)-linear substitution on the \( z^j \)'s, we may assume that

\[
\begin{align*}
(3) & \quad z^j = x^j + \sqrt{-1} y^j, \quad j = 1, \ldots, m - 1 (= n), \\
(4) & \quad z^m = x^m + \sqrt{-1} \phi(x,y),
\end{align*}
\]

with

\[
(5) \quad \phi \text{ real, } \phi(0,0) = 0, \quad d\phi(0,0) = 0.
\]

(see [5], Ch. I, p.20).

Henceforth we write \( z^j = x^j + iy^j \) \((j = 1, \ldots, n)\). But notice that the mapping

\[
(6) \quad (x,y) \mapsto Z(x,y) = (z^1(x,y), \ldots, z^m(x,y))
\]

defines a \textit{diffeomorphism} on the (real) hypersurface \( Z(\Omega) \) of \( \mathbb{C}^m \) defined by the equation

\[
(7) \quad y^m = \phi(x,y'), \quad y' = (y^1, \ldots, y^{m-1}).
\]

This justifies that we call (6) a (local) embedding.

Next we introduce the \textit{Levi form} of the structure, at the origin (without attempting to give here an invariant definition):

\[
(8) \quad Q(\zeta) = \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k}(0,0) \zeta^j \zeta^{\bar{k}} \quad (\zeta \in \mathbb{C}^n).
\]

Note that

\[
(9) \quad \phi(x',0,y') = \text{Re} \left( \sum_{j,k=1}^{n} b_{jk} z^j z^k \right) + Q(z) + O(|z|^3).
\]

It is convenient to introduce the function

\[
W = z^m - \sqrt{-1} \sum_{j,k=1}^{n} b_{jk} z^j z^k,
\]
and to use the new coordinate \( s = \text{Re}W \) in the place of \( x^n \). Instead of \( z^n \) (see (4)) we shall reason with

\[
W = s + i\varphi(z,s),
\]

noting that

\[
\varphi(z,s) = \varphi(z) + O(|z|^3 + |s||z| + |s|^2).
\]

Our basic hypothesis will be:

\[
\text{(12) The Levi form } Q \text{ is non degenerate and it has exactly } n - 1 \text{ eigenvalues of a given sign, and one of the opposite sign (i.e. it has signature } n-2).\]

We shall assume that one eigenvalue of \( Q \) is \( > 0 \) and \( n - 1 \) are \( < 0 \). After a linear substitution on the \( Z \)'s we may assume that

\[
\text{(13) } Q(z) = |z|^2 - |z''|^2,
\]

where \( z'' = (z^2, \ldots, z^n) \). By (11) we see that, in a suitable neighborhood of the origin, \( U \subset \Omega \),

\[
\varphi(z,s) \leq 2|z|^2 - \frac{1}{2} |z''|^2 + C|s| (|z| + |s|),
\]

The orthogonal \( T'^{\perp} \) of \( T' \), for the natural duality between vectors and covectors, is generated over \( \Omega \) by the following \( n \) vector fields:

\[
L_j = \frac{\partial}{\partial z^j} - i\lambda_j(z,s) \frac{\partial}{\partial s} , \ j = 1, \ldots, n,
\]

where the coefficients \( \lambda_j \) are computed by writing that \( L_j W = 0 \):

\[
\lambda_j = (1 + i \varphi \frac{\partial}{\partial s})^{-1} \frac{\partial \varphi}{\partial z^j}, \ j = 1, \ldots, n.
\]

(Incidentally the fact that \( T' \) is closed is equivalent to the property that the commutation bracket of any two smooth sections of \( T'^{\perp} \) is a section of \( T'^{\perp} \)).

We may now state our result:

Theorem: Suppose (13) holds. Then there is a function \( g \in C^\infty(\Omega) \), vanishing to infinite order at the origin, such that the following is true:

(17) There is an open neighborhood \( U \) of the origin in \( \Omega \) such that, for every \( j = 1, \ldots, n \),
\[
\lambda_j^\# = \lambda_j (1 + g/z^1)
\]
is a \( C^\infty \) function in \( U \);

(18) the vector fields \( L_j^\# = \frac{\partial}{\partial z^j} - i\lambda_j^\# \frac{\partial}{\partial s} \) in \( U \) \((j = 1, \ldots, n)\) commute pairwise;

(19) given any open neighborhood \( V \subset U \) of the origin, any solution \( h \in C^1(V) \) of the equations
\[
L_j^\# h = 0, \ j = 1, \ldots, n,
\]
has the property that \( dh|_0 \) is a linear combination of \( dz^1, \ldots, dz^n \).

The meaning of this theorem is, roughly, the following:

Let \( T' \) be a CR structure on a manifold \( \Omega \) of dimension \( 2n+1 \). Suppose that \( T' \cap \overline{T'} \) is a line bundle (i.e., the structure has "codimension one"). Suppose that, in the neighborhood of a point \( \omega_0 \) of \( \Omega \), the CR structure \( T' \) is embeddable, and has a non degenerate Levi form whose signature is equal to \( n - 2 \). Then there is another CR structure \( T'^\# \) in the neighborhood of \( \omega_0 \), tangent at \( \omega_0 \) to \( T' \) to infinite order, which is not locally embeddable (at \( \omega_0 \)).

Proof of Theorem: Inspired by Nirenberg [4] we select a sequence of compact subsets \( K_\nu \ (\nu = 1, 2, \ldots) \) in the upper half-plane \( \text{Im } w > 0 \) \((w = s + it \) will denote the variable in \( \mathbb{C}^1)\) having various properties:

(21) as \( \nu \to +\infty \), \( K_\nu \) converges to \( \{0\} \);

(22) the projections of the \( K_\nu \) into the real axis are pairwise disjoint;

(23) there is a number \( \epsilon > 0 \) such that
\[
K \subset \cap_{\nu} r^\epsilon = \{s + it ; |s| < \epsilon t\}.
\]

We shall furthermore assume that the interior \( K^\circ_\nu \) of \( K_\nu \) is not empty, whatever \( \nu \).
We note that, if \( s + i\varphi(z,s) \in \Gamma^\varepsilon \), we derive from (14):

\[
(e^{-1} - c(|z| + |s|))|s| + \frac{1}{2} |z''|^2 \leq 2|z'|^2,
\]

and therefore, by choosing \( \varepsilon > 0 \) small enough,

\[
(24) \quad e^{-1}|s| + |z'|^2 \leq 4|z'|^2, \quad (s,z) \in U, W \in \Gamma^\varepsilon.
\]

According to (11) we have

\[
(25) \quad \frac{\partial \varphi}{\partial z^j} = \pm z^j + O(|z|^2 + |s|).
\]

We note that, by (16), we have:

\[
(26) \quad \frac{\lambda_j}{z^1} = \left[ \pm z^j + O(|z|^2 + |s|) \right]/z^1.
\]

We select, for each \( v \), a function \( f_v \in C^\infty(\mathbb{R}^2) \) having the following properties:

\[
(27) \quad f_v \geq 0 \text{ everywhere, } \text{supp } f_v \subset K_v, f_v(W_v) > 0 \text{ for some } W_v \in K_v;
\]

\[
(28) \quad f = \sum_{v=1}^{\infty} f_v \in C^\infty(\mathbb{R}^2);
\]

\[
(29) \quad \lambda_j g/z^1 \in C^\infty(U),
\]

where

\[
g(f \circ W)/[1 + (f \circ W) (\log W_s)/z^1].
\]

Let us show that (29) can be achieved (in particular by taking \( U \) small enough).

Recalling that \( W = s + i\varphi(z,s) \), we see that \( \log(1 + i\varphi_s) \) is well defined provided \( U \) is small; furthermore \( \log(1 + i\varphi_s) = O(|z| + |s|) \), hence is \( O(|z'|) \) on \( \text{supp } (f \circ W) \), by (23) and (24). Since \( f \) is flat at the origin, both \((f \circ W)(\log W_s)/z^1\) and \( \lambda_j(f \circ W)/z^1 \) (cf. (26)) are \( C^\infty \) in \( U \), and flat at the origin, whence easily (29).

By differentiating \( L_j W = 0 \) with respect to \( s \) and dividing by \( W_s \) we get

\[
(30) \quad L_j (\log W_s) = i\lambda_j, \quad j = 1, \ldots, n.
\]
A straightforward computation yields

$$L_j g + i\lambda_j g^2/z^1 = \lambda_j h, \ j = 1, \ldots, n,$$

where $h$ is a certain function of $(z,s)$. We have used the fact that $L (f \cdot W) = L (\bar{W} g) \frac{\partial f}{\partial W}$, and $L_j \bar{W} = L_j (W + \bar{W}) = 2L_j z = -2i\lambda_j$.

$$\lambda_j = \frac{i}{2} L_j \bar{W}, \ j = 1, \ldots, n.$$

Note that $L_k \lambda_j = L_j \lambda_k$ (hence $[L_j, L_k] = 0$). We have

$$[L_j^#, L_k^#] = [L_j - i\lambda_j z \frac{\partial}{\partial s}, L_k - i\lambda_k z \frac{\partial}{\partial s}] = -i\delta \frac{\partial}{\partial s},$$

where

$$z^1 q = L_j (\lambda_k g) - L_k (\lambda_j g) - i\lambda_j z \frac{\partial}{\partial s} (\lambda_k g)$$

$$+ i \lambda_k z \frac{\partial}{\partial s} (\lambda_j g)$$

$$= \lambda_k (L_j g + i \frac{q^2}{z} \lambda_j s) - \lambda_j (L_k g + i \frac{q^2}{z} \lambda_k s)$$

$$= 0 \quad \text{according to (31)}.$$

This proves (18).

Finally suppose that $h \in C^1 (V)$ is a solution of (20). In particular it is a solution of $L^# h = 0$ on the plane $z^2 = \ldots = z^n = 0$. We shall prove below that this implies $h \in C^1 (0,0) = 0$. Because of the special form of the equations (20) (see (18)) this implies $\frac{\partial h}{\partial z}(0,0) = 0$, whence (19).

The proof is reduced to the case where $n = 1$. We content ourselves with sketching the argument, which is essentially the same as that given, with full details, in [2]. Let us write $x, y, z = x + iy$, rather than $x^1, y^1, z^1$, and $L = \frac{\partial}{\partial z} - i\lambda (z,s) \frac{\partial}{\partial s}$ rather than $L_1$. We have

$$\varphi(z,s) = |z|^2 + O(|z|^3 + |s||z| + |s|^2).$$

By the implicit function theorem there is a $C^\infty$ function, in a neighborhood of zero, $s \rightarrow z(s)$, with $z(0) = 0$, such that, if we set $\varphi_0 (s) = \varphi(z(s),s)$, we have
Furthermore \( \varphi_0(0) = 0 \). We may therefore assume that the intersection of the cone \( \Pi^* \) (see (23)) with a small open disk centered at the origin, in the \( w = s + it \) plane, is entirely contained in the region

\[ t > \varphi_0(s). \]  

We may and shall assume that all the compact sets \( K \) are contained in the open set (34), and we shall denote by \( \mathcal{R}_0 \) the complement of \( \bigcup_{\nu} K \) in (34), by \( \mathcal{R} \) the set of points \( (z,s) \in \Omega \) such that \( \nu = s + i\varphi(z,s) \in \mathcal{R}_0 \). Notice that we have, in \( \mathcal{R} \):

\[ \text{Lh} = 0. \]

Because of (33), when \( w = s + it \in \mathcal{R}_0 \), the equation \( \varphi(z,s) = t \) defines a smooth closed curve in the \( z \)-plane, \( \gamma(w) \), winding around \( z(s) \). We can use the parameter \( \theta = \text{Arg}(z - z(s)) \) on \( \gamma(w) \). This defines a smooth map

\[ S^1 \times \mathcal{R}_0 \ni (\theta,w) \longmapsto (z,s) \in \mathcal{R}. \]

By virtue of (35) we have \( dh = A\, dw + B\, dz \) in \( \mathcal{R} \), hence

\[ \frac{\partial}{\partial w}(h \frac{\partial z}{\partial \theta}) = \frac{\partial}{\partial \theta}(h \frac{\partial z}{\partial w}) \quad \text{since} \quad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial \theta} = 0. \]

This implies that the integral \( I(w) = \int_{\gamma(w)} h \, dz \) is a holomorphic function of \( w \) in \( \mathcal{R}_0 \). Since \( \gamma(w) \) contracts to the point \( z(s) \) when \( t = \varphi_0(s) \), we have \( I(w) = 0 \) on this curve, therefore everywhere in \( \mathcal{R}_0 \) (note that there is "enough room" for the zeros to propagate around the sets \( K_\nu \), thanks to (22)). We select then a smooth closed curve \( c_\nu \) in \( \mathcal{R}_0 \) winding around \( K_\nu \), such that no point of any set \( K_\nu, \forall \nu' \neq \nu \), lies inside or on \( c_\nu \). We derive

\[ \int_{c_\nu} \int_{\gamma} h(z,s) dz \wedge dw = 0. \]

The 2-chain \( \Sigma_\nu = \{(z,s);(W,z) \in c_\nu \times \gamma\} \) is a kind of torus whose inside we call \( \Omega_\nu \). Stokes theorem implies

\[ \iiint_{\Omega_\nu} dh \wedge dz \wedge dw = 0. \]
But $dh = A dW + B dZ + Lh d\bar{Z}$, hence (38) reads

\[ (39) \quad \int \int \int_{\mathcal{V}} (\lambda/|z|^1) g h_s d\bar{Z} \wedge dz \wedge dw = 0 \]

since $Lh = i \lambda g h_s /|z|^1$. Near the origin, on supp $g$ (cf. (24))

\[ \lambda/|z|^1 \sim 1, \quad g \sim f \circ W. \]

If $h_s(0,0)$ were $\neq 0$, in (39) the argument of the integrand would have a well-defined limit as $\nu \to +\infty$; (39) could not hold true for $\nu$ large enough. \qed

REFERENCES


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