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0. Introduction.

We shall consider the Schrödinger operator $H_v = -\Delta + v(x)$ in $\mathbb{R}^n$, $n=3,5,\ldots$. The potential $v(x)$ will have a short range and for simplicity we assume that there are positive constants $C_\alpha$ and $\varepsilon$ such that

\begin{equation}
|D^\alpha v(x)| \leq C_\alpha (1+|x|)^{-2-\varepsilon-|\alpha|}, \quad x \in \mathbb{R}^n.
\end{equation}

By $\|v^{(n-2)}\|$ we denote the $L^1$ norm of the differential of order $n-2$ of $v$, and we shall say that $v$ is small if this norm is small. Throughout operators in $\mathbb{R}^n$ will be identified with their distribution kernels, and the intertwining relation

\begin{equation}
H_v A = AH_0
\end{equation}

may then be expressed in the form of a differential equation

\begin{equation}
(\Delta_x - \Delta_y - v(x))A(x,y) = 0.
\end{equation}

We shall see that these equations always have solutions which are invertible operators in some weighted $L^p$-spaces. In the case of a small potential at least this gives us rather explicit expressions for the wave operators

\[
W_x^+ = \lim_{t \to +\infty} e^{itH_v} e^{-itH_0}
\]

and also for the scattering operator

\[
S = W_x^+ W_x^-.
\]

From these expressions it will easily follow that (in contrast to the one-dimensional situation) the potential is always uniquely determined from the scattering operator. We shall also discuss some equations of Gelfand-Levitan-Marchenko type which give relations between $A$ and $S$.

In the paper [4] the author studied the inverse scattering problem on the real line and also the characterization problem for scattering matrices. This was carried out by deriving some sharp estimates for the intertwining operators. That research, as well as its generalization to higher dimensions, was to a large extent inspired by Faddeev's papers about one- and multi-dimensional scattering [1],[2]. However, in contrast to Faddeev, we use explicit formulas for fundamental solutions
of $\Delta_x - \Delta_y$ with support conditions. Using this we may construct intertwining operators in a rather straightforward and elementary way, where the main problem consists of estimating some big integrals. For other methods in inverse scattering we refer to Saito [7], [8] and Newton [5], [6]. We also remark here that the method of using intertwining relations is fairly old (cf. Marchenko [3]).

1. Fundamental solutions of the ultrahyperbolic operator.

We consider $\Delta_x - \Delta_y$ in $\mathbb{R}^n$ where $n = 3, 5, \ldots$. For $\omega$ a unit vector in $\mathbb{R}^n$ we define $L_\omega \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

\[(1.1) \quad L_\omega (u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( \langle \omega, \partial_x + \partial_y \rangle \right)^{n-2}(x, x : 2x, \omega \cdot \omega) dx.
\]

We observe that $|y| = |x|, y - x \in \mathbb{R}^n$ in $\text{supp}(L_\omega)$ and that

\[(1.2) \quad L_\omega ((\Delta_x - \Delta_y)u) = \int_{\mathbb{R}^n} \left( \Delta (n-1)/2 \right) U(x) dx,
\]

where $U(x) = u(x, x)$. When proving (1.2) we may assume $\omega = e_n$ and setting $x^\prime = (x_1', \ldots, x_{n-1}')$ we have

\[\{(\partial_x + \partial_y)^{n-2}(\Delta_x - \Delta_y)(x', x; x^\prime, -x)\} = \sum_{j=1}^{n} \partial w_j / \partial x_j + \partial w / \partial x_n,
\]

where $w_j(x) = \{(\partial_x + \partial_y)^{n-2}(\partial_{x_j} - \partial_y)u(x', x_n; x', -x_n)$ and

\[w(x) = ((\partial_x + \partial_y)^{n-1}u(x', x_n; x', -x_n).
\]

The $w_j$ do not contribute to the right-hand side of (1.1) with $u$ replaced by $(\Delta_x - \Delta_y)u$ and the contribution from $w$ equals

\[\int_{x_n = 0} \partial x_n^{n-1} U(x) dx = \int_{x_n = 0} \Delta (n-1)/2 U(x) dx.
\]

Now we note that there are constants $c'_n$ and $c''_n$ such that

\[(1.3) \quad c'_n \int_{\mathbb{S}^n-1} d\omega \int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{S}^n-1} |x|^{-1} f(x) dx,
\]

and

\[(1.4) \quad c''_n \int_{\mathbb{R}^n} |x|^{-1} \Delta (n-1)/2 g(x) dx = g(0),
\]

This together with (1.2) shows that
\[ E_\theta = 2c_n c_n'' \int_{\omega, \theta \geq 0} L_\omega d\omega \]
is a fundamental solution of \( \Delta_x - \Delta_y \) when \( \theta \in S^{n-1} \), and
\[ |x| = |y|, <y-x, \theta > > 0 \text{ in supp}(E_\theta). \]

In addition to these conditions \( E_\theta \) satisfies a growth condition at infinity which makes \( E_\theta \) unique.

**Remark.** Let \( \hat{E}_\theta \) be the Fourier transform of \( E_\theta \) and \( Y_+ \) denote the characteristic function for \( \mathbb{R}_+ \). Then
\[ \hat{E}_\theta(\xi, \eta) = \lim_{\varepsilon \to 0} \left( Y_+ <\xi+\eta, \theta> (|\eta|^2 - |\xi|^2 - i\varepsilon)^{-1} + Y_- <\xi+\eta, \theta> (|\eta|^2 - |\xi|^2 + i\varepsilon)^{-1} \right). \]

2. **Construction of intertwining operators.**

Let \( M \) denote the set of all measurable functions \( R(x,y) \) in \( \mathbb{R}^n \times \mathbb{R}^n \) such that
\[ ||R||_M = \max \left( \sup_x \int |R(x,y)| dy, \sup_y \int |R(x,y)| dx \right) < \infty. \]

Any such \( R \) defines a continuous operator on \( L^p \) when \( 1 \leq p \leq \infty \) and
\[ ||R||_{L^p \to L^p} \leq ||R||_M. \]

**Theorem 2.1.** Let \( \theta \in S^{n-1} \). Then (0.2)' has a unique solution \( A = A_\theta = \delta(x-y) + R(x,y) \) such that

(i) \( <y-x, \theta> > 0 \) in supp(\( A \)).
(ii) There is a constant \( \lambda = \lambda_\nu \), with \( \lambda = 0 \) for \( \nu \) small, such that
\[ ||e^{-\lambda <y-x, \theta>} R ||_M < 1/2. \]

(iii)' \[ \int e^{-\lambda <y-x, \theta>} |R(x,y)| dx \to 0 \text{ as } y \to \infty, y/|y| \to -\theta, \]
(iii)'' \[ \int e^{-\lambda <y-x, \theta>} |R(x,y)| dy \to 0 \text{ as } x \to \infty, x/|x| \to \theta. \]

There are several consequences of this result. If \( \nu \) is small, then \( A \) is an isomorphism on any \( L^p \) and \( H_\nu \) is conjugated to \( H_0 \) under \( A \) and we have no bound states then. When \( \nu \) is big \( A \) will still be an isomorphism if we replace \( L^p \) by
\[ L^p_{\lambda, \theta} = \{ f; e^{\lambda <x, \theta>} f \in L^p \}. \]
From this fact one may obtain information about the eigenfunctions and eigenvalues of $H_v$. In particular, $-\lambda^2$ gives a lower bound for the spectrum.

To construct $A_\theta$, one makes use of the operator $T_u = \Theta(x_v)u(x,y)\), where $(v(x),u)=v(x)u(x,y)$. With $u_0(x,y)=\delta(x-y)$ one finds that the series

$$R(x,y) = \sum_1^\infty T^N u_0$$

converges in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ and that $A = I + R$ solves (0.2). In fact, setting $R_N = T^N u_0$, one has the estimates

$$\|e^{x-y,\Theta}R_N\|_M \leq 3^{-N}$$

with $\lambda$ as above. Moreover, one can prove inequalities of the form

$$\|R_No(I-\chi(\varepsilon D))\|_M \leq 3^{-N} \quad N > N_0,$$

where $\chi \in C^\infty_0(\mathbb{R}^n)$ equals 1 near the origin and $\varepsilon$ is small, and also that $I + R(I-\chi(cD))$ is invertible in any $L^p$ then.

To derive estimates of the form (2.2) for example one has to express $R_N(x,y)$ as the integral over $(S^{n-1})^N$ of expression of the following form:

$$J(\omega_1,\ldots,\omega_N) \sum_{\langle x_j,\omega_j \rangle \geq 0} v_1(x+X_1,\omega)\ldots v_N(x+X_1+\ldots+X_N,\omega)\delta(y-x-2\sum X_j,\omega), \frac{dX_1}{dX_N}$$

Here $\omega_j \in S^{n-1}$ and the norm of $J(\omega_1,\ldots,\omega_N)$ in $L^1((S^{n-1})^N)$ grows at most as $C_n$ as $N$ tends to infinity, while each $|v_j|$ may be majorized by $|v(n-2)|$.

Remark. Using (2.3) and the fact that $I + R(I-\chi(cD))$ is invertible one can find an operator $B$ which is bounded and invertible in any $L^p$ such that

$$H_vB = BH_0 + v(x)\chi(cD)$$

with $\chi$ as above. Thus when $v$ is sufficiently small at infinity the operator $H_v$ is then conjugated to $-\Delta + K$, where $K$ is of trace class for example.

3. Formulas for the wave operators and the scattering matrix.

We set

$$u_+(x,\eta) = \int A_+ \eta/|\eta|(x,y)e^{i<y,\eta>}dy.$$
the origin: $|\eta| \geq C_\nu$ where $C_\nu$ depends on $\nu$. The following result shows that
the $u_\pm$ are the generalized eigenfunctions (solutions of $H_\nu u = |\eta|^2 u$) that
enter in the distorted Fourier transforms.

**Theorem 3.1.** The functions $u_\pm(x,\eta)$ are the kernels of the generalized (adjoint)
Fourier transforms $\mathcal{F}_- F^\pm$, where $F$ denotes the standard Fourier transform.

We shall give a heuristic motivation for this result by considering wave packets.
For the case of simplicity we also assume that $\nu$ is small. Let $\widehat{u} \in C_0^\infty(\mathbb{R}^n)$ be
supported near $R_+\theta$. Then $e^{-i t H_0} u$ is concentrated near $\infty$ in the direction of
$-\theta$ as $t$ tends to $-\infty$. By the condition (iii) of Theorem 2.1 we may expect that
$A e^{-i t H_0} u$ is close to $e^{-i t H_0} u$ for such $t$. Hence we have

$$\mathcal{F}_- u \sim e^{i t H_\nu} e^{-i t H_0} u \sim e^{i t H_\nu} A e^{-i t H_0} u = A u$$

if we apply (0.2). The complete proof of the theorem requires a partition of unity
on the Fourier transform side together with more precise estimates.

Since $(\Delta_x - \Delta_y) A^{\theta} = v A^{\theta}$ we may expect that $A^{\theta}$ is essentially determined from
the behaviour of $(v A^{\theta})^\wedge (\xi,\eta)$ restricted to the set $|\xi| = |\eta|$. This motivates us
(apart from the normalization factor) to introduce the integral operators $K^{\theta}(r)$
on $L^2(S^{n-1})$ depending on $r \in \mathbb{R}_+$ and with kernels

$$K^{\theta}(r;\phi,\psi) = (-i\pi)^{n-1} r n^{-2} (v A^{\theta})^\wedge (r\phi, -r\psi).$$

These are defined for all $r$ when $\nu$ is small and for large $r$ in the general case.

The scattering operator $S$ commutes with $H_0$. In order to describe it w.r.t.
polar coordinates on the Fourier transform side we define the unitary map

$$\gamma: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+;L^2(S^{n-1}))$$

by the formula

$$\gamma f(r,\omega) = r^{(n-1)/2} f(r\omega)/(2\pi)^{n/2}.$$

For any operator $Q$ on $L^2(\mathbb{R}^n)$ we set $Q^\gamma = \gamma Q \gamma^*$. 

**Theorem 3.2.** The kernel of $S^\gamma$ (considered as an operator on $L^2(\mathbb{R}_+)$ acting on vector
valued functions) has the following form

$$(3.2) \quad S^\gamma(r,\rho) = \delta(r-\rho) S(r) = \delta(r-\rho)(1 + T(r)),$$

where $T(r;\phi,\psi) = K^{\phi}(r;\phi,\psi)$. (We assume $r$ to be large if $\nu$ is not small.)
Remark. \( S(r) \) is the scattering matrix and \( T(r) \) the scattering amplitude.

Proof. We assume \( v \) is small for simplicity and then we use the fact that

\[
((S-I)f,g) = \lim_{\varepsilon \to 0} -i \int_{-\infty}^{\infty} e^{-\varepsilon |t|} (vW_0 e^{-itH_0 f}, e^{-itH_0 g}) dt
\]

for a dense set of \( f \) and \( g \) in \( L^2 \). Hence

\[
(F(S-I)F^*_f,g) = \lim_{\varepsilon \to 0} -i \int_{-\infty}^{\infty} e^{-\varepsilon |t|} (G_t f,g) dt,
\]

where \( G_t = e^{itM}(vW_0)^* F^* e^{-itM} \) and \( M = \) multiplication by \( |\xi|^2 \). The kernel of \( G_t \) equals

\[
e^{it(|\xi|^2 - |\eta|^2)} (v\mathcal{A}|\eta|/|\eta|) (\xi,\eta)
\]

by Theorem 3.1. The theorem follows therefore from the definition of \( \mathcal{K}_\Theta \) and the fact that

\[
\int_{-\infty}^{\infty} e^{-\varepsilon |t|} e^{it(\xi^2 - \eta^2)} dt \to \pi |\xi|^{-1} \delta(|\xi| - |\eta|) \quad \text{as} \quad \varepsilon \to 0.
\]

Theorem 3.3. The potential is determined from the scattering matrix if \( v \in L^1 \) (in addition to \((0,1)\)).

Proof. Set \( R^\varepsilon = R_\psi \varepsilon (1-\chi(\varepsilon D)) \) with \( \varepsilon \) and \( \chi \) as above. For large \( r \) we have

\[
(v\mathcal{A})^\wedge (r\phi,-r\psi) = (vu_0^\wedge) (r\phi,-r\psi) + (vR^\varepsilon_\psi) (r\phi,-r\psi).
\]

Since \( v(x) R^\varepsilon_\psi(x,y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \) it follows from the Riemann-Lebesgue lemma that

\[
(vR^\varepsilon_\psi) (r\phi,-r\psi) \to 0 \quad \text{as} \quad r \to \infty \quad \text{(uniformly w.r.t.} \phi \text{and} \psi).\]

On the other hand

\[
(vu_0^\wedge) (r\phi,-r\psi) = \hat{v}(r(\phi,-\psi)),
\]

and any \( \zeta \in \mathbb{R}^n \) can be obtained as a limit as \( r_j \to \infty \) of \( r_j(\phi_j,-\psi_j) \) where \( \phi_j \) and \( \psi_j \) are sequences of unit vectors. Hence \( \hat{v} \) is determined from \( S \).

4. The fundamental identity and a Gelfand-Levitan equation (in the case of small \( v \)).

In this section we shall restrict to the case of a small potential. What happens in the general case is not yet clear for us.

Theorem 4.1. (The fundamental identity) Let \( \mathcal{K}_\Theta \) be as in Section 3. Then
When proving this formula one may consider both sides of (4.1) as analytic functions of \( z \) after \( \nu \) has been replaced by \( \nu z \). It suffices then to show that (4.1) holds for the Taylor expansion at the origin, and it turns out then that (4.1) follows from (1.6) and the equation \( A_\Theta = E_\Theta x(vA_\Theta) + \delta(x-y) \). The fundamental identity is also true for large \( \nu \) as long as we take \( r \) large.

Let \( Y_\pm(s) \) be as in (1.6) and set
\[
K^\pm_\Theta(r;\phi,\psi) = Y_\pm(<\phi,\theta>)K_\Theta(r;\phi,\psi).
\]

Then \( K_\Theta = K^+_\Theta + K^-_\Theta \) is the decomposition of \( K_\Theta(r) \) considered as a matrix into upper and lower triangular ones w.r.t. the ordering of \( S^{n-1} \) defined by \( \Theta \). The equation (4.1) now takes the form
\[
(4.1)' \quad I + K_\Theta^+(r) = \tilde{S}(r)(I-K_\Theta^-(r)), \quad r > 0.
\]

This gives a factorization of \( \tilde{S}(r) \) into a lower and upper triangular matrix. Note also that
\[
(4.2) \quad (I+K^+_{\Theta})(I+K^-_{\Theta}) = (I-K^-_{\Theta})(I-K^+_{\Theta})
\]
in view of the unitarity of the scattering matrix.

Remark 4.2. In the one-dimensional case the operator in (4.2) takes the form
\[
\begin{pmatrix}
0 & \rho \\
\rho & 0
\end{pmatrix}, \quad \text{where } \rho \text{ is the reflection coefficient.}
\]

We shall next show that it is possible to describe the operator in (4.2) without working on the Fourier transform side. When doing this we use the identity
\[
(4.3) \quad (A_{\Theta}^\pm)^{-1} = A_{-\Theta}.
\]

This is a consequence of the uniqueness assertion of Theorem 2.1 since both sides of (4.3) are intertwining operators. The Gelfand-Levitan-Marchenko equation (w.r.t the direction \( -\Theta \) ) may be written as
\[
(4.4) \quad A_{-\Theta}Q_\Theta A_{-\Theta}^{1/2} = I.
\]

Here \( Q_\Theta \) is a positively definit operator and \( A_{-\Theta}Q_\Theta^{1/2} \) is unitary. By (4.3)
\[
(4.5) \quad Q_\Theta = A_{-\Theta}^{-1}A_\Theta.
\]
and $Q_\theta$ commutes with $H_0$. It is natural therefore to consider $Q_\theta^\chi$.

**Theorem 4.3.** Assume that $\phi$ is small. Then

$$Q_\theta^\chi(r,\rho) = \delta(r-\rho)\tilde{Q}_\theta(r), \quad r > 0,$$

where

$$\tilde{Q}_\theta(r) = (I+K_\theta^+(-r))\chi(I+K_\theta^+(r)) = (I-K_\theta^-(r))\chi(I-K_\theta^-(r)).$$

In principle this gives a method to compute $\phi$ from the scattering matrix. When doing so $Q_\theta$ is first computed from (4.7) and the fundamental identity. After that (4.4) is converted into a linear equation for $A_{-\theta}$ as in the one-dimensional case by using the support condition on $A_{-\theta}$. Finally $\phi$ is computed from $A_{-\theta}$.

**REFERENCES**