The Poisson summation formula for a Dirichlet problem with gliding and glancing rays


<http://www.numdam.org/item?id=JEDP_1982___A21_0>
1. Introduction. Let $M$ be a compact Riemannian manifold with smooth boundary, and let $\Delta$ be the Laplacian on $M$. This is an unbounded operator on $L^2(M)$ which has a self adjoint extension with domain the Sobolev space $\{ u \in H^2(M) : u|_{\partial M} = 0 \}$, whose spectrum subset of $\mathbb{R}^+$, say $\{0 > \mu_1 \geq \mu_2 \geq \cdots \to \infty\}$. The corresponding eigenfunctions $e_j$ are a complete orthonormal set in $L^2(M)$; in fact, they are $C^\infty$ and satisfy

\begin{equation}
\Delta e_j + \mu_j e_j = 0, \quad e_j|_{\partial M} = 0, \quad j = 1, 2, \ldots
\end{equation}

in the classical sense. So the $\mu_j$ and $e_j$ are, respectively, the eigenvalues and the eigenfunctions of the Dirichlet problem for $\Delta$ on $M$.

The eigenvalues $\mu_j$, which we take to be positive, satisfy Weyl's estimate $\#\{\mu_j \leq \tau\} = O(\tau^\dim M)$ as $\tau \to \infty$. Hence the spectral measure

\begin{equation}
\sigma(\tau) = \sum_{j=1}^{\infty} \delta(\tau - \mu_j)
\end{equation}

is a tempered distribution.

Consider now the following initial value problem for the wave equation on $M \times \mathbb{R}$:

\begin{equation}
\begin{cases}
(\partial_t^2 - \Delta)u = 0, & u|_{t=0} = f \in C^\infty_0(M), \quad \partial_t u|_{t=0} = 0, \\
& u|_{\partial M \times \mathbb{R}} = 0.
\end{cases}
\end{equation}

For any $t$, this defines a map $C^\infty_0(M) \ni f \mapsto u(.,t) \in C^\infty_0(M)$ whose Schwartz kernel is a function $K: \mathbb{R} \to \mathcal{S}'(M \times M)$ which can be expanded as

\begin{equation}
K(x,y,t) = \sum_{j=1}^{\infty} e_j(x) e_j(y) \cos \mu_j t.
\end{equation}

One can also look upon this as a function $M \times M \to \mathcal{S}'(\mathbb{R})$, and as such it has a trace given by
\( \text{tr } K = \int K(x,x,t) \, dg_x = \sum_{j=1}^{\infty} \cos \mu_j t, \)

where \( dg_x \) is the Riemann measure on \( \mathcal{M} \). By (2) one can write this identity also as

\( \text{tr } K = \hat{\mathcal{S}}_e(t) \)

where \( \hat{\mathcal{S}}_e \) is the even part of the Fourier transform of \( \mathcal{S} \),

\( \hat{\mathcal{S}}_e(t) = \frac{1}{2} (\hat{\mathcal{S}}(t) + \hat{\mathcal{S}}(-t)) = \frac{1}{2} (\mathcal{S}(T) + \mathcal{S}(-T))^\Lambda. \)

Andersson and Melrose \([1]\) have shown that, if \( \mathcal{M} \) is everywhere geodesically concave or convex, then (6) extends the Poisson formula for compact boundaryless manifolds due to Chazarain \([3]\) and Duistermaat and Guillemin \([4]\), to the Dirichlet problem for \( \Delta \). In particular, the singular support of \( \hat{\mathcal{S}}_e \) is contained in the set

\[ \{ T \in \mathbb{R} : |T| \text{ is the length of a closed broken geodesic on } \mathcal{M} \text{ or of a closed boundary geodesic} \}. \]

Here, the broken geodesic flow includes reflection, with the usual 'equal angles' law, at the boundary, and the boundary is equipped with the induced Riemann metric. Furthermore, if \( |T| \) is the length of a closed broken geodesic which meets \( \mathcal{M} \) transversally a finite number of times, and satisfies a certain non-degeneracy condition, then Guillemin and Melrose \([5]\) have established an extension to manifolds with boundary of the asymptotic expansions of \([3]\) and \([4]\) for the restriction of \( \hat{\mathcal{S}}_e(t) \) to a sufficiently small neighbourhood of \( T \).

This leaves two open questions. The first is that of the contribution of closed broken geodesics which graze the boundary; this can happen if \( \mathcal{M} \) has a geodesically concave connected component. The second one, which may be called the gliding ray problem, concerns the behaviour of \( \hat{\mathcal{S}}_e \) in the neighbourhood of \( T \) when \( T \) is the length of a boundary geodesic.
We shall discuss a simple two-dimensional example which throws some light on these questions. The results are primarily due to the first author.

2. The eigenvalue problem. The manifold is a portion of a cylinder, $M = (0,d) \times (\mathbb{R}/2\pi \mathbb{Z})$, where $\mathcal{Y} > 0$ and $d > 0$, equipped with the metric $(1+x)(dx^2 + dy^2)$. So the eigenvalue problem (1) for our example can be put into the form

$$
\begin{align*}
(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \phi + \mu^2 (1+x) \phi &= 0 \quad \text{on} \quad (0,d) \times \mathbb{R}, \\
\phi|_{x=0} = \phi|_{x=d} &= 0, \quad y \to \phi \quad \text{has period} \quad 2\mathcal{Y},
\end{align*}
$$

and we take $\mu > 0$.

**Proposition.** With $x \in \mathbb{R}$, $\mu \in \mathbb{R}^+$, and $\eta \in \mathbb{R}$, write

$$
\begin{align*}
\xi_x &= \xi_x(\mu, \eta) = \mu^{-4/3}(\eta^2 - (1+x)\mu^2),
\end{align*}
$$

and let $\text{Ai}(z)$, $\text{Bi}(z)$ be the standard solutions of Airy's equation $F'''(z) = z F(z)$. (See [9], for example.) For each $m = 0, 1, \ldots$, let $\mu_{mj}$, where $j = 1, 2, \ldots$, be the roots of

$$
\begin{align*}
\text{Ai}(z^m_d) \text{Bi}(z^m_0) - \text{Ai}(z^m_0) \text{Bi}(z^m_d) &= 0,
\end{align*}
$$

arranged in ascending order; here $z^m_x = \xi_x(\mu, m\mathcal{Y}/\pi^2)$. Then the $\mu_{mj}$ are the eigenvalues of (9); they are simple if $m = 0$, and of multiplicity 2 if $m > 0$.

The proof is straightforward, and omitted. It is convenient to let $m$ range over $\mathbb{Z}$ and put

$$
\mu_{-m,j} = \mu_{mj}, \quad m < 0, \quad j = 1, 2, \ldots ;
$$

this takes care of the multiplicities. The spectral measure (2) is then

$$
\sigma(\tau) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \delta(\tau - \mu_{mj}),
$$
and the even part of its Fourier transform, (7), becomes

$$\hat{\sigma}_e(t) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \cos \mu_m \sigma_j t.$$  

3. The broken geodesic flow. For our example, the wave equation is

$$Pu = (1+x) \partial^2_t u - \partial^2_x u - \partial^2_y u.$$  

The geodesic flow on $T^*M$ is just the biocharacteristic flow of $P$. Leaving aside the zero section ('geodesics of zero length'), one can restrict this to $S^*M = \{(x,y,\xi,\eta) \in T^*M : \xi^2 + \eta^2 = 1 + x^2, and t then gives the (signed) length of the geodesics, which are the biocharacteristic curves. On the covering manifold $\tilde{M} = (0,d) \times \mathbb{R}$, one can visualize these as the trajectories of a billiard ball on an infinitely long inclined billiard table whose (parallel) edges are horizontal, and perfectly reflecting.

From now on, we shall refer to the broken geodesics, both on $\tilde{M}$ and on $M$, as geodesics. A closed geodesic on $M$, of length $T \neq 0$, is the image under $\tilde{M} \to M$ of a geodesic on $M$ such that $x(T) = x(0), y(T) = y(0) + 2\pi n$, where $n \in \mathbb{Z}$, consisting of parabolic arcs reflected or grazing at the boundary. Here $n$ is the winding number; one must also associate an integer $k \neq 0$ with the geodesic, where $|k|$ is the number of reflections at $x = d$, with $k > 0$ if $T > 0$, and $k < 0$ if $T < 0$. We denote such a geodesic by $\gamma_{nk}$. It will be said to be of type I if it does not meet $x = 0$, of type II if it is reflected alternately at $x = d$ and at $x = 0$, and grazing if it is tangent to $x = 0$. Geodesics of type II are of no interest for the problem in hand, and will be ignored. Elementary computations give the following:
Proposition. Let \( \lambda \) be a real number, and put

\[
Y_\lambda = 2\lambda (1+d-\lambda^2)^{1/2}, \quad T_\lambda = \frac{2}{3} (1+d-\lambda^2)^{1/2} (1+d+2\lambda^2);
\]

let \( n \) and \( k \) be nonzero rational integers. There is a closed geodesic \( y_{nk} \) of type I, with length \( 2kT_\lambda \), if there is a \( \lambda \) such that \( 1 < \lambda^2 < 1+d \) and

\[
kT_\lambda = nY.
\]

This has no (real) solutions if \( |n/k| > (1+d)/Y \). If \( |n/k| \leq (1+d)/Y \), then (18) has one solution \( \lambda_{nk} \) such that \( \lambda_{nk}^2 \geq \frac{1}{2}(1+d) \); if also \( |n/k| > 2d^{1/3}/Y \), then the second solution \( \lambda'_{nk} \) of (18), for which \( \lambda'_{nk}^2 < \frac{1}{2}(1+d) \), is also admissible. If \( d^{1/3}/Y \) is a rational number, and \( |n/k| = 2d^{1/3}/Y \), then (18) holds for \( \lambda = 1 \) or for \( \lambda = -1 \), and the corresponding \( y_{nk} \) is grazing.

Remark. Let \( F^t : S^*M \to S^*M \) be the map obtained by letting every point of \( S^*M \) move for a time \( t \) along the lifted (broken) geodesic issuing from it, with a suitable convention for points lying above \( \partial M \). If \( \gamma \subset M \) is a closed geodesic of (signed) length \( T \), then it is clear that the points of \( \gamma \), lifted to \( S^*M \), and their \( y \)-translates, are the fixed point set of \( F^T \). So this set has dimension 2. One can show that it is clean, in the sense of [4] and of [5], unless \( \gamma \) is of type I and \( |\lambda| = (\frac{1}{2}(1+d))^{1/3} \). Such a geodesic will be called degenerate; it occurs when the roots of (18) coincide, and one then also has

\[
\partial Y_\lambda / \partial \lambda = 0.
\]

4. The trace formula. In our example, the first member of (5) can be obtained without explicitly determining \( K \) by solving the initial value problem (3). One needs a technical lemma.
Lemma. Let $z \in \mathbb{R}$, and put

$$
\chi(z) = \frac{1}{\pi} \int_0^\infty \frac{dt}{A_t^2(t) + B_t^2(t)}.
$$

Then $\chi \in C^\infty(\mathbb{R})$ is positive and strictly decreasing, and one has

$$
\tan \chi(z) = \frac{\text{Ai}(z)}{\text{Bi}(z)} \quad \text{if } \text{Bi}(z) \neq 0.
$$

Furthermore, $-\chi'(z)$ is also strictly decreasing. For $z$ large and positive, one has $\chi(z) = O(\exp(-4z^{3/2}/3))$ and

$$
\overline{\chi}(z) = \frac{2}{3} z^{3/2} + O(z^{-3/2}).
$$

This follows from standard properties of the solutions of Airy's equation [9]. One can now reformulate Proposition (10). With $z_x$ defined by (11), put

$$
\tilde{z} = \chi_{x_0}^x(\tau, \eta) - \chi_{x_0}^x(\tau, \eta), \quad (\tau, \eta) \in \mathbb{R}^+ \times \mathbb{R}.
$$

Then $\tilde{z} > 0$, and $\tau \to \tilde{z}$ is strictly increasing. One can therefore invert (24) to obtain

$$
\tau = \mu(\tilde{z}, \eta) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}),
$$

and infer from (12) that the eigenvalues of the Dirichlet problem (9) are given by $\lambda_{mj}^x = (j, m\gamma/Y)$. So one can write (15) as

$$
\hat{\sigma}_e(t) = \sum_{m, j = -\infty}^\infty \rho(j) \cos(\mu(j, m\gamma/Y)t),
$$

where $\rho(\tilde{z}) \in C^\infty(\mathbb{R})$ is such that

$$
\rho = 0 \quad \text{if } \tilde{z} \leq \delta, \quad \rho = 1 \quad \text{if } \tilde{z} \geq \delta', \quad 0 < \delta < \delta' < 1.
$$

The second member of (26) converges in $\mathcal{J}'(\mathbb{R})$. So, if $\phi \in \mathcal{J}(\mathbb{R})$ is real valued, one has

$$
\langle \hat{\sigma}_e, \phi \rangle = \Re \sum_{m, j = -\infty}^\infty \rho(\tilde{z}) \phi(\mu(j, m\gamma/Y)).
$$
It is not hard to show that $\rho(\frac{\hat{\mu}}{\hat{\mu}}) \mu(\frac{\hat{\mu}}{\hat{\mu}}) \in \mathcal{J}(\mathbb{R}^2)$. One can therefore appeal to the classical Poisson summation formula, and after some manipulations, one obtains:

(28) Proposition. Let $\phi \in \mathcal{J}(\mathbb{R})$ be real valued. Then

(29) $\langle \hat{\sigma}_\epsilon, \phi \rangle = \text{Re} \sum_{\eta, k=-\infty}^{\infty} \int \sigma_{nk}(\tau) \hat{\phi}(\tau) d\tau = \text{Re} \sum_{\eta, k=-\infty}^{\infty} \langle \hat{\sigma}_{nk}, \phi \rangle$,

where

(30) $\sigma_{nk}(\tau) = \int A_{nk}(\tau, \lambda) \exp(iS_{nk}(\tau, \lambda)) d\lambda$,

(31) $S_{nk} = 2\pi k \hat{\chi}(\tau, \lambda) + 2n \lambda \tau$,

(32) $\hat{\chi}(\tau, \lambda) = \chi(\tau^{2/3}(\lambda^2 - 1)) - \chi(\tau^{2/3}(\lambda^2 - 1))$,

(33) $(3\pi/2\lambda)A_{nk} =

\rho * \hat{\chi}(\sigma_{nk}((2\lambda^2 + 1) \chi'(\tau^{2/3}(\lambda^2 - 1)) - (2\lambda^2 + 1 + d) \chi'((\tau^{2/3}(\lambda^2 - 1))).

Also, $A_{nk} = 0$ for $\tau \leq \delta^h$, where $\delta^h > 0$ depends on the choice of $\rho$.

5. The singularities of $\hat{\sigma}_\epsilon$. These can now be examined by analysing the behaviour of $\sum_{\eta} \sigma_{nk}(\tau)$ as $\tau \to \infty$. Roughly speaking, the terms with $k = 0$ are related to the singularity at $t = 0$. As this is now well understood in the general case([10], [8], [6]), it will not be discussed here.

For $k \neq 0$, it is found that the asymptotic behaviour of $\sigma_{nk}$ yields information on the singularity of $\hat{\sigma}_\epsilon$ near $t = T_{nk}$, the length of the geodesic $\gamma_{nk}$ of Proposition (16). We now go on to state the principal results obtained; the proofs will be published elsewhere [23]. As $\sigma_{nk}$ is even, we take $t > 0$. We write

(34) $\sum_{\ell} T \in \mathbb{R}$: there is a closed geodesic on $M$ of length $|T|$.
We shall use the notation, for any real number \( s \),
\[
H^{s-}_{100} = \{ f : f \in H^{s}_{100}(\mathbb{R}) \text{ for } t < s \}.
\]
We begin with the 'regular' case.

**Theorem.** Let \( \gamma_{nk} \) be a non degenerate closed geodesic of type I, with \( n \) and \( k \) as in Proposition (16), \( k > 0 \). Let \( T_{nk} \) be the length of \( \gamma_{nk} \), and \( J \subset \mathbb{R} \) an open interval such that \( J \cap \Sigma = \{ T_{nk} \} \). Then there are complex numbers \( a_{nk}^{(m)} \), \( m = 0, 1, \ldots \) such that, for any \( N \geq 0 \),
\[
(37) \quad \hat{\sigma}_{e}(t)\vert_{J} = \text{Re} \sum_{m=0}^{N} a_{nk}^{(m)}(t-T_{nk})^{m} + r_{N}, \quad r_{N} \in H^{N-}_{100}.
\]
Also,
\[
(38) \quad a_{nk}^{(\sigma)} = i^{k+\varepsilon} \frac{T\gamma_{nk}}{2\pi k^{3/2}} |\psi_{nk}|^{3/2},
\]
where \( \lambda \) is the appropriate solution of (18), and \( \varepsilon = 1 \) if \( \lambda^{2} < \frac{1}{2}(1+d) \), \( \varepsilon = 0 \) if \( \lambda^{2} > \frac{1}{2}(1+d) \).

The proof is in effect an application of the method of stationary phase to (30). The result is essentially that of [5], allowing for the observation made in the remark following Proposition (16). The factor \( \frac{T\gamma_{nk}}{2\pi k^{3/2}} |\psi_{nk}|^{3/2} \) incorporates the Maslov index and the changes of sign due to reflection at the boundary. The other factor in (38) is proportional to the so-called invariant volume of the relevant fixed point set of the geodesic flow on \( S^{*M} \).

It is clear from (19) and (38) that (37) cannot hold when the closed geodesic \( \gamma_{nk} \) is degenerate. In fact, the phase function which comes from (31) and (32) is then degenerate. However, this case is easy to handle. We only remark that, whereas in the non-degenerate case \( \sigma_{nk} \) is a classical symbol of order \( \frac{1}{2} \), it is the sum of two such in the degenerate case, of orders \( \frac{3}{2} \) and \( \frac{1}{2} \) respectively, and omit the detailed formulae.
Theorem. Suppose that $\frac{d}{Y}$ is a rational number, and that
$m/k = 2d/k$, $k > 0$. Then there is a closed grazing geodesic $\gamma_{nk}$ of
length $T_{nk} = \frac{2n(2+d)}{3Y}$. Let $J \subset \mathbb{R}$ be an open interval such that
$J \cap \sum = \{ T_{nk} \}$. Then $\sum_{ \gamma_{nk}} J$ is the sum of two terms, one of which has the
expansion (37), while the other one can be expanded as

$$
(40) \quad \text{Re} \sum_{m=0}^{N} g_m (t-T_{nk} - \omega) (m-4)/3 + r_N, \quad r_N \in H^{(2N-3)/6} , \quad N = 0, 1, \ldots
$$

The $g_m$ involve the (oscillatory) integrals

$$
c_{km} = \frac{1}{2\pi i} \int w^m \frac{A_{-}^{k-1}(w)}{A_{+}^{k+1}(w)} \, dw
$$

where $A_{+}(w) = \text{Ai}(e^{2\pi i/3} w)$ and $A_{-}(w) = \text{Ai}(e^{-2\pi i/3} w)$; in particular, $g_0$ is a multiple of $c_{0,k}^{-k} \gamma_{nk}$.

In this case, the significant contribution to (30) comes from a neigh-
bourhood of $\lambda = 1$ or $\lambda = -1$, and the term $\gamma(t^2/3(\lambda^2 - 1))$ in $S_{nk}$ cannot
be handled by means of (23). However, it also follows from Lemma (20)
that, if $k \in \mathbb{Z}$, then

$$
\exp ik(\gamma(z) - \frac{2\pi}{3}) = A_{-}^{k}(z)/A_{+}^{k}(z).
$$

This gives an alternative form of $\sum_{nk}$ which, with appropriate asymptotic
analysis, gives (40). The 'strange constants' $c_{km}$ resemble those which
appear in the problem of forward scattering [7] and, like them, are no
doubt related to the fact that Airy operators are needed for the construc-
tion of microlocal parametrices near diffractive points of the boundary.

Finally, we consider the gliding ray problem, perhaps the most interest-
ing feature. Write $\mathcal{D}^0 = \{ \tilde{d}^1(x) \in \mathbb{R}/2Y \mathbb{Z} \}$ for the geodesically convex con-
nected component of $\mathcal{D}M$. Its (Riemannian) length is $L = 2Y(1+d)^{\frac{1}{2}}$. It is
not a geodesic, but a limit of (broken) geodesics. Indeed, the following
is easily deduced from Proposition (16):
Proposition. The set of accumulation points of $\Sigma$ is $\{ \mathbb{ZL} \}$.

For any $n > 0$, there is a $k_0 > 0$ and a sequence $\gamma_{nk}, k = k_0, k_0 + 1, \ldots$ of non-degenerate closed type I geodesics such that $\lambda_{nk} \rightarrow (1+d)^{\frac{1}{2}}$, $T_{nk} \rightarrow nL$, and these $\gamma_{nk}$ converge to $\gamma_0^M$ described $n$ times with positive orientation. Similar statements are true for $n < 0$.

Theorem (36) holds for each $\gamma_{nk}$, but one cannot simply add the asymptotic expansions (37) in order to obtain the behaviour of $\hat{\sigma}_e(t)$ in the neighbourhood of $t = nL$. However, one easily sees from (38) and (17) $a^{(0)}_{nk} = O(k^{-2})$, so that the sum of the top order terms converges. Put

$$K_n(t) = \Re \sum_{k=k_0}^{\infty} a^{(0)}_{nk}(t-T_{nk}-i0)^{-3/2}$$

Then one has

Theorem. Let $n$ be a positive integer, and let $J$ be an open interval such that $J \cap \Sigma = \{ T_{nk} : k > k_0 \}$, with $k_0$ and $T_{nk}$ as in Proposition (41). Then

$$\hat{\sigma}_e(t)|_J = K_n(t) + O(H^{-3/4}_{\infty})$$

Observe that this is a genuine error estimate, as $K_n \in H^{-1}_{\infty}$; we do not know if it is the best possible.

As in the case of Theorem (39), the difficulty is that one has to work in a range of $\lambda$ (a neighbourhood of $(1+d)^{\frac{1}{2}}$ or of $-(1+d)^{\frac{1}{2}}$) where the application of (23) to the phase function $S_{nk}$ of Proposition (28) is problematical. There is a constant $c$ such that, for any $T > 0$, the $\sigma_{nk}$ with $k > cT^{1/3}$ are smooth; but one cannot control the error terms for the sum over $k < cT^{1/3}$. However, it turns out that one can do so for the sum of the $\sigma_{nk}$ over $k < cT^{1/4}$, and obtain another estimate for the range $cT^{1/4} < k < cT^{1/3}$.
REFERENCES


