LINDA P. ROTHSCHILD

Existence of solutions for transversally elliptic left invariant
differential operators on nilpotent Lie groups


<http://www.numdam.org/item?id=JEDP_1983_____A12_0>
Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups.

Linda Preiss Rothschild
University of Wisconsin - Madison

1. Introduction and notation. We describe here some recent results, obtained jointly with Lawrence Corwin [3] on solvability of left invariant differential operators on nilpotent Lie groups. For related results see [2], [8], [9], [10], [11], [14], [15], [16], [17].

We consider first operators on a 2-step nilpotent Lie group \( G \), i.e. we assume that the Lie algebra \( \mathfrak{g} \) is a vector space direct sum \( \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 \) with \( [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \) and \( [\mathfrak{g}_2, \mathfrak{g}] = (0) \). For \( \eta \in \mathfrak{g}_2^* \) let \( B_\eta \) be the bilinear form \( B_\eta(X_1, X_2) = \eta([X_1, X_2]) \) for \( X_1, X_2 \in \mathfrak{g}_1 \). \( B_\eta \) assumes its maximal rank on a Zariski open subset in \( \mathfrak{g}_2^* \). Recall that to every \( \ell \in \mathfrak{g}_1^* \) we may associate, by the Kirillov theory, an irreducible unitary representation \( \pi_\ell \) of \( G \), realized on a Hilbert space of the form \( L^2(\mathbb{R}^k) \) for some \( k \).

By a transversally elliptic operator on \( G \) we shall mean a left invariant differential operator \( L \) on \( G \) which is an elliptic polynomial on \( \mathfrak{g}_1 \), i.e.

\[
L = L_m + L_{m-1} + \cdots + L_0,
\]

with \( L_j \) homogeneous of degree \( j \), and \( L_m \) an elliptic polynomial on \( \mathfrak{g}_1 \).

2. Necessary conditions for local solvability. We give the following criterion, which generalizes known results [2] for homogeneous operators i.e. those for which \( L_j \equiv 0, \ j < m \).
Theorem 1. Let $L$ be a left invariant operator on $G$ which is transversally elliptic. Assume that there is a non-empty open set $V \subset \mathcal{O}$ such that

\[(2.1)\quad \ker \pi_L(L^T) \neq 0 \quad \text{for all} \quad l \in V,
\]
or, equivalently,

\[(2.2)\quad \ker L^T \cap L^2(G) \neq 0.\]

Then $L$ is not locally solvable.

The idea of the proof is as follows. First, if $B_\eta$ is degenerate for all $\eta$, then [1] may be applied to show that the hypothesis is vacuous. So assume $B_\eta$ nondegenerate for $\eta$ in a Zariski open set. We show, using microlocal constructions as in [6] that there is a pseudo-differential operator $\Pi$ not of order $-\infty$ such that $L^T \Pi$ is of order $-\infty$. Now for any distribution $\sigma$ for which $Lv - \sigma = 0$ in an open set $U$ for some distribution $v$, $\Pi^T(Lv - \sigma)$ is smooth, and hence $\Pi^T \sigma$ is smooth. Hence $\sigma$ cannot be arbitrary.

3. Sufficient conditions for solvability on H-groups. $G$ is called an H-group if $B_\eta$ is nondegenerate for $\eta \neq 0$. We prove the following converse to Theorem 2 for H-groups. A globally defined differential operator $P$ is uniformly semi-globally solvable if there is an integer $r$ such that for every bounded open neighborhood $U$ of $0$ there exists a distribution $\sigma_U$ of order at most $r$ such that $L\sigma_U = \delta$ in $U$.

Theorem 2. If $G$ is an H-group and $L$ a left invariant transversally elliptic operator on $G$ then $L$ is uniformly semi-globally solvable if
(2.1) and (2.2) do not hold.

The proof of Theorem 2 is somewhat similar to that of the corresponding result [17] in the case where $L$ is homogeneous. Both rely on the theorem of Lojasiewicz which says that one can divide a distribution by a non-zero analytic function.

**Corollary.** If $L_m$ is locally solvable, then $L$ is locally solvable.

4. **Existence of global fundamental solutions.** Here we allow $G$ to be any connected Lie group, not necessarily nilpotent.

**Theorem 3.** Let $L$ be a left invariant differential operator on $G$ which is uniformly semi-globally solvable. Suppose that $G$ is $L$-convex. Then $L$ has a global fundamental solution; i.e. there is a distribution $\sigma$ on $G$ for which $L\sigma = \delta$.

The proof of Theorem 3 involves a construction similar to that used in proving that $L$-convexity implies global solvability. The theorem gives a new result even for homogeneous operators.

**Corollary 1.** Let $L$ be a homogeneous left invariant differential operator on a nilpotent Lie group $G$ with dilations. If $L$ is locally solvable at 0 then $L$ has a global fundamental solution.

**Corollary 2.** If $L$ is a transversally elliptic operator on an $H$-group which satisfies the hypothesis of Theorem 2 then $L$ has a global fundamental solution.
5. Global criteria for hypoellipticity. The various global criteria for local solvability for homogeneous differential operators on nilpotent groups, e.g., \( \ker L^T \cap L^2(G) = (0) \), suggest that the representation-theoretic criterion of Helffer-Nourrigat [5] may be reformulated. Indeed, using a recent Liouville-type theorem of Geller [4] one may obtain the following.

**Theorem 4.** (Geller, Helffer-Nourrigat). Let \( G \) be a stratified nilpotent Lie group and \( L \) a homogeneous left invariant differential operator on \( G \). Then \( L \) is hypoelliptic if and only if there is no non-constant bounded function \( f \) on \( G \) such that \( Lf = 0 \).
References


