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Analyticity for certain solutions of non-hypoelliptic differential operators on the Heisenberg group

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We consider left invariant differential operators on the Heisenberg group $G$ with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ where $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, $[\mathfrak{g}_2, \mathfrak{g}_2] = 0$, and $X_1, X_2, \ldots, X_{2n}$ is a basis of $\mathfrak{g}_1$ and $T$ a basis of $\mathfrak{g}_2$. Let $p$ be an elliptic, homogeneous non-commuting polynomial in $2n$ variables, i.e., $p(\xi_1, \xi_2, \ldots, \xi_{2n}) \geq C |\xi|^d$, $C > 0$. An operator of the form $L = p(X_1, X_2, \ldots, X_{2n})$ will be said to be homogeneous and elliptic in the generating directions. It is known that $L$ is analytic-hypoelliptic and $C^\infty$ hypoelliptic if and only if the $L^2$ nullspace of $L$ is nontrivial (see [10], [9], [7], [8], and [5]). The results announced here show that even if $L$ is not hypoelliptic, it has a left inverse, modulo the projection onto its kernel, which preserves real analyticity, locally. More precisely, our main result is the following.

**Theorem 1.** Let $L$ be a homogeneous, left invariant differential operator on the Heisenberg group $G$ elliptic in the generating directions. Then there are distributions $k_1$ and $k_2$ such that

1. $Lf * k_1 = f - \mathbb{P}_1 f$
2. $L(f * k_2) = f - \mathbb{P}_2 f$
for $f \in C_0^\infty(G)$, where $\Pi_1$ and $\Pi_2$ are the orthogonal projections onto the $L^2$ nullspaces of $L$ and its adjoint $L^*$, respectively, and

$(\ast)$ denotes group convolution. Furthermore, the operators $f + f \ast k_i$

and $f \rightarrow \Pi_i f$, $i = 1, 2$, all preserve analyticity locally.

Corollary. If $u$ and $f$ are smooth functions of compact support on $G$ and

$$Lu = f \quad \text{in} \quad U,$$

where $U$ is an open set, then $u_1 = (I - \Pi_1)u$ is analytic in every
subset of $U$ where $f$ is, and $u_1$ also satisfies (3).

In the special case where $L = \square_b$, the boundary Laplacian operator acting on 0-forms (see [2]) the analog of Theorem 1 was given by Greiner, Kohn, and Stein [4], who derived explicit formulas for $k_i$ and $\Pi_i$. The analyticity of the projections $\Pi_i$ was proved by Geller [3], who also proved the existence of distributions $k_i$, satisfying (1) and (2) and preserving local smoothness. The general result was conjectured by Stein [3]. See also Melin [6] for related results.

To prove Theorem 1, we use a standard reduction to the case where $L$ is self-adjoint and of high degree, in addition to satisfying the conditions of Theorem 1. The following is partly based on an idea of Beals and Greiner [1].

Theorem 2. Let $L$ be a self-adjoint differential operator of high
homogeneous degree $d$ satisfying the conditions of Theorem 1. Then there
is a closed contour $\Gamma$ around 0 in $\mathcal{C}$ such that $L_\alpha = L - \alpha(-it)^{d/2}$
is hypoelliptic for all $\alpha \in \Gamma$. There exist distributions $k_\alpha$, $\alpha \in \Gamma$,
such that \( L_k \alpha = \delta \) and for any \( f \in C_0^\infty(G) \) and any multi-index \( \beta \) the function \( \alpha \to \| \partial^\beta (f * k_\alpha) \|_L^\infty \) is bounded for \( \alpha \) on \( \Gamma \). Hence define \( K, S : C_0^\infty(G) \to C_0^\infty(G) \) by

\[
Kf = \frac{1}{2\pi i} \int_\Gamma f \alpha^{-1} f * k_\alpha d\alpha
\]

and

\[
Sf = \frac{1}{2\pi i} \int_\Gamma T^{d/2} f * k_\alpha d\alpha
\]

Then

\[
(4) \quad LKf = K^* Lf = \mathcal{F} - Sf, \quad f \in C_0^\infty(G),
\]

and \( S = \mathbb{I} \), the orthogonal projection onto the \( L^2 \) kernel of \( L \).

Furthermore, \( K \) and \( S \) preserve real analyticity, locally.

The proof of Theorem 2 first requires constructing the \( k_\alpha \). For this we follow the method given by Metivier [7], checking that the \( k_\alpha \) so obtained vary well with \( \alpha \). The first identity in (4) follows from the self adjointness of \( L \), while the second is immediately obtained by writing \( L = L_\alpha + \alpha(-iT)^{d/2} \). The proof that \( S = \mathbb{I} \) is accomplished by applying the irreducible unitary representations to both operators. Then the equality reduces to a resolvent identity, and the original identity follows by the Plancherel formula for \( G \).

Finally, to show that \( K \) and \( S \) preserve real analyticity, it suffices to show that the operators \( f \to f * k_\alpha \), each of which preserves real analyticity, satisfy estimates uniform in \( \alpha \) for \( \alpha \) on \( \Gamma \). For this, we use the methods of the second author [9] to estimate the \( L^2 \) norms of derivatives of \( f * k_\alpha \), checking again that the constants obtained may be chosen independent of \( \alpha \).
References


