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Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups.

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1. Introduction and notation. We describe here some recent results, obtained jointly with Lawrence Corwin [3] on solvability of left invariant differential operators on nilpotent Lie groups. For related results see [2], [8], [9], [10], [11], [14], [15], [16], [17].

We consider first operators on a 2-step nilpotent Lie group $G$, i.e., we assume that the Lie algebra $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ with $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_2, \mathfrak{g}] = (0)$. For $\eta \in \mathfrak{g}_2^*$ let $B_\eta$ be the bilinear form $B_\eta(X_1, X_2) = \eta([X_1, X_2])$ for $X_1, X_2 \in \mathfrak{g}_1$. $B_\eta$ assumes its maximal rank on a Zariski open subset in $\mathfrak{g}_2^*$.

Recall that to every $\iota \in \mathfrak{g}^*$ we may associate, by the Kirillov theory, an irreducible unitary representation $\pi_\iota$ of $G$, realized on a Hilbert space of the form $L^2(\mathbb{R}^k)$ for some $k$.

By a transversally elliptic operator on $G$ we shall mean a left invariant differential operator $L$ on $G$ which is an elliptic polynomial on $\mathfrak{g}_1$, i.e.,

$$L = L_m + L_{m-1} + \cdots + L_0,$$

with $L_j$ homogeneous of degree $j$, and $L_m$ an elliptic polynomial on $\mathfrak{g}_1$.

2. Necessary conditions for local solvability. We give the following criterion, which generalizes known results [2] for homogeneous operators i.e. those for which $L_j \equiv 0, \ j < m$. 
Theorem 1. Let $L$ be a left invariant operator on $G$ which is transversally elliptic. Assume that there is a non-empty open set $V \subset \mathcal{O}$ such that

$$(2.1) \quad \ker \pi_I(L^T) \neq 0 \quad \text{for all } l \in V,$$

or, equivalently,

$$(2.2) \quad \ker L^T \cap L^2(G) \neq 0.$$  

Then $L$ is not locally solvable.

The idea of the proof is as follows. First, if $B_\eta$ is degenerate for all $\eta$, then [1] may be applied to show that the hypothesis is vacuous. So assume $B_\eta$ nondegenerate for $\eta$ in a Zariski open set. We show, using microlocal constructions as in [6] that there is a pseudo-differential operator $\Pi$ not of order $-\infty$ such that $L^T \Pi$ is of order $-\infty$. Now for any distribution $\sigma$ for which $Lv - \sigma = 0$ in an open set $U$ for some distribution $v$, $\Pi^T(Lv - \sigma)$ is smooth, and hence $\Pi^T \sigma$ is smooth. Hence $\sigma$ cannot be arbitrary.

3. Sufficient conditions for solvability on $H$-groups. $G$ is called an $H$-group if $B_\eta$ is nondegenerate for $\eta \neq 0$. We prove the following converse to Theorem 2 for $H$-groups. A globally defined differential operator $P$ is uniformly semi-globally solvable if there is an integer $r$ such that for every bounded open neighborhood $U$ of $0$ there exists a distribution $\sigma_U$ of order at most $r$ such that $L \sigma_U = \delta$ in $U$.

Theorem 2. If $G$ is an $H$-group and $L$ a left invariant transversally elliptic operator on $G$ then $L$ is uniformly semi-globally solvable if
(2.1) and (2.2) do not hold.

The proof of Theorem 2 is somewhat similar to that of the corresponding result [17] in the case where \( L \) is homogeneous. Both rely on the theorem of Lojasiewicz which says that one can divide a distribution by a non-zero analytic function.

**Corollary.** If \( L_m \) is locally solvable, then \( L \) is locally solvable.

4. **Existence of global fundamental solutions.** Here we allow \( G \) to be any connected Lie group, not necessarily nilpotent.

**Theorem 3.** Let \( L \) be a left invariant differential operator on \( G \) which is uniformly semi-globally solvable. Suppose that \( G \) is \( L \)-convex. Then \( L \) has a global fundamental solution; i.e. there is a distribution \( \sigma \) on \( G \) for which \( L\sigma = \delta \).

The proof of Theorem 3 involves a construction similar to that used in proving that \( L \)-convexity implies global solvability. The theorem gives a new result even for homogeneous operators.

**Corollary 1.** Let \( L \) be a homogeneous left invariant differential operator on a nilpotent Lie group \( G \) with dilations. If \( L \) is locally solvable at \( 0 \) then \( L \) has a global fundamental solution.

**Corollary 2.** If \( L \) is a transversally elliptic operator on an \( H \)-group which satisfies the hypothesis of Theorem 2 then \( L \) has a global fundamental solution.
5. **Global criteria for hypoellipticity.** The various global criteria for local solvability for homogeneous differential operators on nilpotent groups, e.g., \( \ker L^r \cap L^2(G) = (0) \), suggest that the representation-theoretic criterion of Helffer-Nourrigat [5] may be reformulated. Indeed, using a recent Liouville-type theorem of Geller [4] one may obtain the following.

**Theorem 4.** (Geller, Helffer-Nourrigat). Let \( G \) be a stratified nilpotent Lie group and \( L \) a homogeneous left invariant differential operator on \( G \). Then \( L \) is hypoelliptic if and only if there is no non-constant bounded function \( f \) on \( G \) such that \( Lf = 0 \).
References


