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An example of the Heisenberg group


<http://www.numdam.org/item?id=JEDP_1982___A15_0>
Let us consider the Kohn-Laplacian $\Box b^q$ which acts on appropriate $q$-forms on the Heisenberg group $\mathbb{H}_n$. Then as Kohn showed (more generally), $\Box b^q$ is $C^\infty$ hypoelliptic and locally solvable, when $0 < q < n$; however this is not the case when $q = 0$, or $q = n$, and the latter fact goes back to the fundamental example of Lewy.

If we introduce the coordinates $(z,t) \in \mathbb{C}^n, t \in \mathbb{R}^n$, and the complex vectorfields

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n$$

then one can write

$$\Box b^q = \mathcal{L}_\alpha \otimes I, \quad \text{with} \quad \alpha = n - 2q$$

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_j Z_j \bar{Z}_j + (\bar{Z}_j Z_j) + i\alpha \frac{\partial}{\partial t},$$

and there is an explicit fundamental solution, given by

$$\psi_\alpha C_\alpha^{-1}, \quad C_\alpha = \frac{C}{\Gamma(\frac{n-\alpha}{2}) \Gamma(\frac{n+\alpha}{2})}$$

with

$$\psi_\alpha = (|z|^2 - it)^{-(n+\alpha)/2} (|z|^2 + it)^{-(n-\alpha)/2}.$$ 

All of this holds for $\alpha \neq \pm n, n+2, \ldots$ and incidently shows the analytic hypoellipticity of $\Box b^q$, $0 < q < n$, on the Heisenberg group (see [1]).

Our first question, is what happens when $q = 0$ (i.e. $\alpha = n$)? These is then ([2]) a relative fundamental solution $\xi$ so that

$$\Box b \xi = I - \xi.$$. 
where ζ is the projection operator on the null space of $a^b$ (i.e. the Cauchy-Szego projector) and $\mathcal{K}$ has a description similar to the above fundamental solutions.

Finally what happens to $a^0_b + \mu I$? (From now on, for simplicity of notation $a^b = a^0_b$ and $n = 1$).

**Theorem:** Suppose $\mu \neq 0$, the operator $a^b + \mu I$ is locally solvable, $C^\infty$ and analytic hypoelliptic.

The $C^\infty$ hypoellipticity was already observed by Melin

The idea will be to construct a parametrix (analytic away from the diagonal).

The formal solution to our problem is

$$(a^b + \mu I)^{-1} = \frac{\zeta}{\mu} + \sum_{n=1}^{\infty} \left(-\mu\right)^{n-1} \mathcal{K}^n.$$ 

The main difficulty is to give a meaning to this infinite series, and prove the appropriate properties of the kernel it represents. This requires some definitions.

Let

$$E(u) = \sum_{n=3}^{\infty} \frac{u^n}{n!(n-3)!}.$$ 

This is a Bessel function. What is important for us is that $|E(u)| < e^{-|u|^{1/2}}$, $u$ complex. Next let $w = |z|^2 - it$. Write

$$P_\mu = \frac{C}{\mu} + K^1 - \frac{\bar{w}}{2\mu \pi^2} \int_0^\infty E\left(2\mu(w+\bar{w}s)\frac{\log s}{s-1}\right) \log (w+\bar{w}s) \times \frac{s-1}{(w+\bar{w}s)^3} \ ds$$

with $C$ the Cauchy-Szego kernel = $\frac{1}{\pi^2 w^2}$

$$K = \frac{1}{2\pi} (\log w - \log \bar{w})w^{-1},$$

and $K^2 = \frac{1}{\pi^2 \bar{w}} \int_0^\infty (w + \bar{w}s)^{-1} \frac{\log s}{s-1} \ ds$.

**Proposition:**

1. $P_\mu$ is real-analytic away from the origin
2. $(a^b + \mu I)P_\mu = S + R_\mu$, where $R_\mu$ is everywhere real-analytic,

with $\delta = \text{Dirac function of the origin}$.

The proof of the proposition is somewhat complicated; its details will appear elsewhere [3].
Bibliography


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