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A GENERAL CLASS OF GEVREY-TYPE
PSEUDO DIFFERENTIAL OPERATORS

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Recently much attention has been paid to the study of new classes of analytic and Gevrey-type pseudo differential operators; see for example Matsuzawa [8], Ifrimie [5], Bolley-Camus-Métivier [2].

We shall consider here symbols $a(x,\xi)$ of general Gevrey type for which

$$
(1) \quad |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C|\alpha|+|\beta|+1 \alpha!\beta! \varphi(\xi)^m-|\alpha| \varphi(\xi)^m'-|\alpha|-|\beta| \\
\text{if } \quad C'|\beta| \leq \varphi(\xi).
$$

The weight functions $\varphi, \psi$ are continuous in $\mathbb{R}^n$ and satisfy for suitable positive constants $\varepsilon, \varepsilon'$ independent of $\xi, \eta \in \mathbb{R}^n$:

$$
(2) \quad \varepsilon (1 + |\xi|)^C \leq \varphi(\xi) \leq \varepsilon' \psi(\xi) \\
(3) \quad \varepsilon \leq \varphi(\xi) \frac{\psi(\eta)}{|\xi-\eta|} \leq \varepsilon', \quad \varepsilon \leq \varphi(\xi) \frac{\psi(\eta)}{|\xi-\eta|} \leq \varepsilon' \\
\text{if } \quad |\xi-\eta| \leq \varepsilon \varphi(\xi).
$$

To these conditions, which are quite common for general pseudo differential operators, we add the technical assumption:

$$
(4) \quad \text{for every } \delta \text{ there exists } \delta' \text{ such that } \varphi(\eta) \leq \delta |\xi-\eta| \\
\text{implies } \varphi(\eta) \frac{\varphi(\xi)}{|\xi-\eta|} \leq \delta' |\xi-\eta| \varphi(\xi).
$$

From the $C^\infty$ point of view our symbols can be regarded as elements of a class of Beals [1] $S^\lambda_{q,\varphi}$, with $\varphi = \psi$, $\varphi = \varphi/\psi$. The reason why we prefer here to refer to the function $\varphi = \varphi/\psi$ for the estimates in (1) is that a peculiar property of the pseudo differential operator

$$
(5) \quad a(x,\partial)f(x) = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi) \hat{f}(\xi) \, d\xi
$$

associated with $a(x,\xi)$ turns out to be the continuity from
G to $G$, where $G, G$ are the inhomogeneous Gevrey classes related to the weight functions $\psi, \varphi$, respectively.

Let us begin by giving a general definition of such classes in terms of Fourier transform. Let $\psi$ (or $\varphi$) be a weight function as in (2), (3). More generally, let $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be Lipschitzian, in the sense that $|\lambda(\xi) - \lambda(\eta)| \leq C|\xi - \eta|$ for some constant $C$ independent of $\xi, \eta$, and assume also $\varepsilon(1 + |\xi|)^\varepsilon \leq \lambda(\xi)$ for some $\varepsilon > 0$. Let $X$ be open in $\mathbb{R}^n$.

**Definition 1.** We say that $f \in \mathcal{D}'(X)$ is of class $G^\lambda_\lambda$ at $x_0 \in X$ if there is a neighborhood $U$ of $x_0$, $U \subset X$, and a bounded sequence $f_j \in \mathcal{E}'(X)$ such that $f = f_j$ in $U$ and

$$|\hat{f}_j(\xi)| < c(cj/\lambda(\xi))^j, \quad j = 1, 2, \ldots$$

We denote by $G^\lambda_\lambda(X)$ the set of all $f \in \mathcal{D}'(X)$ which are of class $G^\lambda_\lambda$ at every $x_0 \in X$.

When $\lambda(\xi) = (1 + |\xi|)^\lambda$, $0 < \rho < 1$, $G^\lambda_\lambda(X)$ is the standard class $G^{1/\rho}(X)$ of all the functions $f \in \mathcal{C}^\infty_\rho(X)$ which satisfy in every $K \subset X$ the estimates

$$|D^\alpha f(x)| < c|\alpha| + 1 (\alpha!)^{1/\rho}$$

(cf. Hörmander [4], Proposition 2.4).

In particular for $\lambda(\xi) = 1 + |\xi|$ we have $G^\lambda_\lambda(X) = \mathcal{O}(X)$, the set of all the real analytic functions in $X$.

Classes $G^\lambda_\lambda(X)$ with inhomogeneous $\lambda$ have been considered by several authors under different definitions; see for example Liess [6] and the references there. The advantage of the present definition is that it can be microlocalized in a natural
way, adapting the procedure used by Rodino [10] in the $C^\infty$ framework. Fix $\Gamma \subset R^n_\xi$ and set for $\varepsilon > 0$

\begin{equation}
\Gamma_{\varepsilon,\lambda} = \{\xi \in R^n, \text{dist } (\xi,\Gamma) < \varepsilon \lambda(\xi)\}.
\end{equation}

**Definition 2.** We shall say that $f$ is $G_\lambda$-smooth at $\{x_o\} \times \Gamma$ and we shall write formally $WF_\lambda f \cap (\{x_o\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{\varepsilon,\lambda}$, for a sufficiently small $\varepsilon > 0$.

It is natural then to introduce the space of the "microfunctions" at $\{x_o\} \times \Gamma$.

**Definition 3.** We denote by $C_{x_o,\Gamma,\lambda}^\infty$ the factor space $C_{x_o}^\infty / \sim$, where $C_{x_o}^\infty$ is the set of the germs of $C^\infty$ functions defined near $x_o$ and $f \sim g$ in $C_{x_o}^\infty$ iff $WF_\lambda (f-g) \cap (\{x_o\} \times \Gamma) = \emptyset$.

It is convenient in certain applications to use also a different kind of microlocalization. Precisely, set for $\varepsilon > 0$

\begin{equation}
\Gamma_{[\varepsilon,\lambda]} = \{\xi \in R^n, \lambda(\xi-\eta) < \varepsilon \lambda(\xi) \text{ for some } \eta \in \Gamma\}.
\end{equation}

**Definition 2'.** We shall say that $f$ is strongly $G_\lambda$-smooth at $\{x_o\} \times \Gamma$ and we shall write formally $WF_\lambda f \cap (\{x_o\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{[\varepsilon,\lambda]}$, for a sufficiently small $\varepsilon > 0$.

For example, if $\Gamma$ is the halfray generated by $\xi_o \neq 0$ and $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho \leq 1$, then $WF_\lambda f \cap (\{x_o\} \times \Gamma) = \emptyset$ means that $(x_o,\xi_o)$ is not in the Gevrey wave front set $WF_{1/\rho} f$ of Hörmander [4].

Note that strong $G_\lambda$-smoothness at $\{x_o\} \times \Gamma$ implies $G_\lambda$-smoothness there, but the converse is not true in general.

Let us now return to pseudo differential operators and give a precise definition of our classes from the microlocal point of view.
Assume \( \varphi \) and \( \psi \) satisfy the conditions (2), (3), (4). Let \( X \) be open in \( \mathbb{R}^n_x \) and fix \( \Gamma \subset \mathbb{R}^n_\xi \).

**Definition 4.** We define \( S^{m,m'}_{\varphi,\psi}(X,\Gamma) \) to be the set of all \( a(x,\xi) \in C^\infty(X \times \Gamma) \) which can be extended for some \( \varepsilon > 0 \) to functions in \( C^\infty(X \times \Gamma_{\varepsilon \psi}) \) such that (1) is satisfied with suitable positive constants \( C, C' \) independent of \( x \in X, \xi \in \Gamma_{\varepsilon \psi} \).

A symbol \( a(x,\xi) \in S^{m,m'}_{\varphi,\psi}(X,\Gamma) \) can be further extended to a function \( \tilde{a}(x,\xi) \in C^\infty(X \times \mathbb{R}^n_\xi) \), by cutting off in the \( \xi \) variables, and \( \tilde{a}(x,\xi) \) from (5) is then defined as a map from \( C^\infty_0(X) \) to \( C^\infty(X) \). The continuity property can now be expressed in the following microlocal form.

**Theorem 5.** Let \( a(x,\xi) \) be in \( S^{m,m'}_{\varphi,\psi}(X,\Gamma) \), and take \( x_0 \in X, \Lambda \subset \Gamma \). Then \( \tilde{a}(x,\xi) \) defines by factorization an operator

\[
(9) \quad a(x,\xi) : C^\infty_{x_0,\Lambda,\psi} \rightarrow C^\infty_{x_0,\Lambda,\varphi}
\]

which depends only on \( a \) and not also on the extensions \( \tilde{a} \) of \( a \).

The symbolic calculus for the operators \( a(x,\xi) \) in (9) follows the lines of the calculus of the \( C^\infty \)-general pseudo differential operators (cf. Beals [1]), with some evident complications in the estimates due to the factor

\[
C|\alpha| + |\beta| + 1 \alpha! \beta! \text{ which we expect in (1).}
\]

From Theorem 5 and from symbolic calculus one deduces by means of a standard argument the following result on existence of parametrices.

**Theorem 6.** Consider \( a(x,\xi) \in S^{m}_{\varphi}(X,\Gamma) = S^{0,m}_{\varphi,\psi}(X,\Gamma) \subset S^{0,m}_{\varphi,\psi}(X,\Gamma) \) and fix \( x_0 \in X, \Lambda \subset \Gamma \). Assume there exist a neighborhood \( U \) of \( x_0, U \subset X \), real numbers \( m, m' \) and positive constants \( c, c', \varepsilon, C \)
such that

\[ |a(x, \xi)| > c \varphi(\xi) \psi(\xi) \]  

for \( x \in U, \xi \in \Lambda \in \psi \) and \( |\xi| > C \)

\[ |D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C |\alpha| + |\beta| |a(x, \xi)| \varphi(\xi) - |a| |\psi(\xi)| c |\alpha| - |\beta| \]

for all \( \alpha \) and all \( x, \xi, \beta \) with \( x \in U, \xi \in \Lambda \in \psi \) and \( |\xi| > C \).

Then there is \( b \in S_{m^1, -m^1}^m (U, \Lambda \) such that \( b(x, D) a(x, D) \):

\[ C_x^\infty \Lambda, \psi \rightarrow C_x^\infty \Lambda, \psi \]  

is the natural inclusion. In particular, for any fixed extension \( \tilde{a} \) of \( a \), we have that \( \mathrm{WF} \tilde{a}(x, D) \cap \{ x \} \times \Lambda \) = \( \phi \) implies \( \mathrm{WF} \phi \cap \{ x \} \times \Lambda \) = \( \phi \).

When \( A = a(x, D) \) is a linear partial differential operator with analytic coefficients in \( X \) there are some obvious simplifications in the statement; namely, if for every \( K \subset X \) we have for large \( |\xi| \) and suitable constants \( |a(x, \xi)| > c |\xi|^r \) and

\[ |D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C |\alpha| + 1 |\alpha| |a(x, \xi)| \varphi(\xi) - |a| |\psi(\xi)| c |\alpha| - |\beta| \]

then \( Af \in G_\psi(X) \) implies \( f \in G_\varphi(X) \) for every \( f \in \mathcal{D}'(X) \); in particular all solutions of \( Af = 0 \) are in \( G_\varphi(X) \).

A simple example is given by the hypoelliptic operators with constant coefficients \( P = p(D) \). Let \( \delta(\xi) \) be the distance from \( \xi \in \mathbb{R}^n \) to the surface \( \{ \xi \in \mathbb{C}^n, p(\xi) = 0 \} \), and set \( \psi(\xi) = 1 + \delta(\xi) \). It is well known that

\[ |D^\beta_\xi p(\xi)| \leq C |p(\xi)| \psi(\xi)^-|\beta| \]

and \( (11)' \) is then satisfied with \( \phi = \varphi \). We conclude that \( P f \in G_\psi(X) \) implies \( f \in G_\psi(X) \) for any \( X \subset \mathbb{R}^n \) and all \( f \in \mathcal{D}'(\mathbb{R}^n) \).
An example of operator for which $\varphi \neq \psi$ (that means a loss of Gevrey regularity for the solutions) is given by

\begin{equation}
A = 1 + |x|^{2k} p(D),
\end{equation}

where $p(D)$ is hypoelliptic and $p(\xi) \geq 0$; the estimates (11)' are satisfied for $\psi(\xi)$ as in preceding example and any $\varphi(\xi)$ for which $p(\xi) < (\psi(\xi)/\varphi(\xi))^{2k}$.

Theorem 6, as well as Theorem 5, can be restated in terms of strong $G_\lambda$-smoothness, according to Definition 2'. A relevant application is given by the choice $\psi(\xi) = (1+|\xi|)^p$, $\varphi(\xi) = (1+|\xi|)^{-\delta}$, $0 \leq \delta < p < 1$, which corresponds to the operators in [2], [5], [8]. Since the related Gevrey wave front sets are invariant under canonical transformations, geometric invariant statements are possible in this case; for example, let us consider a classical analytic symbol $a(x,\xi) = \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$ and assume the principal part $a_m(x,\xi)$ vanishes exactly of order $k$, $k \geq 2$, on an involutive manifold $\Sigma \subset T^*X \setminus 0$. Noting $a_{m-1}^\prime$ the subprincipal symbol, set for any $\gamma \in \Sigma$ and for any $C^\infty$ vector field $Y$ defined in a neighborhood of $\gamma$

\begin{equation}
I_a(\gamma,Y) = (k!)^{-1} (Y^k a_m)(\gamma) + a_{m-1}^\prime(\gamma).
\end{equation}

Theorem 7. Assume $I_a(\gamma,Y) \neq 0$ for every $\gamma$ and $Y$. Then, writing $s = k/(k-1)$, we have $WF_s a(x,D)f = WF_s f$ for all $f \in \mathcal{E}'(X)$. In particular $a(x,D)f \in G^s(X)$ implies $f \in G^s(X)$. In fact, after conjugation by a Fourier integral operator, $a(x,D)$ becomes an operator to which Theorem 6 applies with $\psi(\xi) = \varphi(\xi) = (1+|\xi|)^{1/s}$ (cf. Parenti-Rodino [9], where $C^\infty$-
hyp oellipticity was proved under the same assumptions. Similarly we can prove a $G^2$-hyp oellipticity result for the operators in the classes of Boutet de Monvel-Grigis-Helffer [2].

Another application of Theorem 6 refers the choice
\[ \phi(\xi) = \sum_{j=1}^{n} |\xi_j|^{1/M_j}, \]
where $M = (M_1, \ldots, M_n)$ is a n-tuple of rational numbers $> 1$; the related hyp oellipticity results can be expressed in terms of the anisotropic Gevrey wave front set $WF^M$ of Zanghirati [12], Rodino [11]. Details and proofs of the results announced here will be found in Liess-Rodino [7].

REFERENCES


