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A GENERAL CLASS OF GEVREY-TYPE
PSEUDO DIFFERENTIAL OPERATORS

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Recently much attention has been paid to the study of new classes of analytic and Gevrey-type pseudo differential operators; see for example Matsuzawa [8], Iftimie [5], Bolley–Camus–Métivier [2].

We shall consider here symbols \( a(x,\xi) \) of general Gevrey type for which

\[
|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C |\alpha| + |\beta| + 1 \alpha! \beta! \varphi(\xi)^m - |\alpha| |\psi(\xi)|^{m'} + |\alpha| |\beta|
\]

if \( C' |\beta| \leq \varphi(\xi) \).

The weight functions \( \varphi, \psi \) are continuous in \( \mathbb{R}^n \) and satisfy for suitable positive constants \( \varepsilon, \varepsilon' \) independent of \( \xi, \eta \in \mathbb{R}^n \):

\[
\begin{align*}
(2) & \varepsilon (1 + |\xi|)^{\varepsilon} \leq \varphi(\xi) \leq \varepsilon' \varphi(\xi) \\
(3) & \varepsilon \leq \frac{\varphi(\xi)}{\varphi(\eta)}^{-1} \leq \varepsilon', \quad \varepsilon \leq \frac{\varphi(\xi)}{\varphi(\eta)}^{-1} \leq \varepsilon', \quad |\xi - \eta| \leq \varepsilon \varphi(\xi).
\end{align*}
\]

To these conditions, which are quite common for general pseudo differential operators, we add the technical assumption:

\[
(4) \text{ for every } \delta \text{ there exists } \delta' \text{ such that } \varphi(\eta) \leq \delta |\xi - \eta| \implies \varphi(\eta) \psi(\xi) \leq \delta' |\xi - \eta| \varphi(\xi).
\]

From the \( C^\infty \) point of view our symbols can be regarded as elements of a class of Beals [1] \( S^{\lambda}_{\mathcal{H}, \vartheta} \), with \( \mathcal{H} = \psi, \vartheta = \varphi/\psi \). The reason why we prefer here to refer to the function \( \varphi = \vartheta \psi \) for the estimates in (1) is that a peculiar property of the pseudo differential operator

\[
(5) \quad a(x,D)f(x) = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi) \hat{f}(\xi) \, d\xi
\]

associated with \( a(x,\xi) \) turns out to be the continuity from
$G$ to $G$, where $G$, $G$ are the inhomogeneous Gevrey classes related to the weight functions $\phi$, $\varphi$, respectively.

Let us begin by giving a general definition of such classes in terms of Fourier transform. Let $\varphi$ (or $\phi$) be a weight function as in (2), (3). More generally, let

$\lambda: \mathbb{R}^n \to \mathbb{R}_+$ be Lipschitzian, in the sense that $|\lambda(\xi) - \lambda(\eta)| \leq C|\xi - \eta|$ for some constant $C$ independent of $\xi, \eta$, and assume also $\varepsilon(1 + |\xi|)^\varepsilon \leq \lambda(\xi)$ for some $\varepsilon > 0$. Let $X$ be open in $\mathbb{R}^n$.

**Definition 1.** We say that $f \in \mathcal{D}'(X)$ is of class $G_\lambda$ at $x_0 \in X$ if there is a neighborhood $U$ of $x_0$, $U \subset X$, and a bounded sequence $f_j \in \mathcal{E}'(X)$ such that $f = f_j$ in $U$ and

$$|\hat{f}_j(\xi)| < c(cj/\lambda(\xi))^j, \quad j = 1, 2, \ldots$$

We denote by $G_\lambda(X)$ the set of all $f \in \mathcal{D}'(X)$ which are of class $G_\lambda$ at every $x_0 \in X$.

When $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho < 1$, $G_\lambda(X)$ is the standard class $G^{1/\rho}(X)$ of all the functions $f \in \mathcal{C}^\infty(X)$ which satisfy in every $K \subset X$ the estimates

$$|D^\alpha f(x)| < C|\alpha| + 1 (\alpha!)^{1/\rho},$$

(cf. Hörmander [4], Proposition 2.4).

In particular for $\lambda(\xi) = 1 + |\xi|$ we have $G_\lambda(X) = \mathcal{O}(X)$, the set of all the real analytic functions in $X$.

Classes $G_\lambda(X)$ with inhomogeneous $\lambda$ have been considered by several authors under different definitions; see for example Liess [6] and the references there. The advantage of the present definition is that it can be microlocalized in a natural
way, adapting the procedure used by Rodino [10] in the $C^\infty$ framework. Fix $\Gamma \subseteq \mathbb{R}^n_\xi$ and set for $\varepsilon > 0$

$$\Gamma_{\varepsilon, \lambda} = \{\xi \in \mathbb{R}^n, \text{dist} (\xi, \Gamma) < \varepsilon \lambda(\xi)\}.$$  

**Definition 2.** We shall say that $f$ is $G_\lambda$-smooth at $\{x_0\} \times \Gamma$ and we shall write formally $WF_{\lambda} f \cap (\{x_0\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{\varepsilon, \lambda}$, for a sufficiently small $\varepsilon > 0$.

It is natural then to introduce the space of the "microfunctions" at $\{x_0\} \times \Gamma$.

**Definition 3.** We denote by $C^{\infty}_{x_0, \Gamma, \lambda}$ the factor space $C^{\infty}_{x_0} / \sim$, where $C^{\infty}_{x_0}$ is the set of the germs of $C^\infty$ functions defined near $x_0$ and $f \sim g$ in $C^{\infty}_{x_0}$ iff $WF_{\lambda} (f-g) \cap (\{x_0\} \times \Gamma) = \emptyset$.

It is convenient in certain applications to use also a different kind of microlocalization. Precisely, set for $\varepsilon > 0$

(8)$'$ $\Gamma_{\varepsilon, \lambda} = \{\xi \in \mathbb{R}^n, \lambda(\xi - \eta) < \varepsilon \lambda(\xi) \text{ for some } \eta \in \Gamma\}.$

**Definition 2$'$.' We shall say that $f$ in strongly $G_\lambda$-smooth at $\{x_0\} \times \Gamma$ and we shall write formally $WF_{\lambda} f \cap (\{x_0\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{\varepsilon, \lambda}$, for a sufficiently small $\varepsilon > 0$.

For example, if $\Gamma$ is the halfray generated by $\xi_0 \neq 0$ and $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho \leq 1$, then $WF_{\lambda} f \cap (\{x_0\} \times \Gamma) = \emptyset$ means that $(x_0, \xi_0)$ is not in the Gevrey wave front set $WF_{1/\rho} f$ of Hörmander [4].

Note that strong $G_\lambda$-smoothness at $\{x_0\} \times \Gamma$ implies $G_\lambda$-smoothness there, but the converse is not true in general.

Let us now return to pseudo differential operators and give a precise definition of our classes from the microlocal point of view.
Assume \( \psi \) and \( \psi \) satisfy the conditions (2), (3), (4). Let \( X \) be open in \( \mathbb{R}^n \) and fix \( \Gamma \subset \mathbb{R}^n \).

**Definition 4.** We define \( S_{\psi,\psi}^{m,m'}(X,\Gamma) \) to be the set of all \( a(x,\xi) \in C^\infty(X \times \Gamma) \) which can be extended for some \( \varepsilon > 0 \) to functions in \( C^\infty(X \times \Gamma, \varepsilon^\psi) \) such that (1) is satisfied with suitable positive constants \( C, C' \) independent of \( x \in X, \xi \in \varepsilon^\psi \).

A symbol \( a(x,\xi) \in S_{\psi,\psi}^{m,m'}(X,\Gamma) \) can be further extended to a function \( \hat{a}(x,\xi) \in C^\infty(X \times \mathbb{R}^n) \), by cutting off in the \( \xi \) variables, and \( \hat{a}(x,\xi) \) from (5) is then defined as a map from \( C^\infty(X) \) to \( C^\infty(X) \). The continuity property can now be expressed in the following microlocal form.

**Theorem 5.** Let \( a(x,\xi) \) be in \( S_{\psi,\psi}^{m,m'}(X,\Gamma) \), and take \( x_0 \in X, X \subset \Gamma \). Then \( \hat{a}(x,\xi) \) defines by factorization an operator

\[
(9) \quad a(x,\xi) : C^\infty_{X_0,\Lambda,\psi} \rightarrow C^\infty_{X_0,\Lambda,\psi}
\]

which depends only on \( a \) and not also on the extensions \( \hat{a} \) of \( a \).

The symbolic calculus for the operators \( a(x,\xi) \) in (9) follows the lines of the calculus of the \( C^\infty \)-general pseudo differential operators (cf. Beals [1]), with some evident complications in the estimates due to the factor

\[
C|\alpha|+|\beta| + 1 \alpha!\beta! \text{ which we expect in (1).}
\]

From Theorem 5 and from symbolic calculus one deduces by means of a standard argument the following result on existence of parametrices.

**Theorem 6.** Consider \( a(x,\xi) \in S_{\psi,\psi}^m(X,\Gamma) = S_{\psi,\psi}^{O,m}(X,\Gamma) \subset S_{\psi,\psi}^{O,m}(X,\Gamma) \) and fix \( x_0 \in X, \Lambda \subset \Gamma \). Assume there exist a neighborhood \( U \) of \( x_0, U \subset X, \) real numbers \( m, m' \) and positive constants \( c, c', \varepsilon, \varepsilon \).
such that

\( |a(x, \xi)| \geq c \varphi(\xi)^{m_1} \psi(\xi)^{m_1} \) for \( \xi \in \Lambda \) and

\( |\xi| \geq C \)

\( |D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C |\alpha| + |\beta| |a(x, \xi)| \varphi(\xi)^{-|\alpha|} \psi(\xi)^{|\alpha|} \)

for all \( \alpha \) and all \( x, \xi, \beta \) with \( x \in U, \xi \in \Lambda \), \( |\beta| \leq \varphi(\xi) \)

and \( |\xi| \geq C \).

Then there is \( b \in S^{-m_1, -m_1'}(U, \Lambda) \) such that \( b(x, D) a(x, D) \):

\[ C_{\Lambda, \psi} \rightarrow C_{\Lambda, \psi} \]

is the natural inclusion. In particular, for any fixed extension \( \tilde{a} \) of \( a \), we have that \( WF \tilde{a}(x, D) f \cap

\( \{x \} \times \Lambda = \emptyset \) implies \( WF_f \cap \{x \} \times \Lambda = \emptyset \).

When \( A = a(x, D) \) is a linear partial differential operator with analytic coefficients in \( X \) there are some obvious simplifications in the statement; namely, if for every \( K \subseteq X \) we have for large \( |\xi| \) and suitable constants \( |a(x, \xi)| \geq c|\xi|^r \) and

\( |D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C |\alpha| + |\beta| |a(x, \xi)| \varphi(\xi)^{-|\alpha|} \psi(\xi)^{|\alpha|} \)

then \( A f \in G_\varphi(X) \) implies \( f \in G_\varphi(X) \) for every \( f \in \mathcal{O}'(X) \); in particular all solutions of \( A f = 0 \) are in \( G_\varphi(X) \).

A simple example is given by the hypoelliptic operators with constant coefficients \( P = p(D) \). Let \( \delta(\xi) \) be the distance from \( \xi \in \mathbb{R}^n \) to the surface \( \{\xi \in C^n, p(\xi) = 0\} \), and set \( \psi(\xi) = 1 + \delta(\xi) \). It is well known that

\( |D^\beta_\xi p(\xi)| \leq C |p(\xi)| \psi(\xi)^{-|\beta|} \)

and \((11)'\) is then satisfied with \( \psi = \varphi \). We conclude that

\( P f \in G_\psi(X) \) implies \( f \in G_\psi(X) \) for any \( X \subseteq \mathbb{R}^n \) and all \( f \in \mathcal{O}'(\mathbb{R}^n) \).
An example of operator for which $\varphi \neq \psi$ (that means a loss of Gevrey regularity for the solutions) is given by

$$A = 1 + |x|^{2k} p(D),$$

where $p(D)$ is hypoelliptic and $p(\xi) > 0$; the estimates $(11)'$ are satisfied for $\varphi(\xi)$ as in preceding example and any $\varphi(\xi)$ for which $p(\xi) < (\varphi(\xi)/\varphi(0))^{2k}$.

Theorem 6, as well as Theorem 5, can be restated in terms of strong $G_\lambda^s$-smoothness, according to Definition 2'. A relevant application is given by the choice $\varphi(\xi) = (1 + |\xi|)^p$, $\varphi(\xi) = (1 + |\xi|)^{2-\delta}$, $0 \leq \delta < p < 1$, which corresponds to the operators in [2], [5], [8]. Since the related Gevrey wave front sets are invariant under canonical transformations, geometric invariant statements are possible in this case; for example, let us consider a classical analytic symbol $a(x,\xi) = \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$ and assume the principal part $a_m(x,\xi)$ vanishes exactly of order $k$, $k \geq 2$, on an involutive manifold $\Sigma \subset T^*X \setminus 0$. Noting $a_{m-1}^*$ the subprincipal symbol, set for any $\gamma \in \Sigma$ and for any $C^\infty$ vector field $Y$ defined in a neighborhood of $\gamma$

$$I_a(\gamma, Y) = (k!)^{-1} \left( Y^k a_m \right) (\gamma) + a_{m-1}^*(\gamma).$$

**Theorem 7.** Assume $I_a(\gamma, Y) \neq 0$ for every $\gamma$ and $Y$. Then, writing $s = k/(k-1)$, we have $WF_s a(x,D)f = WF_s f$ for all $f \in \mathcal{E}'(X)$. In particular $a(x,D)f \in G^s(X)$ implies $f \in G^s(X)$. In fact, after conjugation by a Fourier integral operator, $a(x,D)$ becomes an operator to which Theorem 6 applies with $\psi(\xi) = \varphi(\xi) = (1 + |\xi|)^{1/s}$ (cf. Parenti-Rodino [9], where $C^\infty$-
hyp- oellipticity was proved under the same assumptions).
Similarly we can prove a $G^2$-hyp- oellipticity result for the
operators in the classes of Boutet de Monvel-Grigis-Helffer
[2].

Another application of Theorem 6 refers the choice

$$
\psi(\xi) = \sum_{j=1}^{n} |\xi_j|^{1/M_j},
$$

where $M = (M_1, \ldots, M_n)$ is a n-tuple of rational numbers $>1$; the related hyp- oellipticity results
can be expressed in terms of the anisotropic Gevrey wave
front set $WF_M$ of Zanghirati [12], Rodino [11]. Details and
proofs of the results announced here will be found in Liess-
Rodino [7].

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