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RICHARD B. MELROSE

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Semilinear waves with cusp singularities

RICHARD B. MELROSE
Massachusetts Institute of Technology

Let P be a second order strictly hyperbolic operators with $\mathcal{C}^\infty$ coefficients in some open domain $\Omega \subset \mathbb{R}^{1+n}$. The propagation of conormal regularity for bounded solutions of the (weakly) semilinear hyperbolic equation

$$(*) \qquad Pu(z) = g(z,u) \quad z \in \Omega,\ u \in L^\infty_{\text{loc}}(\Omega),\ g \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$$

will be described when the wavefront, a characteristic surface for P, has cusp singularities.

The case where the characteristic surface is smooth, or two characteristic surfaces intersect transversally was treated by Bony [**2**] under somewhat more stringent conditions (see also [**7**]). The present case of a surface with cusp singularities is much closer to that of two characteristic surfaces tangent along a common submanifold of codimension one examined in [**8**]. In particular the methods used here are similar to those in [**8**] except that more explicitly microlocal results are used. The spaces of 'marked Lagrangian' distributions introduced below seem particularly useful for the treatment of these problems and they will be further exploited in a similar (though necessarily more complicated) resolution of the case of a characteristic surface developing a swallowtail singularity (a caustic). In practice it is not possible for a cusp singularity to arise without the appearance of a swallowtail or other singularity which is of effective codimension three (see [**1**]). See also the recent work of Bony and Lerner [**4**].

In order to describe the propagation theorem for (*) precisely we need first to introduce the spaces of conormal functions associated to a hypersurface with cusp singularities. These should be the 'simplest' functions which are singular on the surface in question. Here the space is defined in terms of a resolution of the cusp to normal crossings through (radial) blow up. This also reveals the relationship of the cusp to the case of simple tangency referred to above.

The outline of proof following the statement of the theorem consists of four main elements. First a general propagation theorem for (*) is given. This reduces the proof to the verification of a multiplicative property and a linear propagation property for the spaces involved. The multiplicative property follows directly from the definition in terms of blow up, since it reduces to a from of the Gagliardo-Nirenberg estimates for a Lie algebra of vector fields on the blown up manifold. Next it is shown that the spaces of (finitely) conormal functions have purely microlocal interpretations in terms of spaces of marked Lagrangian distributions. The 'marking' refers to a submanifold of the Lagrangian at which the iterative regularity properties are weaker. This identification is made using (though not in an essential way) the calculus of totally characteristic pseudodifferential operators in [**5**]. Finally the appropriate (microlocal) linear propagation theorems can be proved for these spaces using standard Fourier integral operator techniques.

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In the case of analytic geometry the results described here are contained in recent work of Giles Lebeau.

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§1: Cusps and conormality

Let Y be a $\mathcal{C}^\infty$ manifold without boundary and $\Lambda \subset T^*Y\backslash 0$ be a $\mathcal{C}^\infty$ conic embedded closed Lagrangian submanifold. If $\lambda \in \Lambda$ is a point near which the differential of the projection $\pi : \Lambda \longrightarrow Y$ restricted from T^*Y has constant rank then $H = \pi(N)$ is a $\mathcal{C}^\infty$ submanifold of Y if $N \subset \Lambda$ is a sufficiently small neighbourhood of λ and moreover $\Lambda = N^*H$ near λ. The only stable case of this is when H is a hypersurface in Y and then the rank of π_* is $N-1$, $N = \dim Y$.

The simplest (generic) singular case is where the rank of π_* drops by one across

(1.1) $\quad \Sigma \subset \Lambda$ a $\mathcal{C}^\infty$ hypersurface s. t. $\pi_* : T_\lambda\Lambda \longrightarrow T_{\pi(\lambda)}Y$ has rank $N-2\ \forall\ \lambda \in \Sigma$

and where there is an additional non-degeneracy condition
(1.2)
$\exists\ f \in \mathcal{C}^\infty(Y),\ f = 0$ on $\pi(\Sigma \cap N)$ with $\pi^* f$ vanishing to precisely second order at Σ.

In view of the following result the Lagrangian immersion $\pi : \Lambda \longrightarrow Y$ is then said to have a cusp singularity:

(1.3) Theorem (Arnol'd [**1**]). *If $\Lambda \subset T^*Y\backslash 0$ is a $\mathcal{C}^\infty$ conic Lagrangian submanifold satisfying* (1.1) *and* (1.2) *then there are local coordinates $(x,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-2}$ in Y near $\pi(\lambda)$ in terms of which $\lambda = (1,0,0)$ and*

$$(1.4) \qquad \begin{cases} \Lambda \setminus \Sigma = N^*(C^{\mathrm{reg}})\backslash 0 \ \textit{near}\ \lambda \\ C^{\mathrm{reg}} = \{(x,y,z); y^3 = x^2, x \neq 0\},\ C = \overline{C^{\mathrm{reg}}} \end{cases}$$

The Lagrangian distributions associated to such a Lagrangian, as introduced by Hörmander [**Ho1**], are completely independent of the position of the projection. However such spaces are not multiplicative and one needs some multiplicative properties for spaces which propagate under (*).

To define the conormal functions associated to the model cusp C in (1.4) we shall introduce polar coordinates in (x,y), i.e. consider the $\mathcal{C}^\infty$ (blow down) map

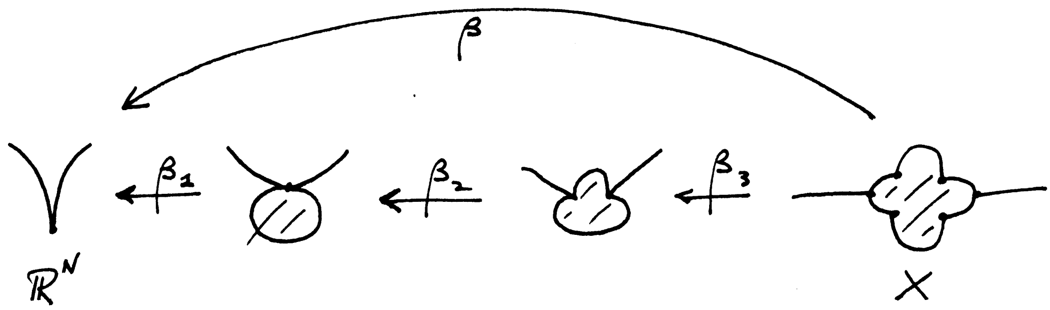
$$(1.5) \qquad \beta_1 : X_1 = [0,\infty)\times \mathbb{S}^1 \times \mathbb{R}^{N-2} \ni (r,\omega,z) \longmapsto (r\omega_1, r\omega_2, z) \in \mathbb{R}^N.$$

This correspond to normal blow up of the singular variety of the cusp

$$(1.6) \qquad L = \{x = y = 0\} \subset C$$

under which the lift

$$(1.7) \qquad \beta_1^*(C) = \overline{\beta_1^{-1}(C^{\mathrm{reg}})} \text{ is } \mathcal{C}^\infty.$$



Resolving the cusp to normal crossing

This is easily seen by noting that on $\beta_1^* C$, $\omega_2 > 0$ so in the manifold X_1 the projective coordinates $s = \frac{x}{y}, y, z$ are valid near $\beta_1^*(C)$ and in terms of these

$$\beta_1^*(C) = \{y = s^2\}. \tag{1.8}$$

Clearly this is a $\mathcal{C}^\infty$ hypersurface simply tangent to the boundary $s = 0$ along the submanifold $\{s = y = 0\}$. Since the boundary is in a natural way the lift of the singular variety $\partial X_1 = \beta_1^*(L)$, this shows the close relationship between the cusp and the case of simply tangent hypersurfaces alluded to above.

To fully resolve the cusp we need to make two further blow ups (see [8]). First introduce polar coordinates in (s, y) giving the new manifold with corner X_2 and blow down map $\beta_2 : X_2 \longrightarrow X_1$. Under this the lift of the cusp $(\beta_1 \cdot \beta_2)^*(C)$ is $\mathcal{C}^\infty$ but still passes through the corner of the manifold, although it is not tangent to any codimension one boundary face. A final blow up of the two components of the corner of X_2 gives a normal resolution.

Let the resulting manifold with corner be denoted X and consider the overall blow down map $\beta : X \longrightarrow \mathbb{R}^N$. Then $\beta^*(C) = \overline{\beta^{-1}(C^{\text{reg}})}$ is a $\mathcal{C}^\infty$ hypersurface in X which meets the boundary transversally and only in codimension one boundary components. Moreover if μ is Lebesgue measure on $\mathbb{R}^N$ then there is a unique $\mathcal{C}^\infty$ measure ν on X such that $\beta_*(\nu) = \mu$. On X consider the Lie algebra of vector fields

$$\mathcal{V}_b(\beta^*(C)) = \{V \in \mathcal{C}^\infty(X, TX); V \text{ is tangent to } \partial X \text{ and } \beta^*(C)\}. \tag{1.9}$$

The relative divergence of such vector fields

$$\nu - \operatorname{div}(V) \in \mathcal{C}^\infty(X) \quad \forall\, V \in \mathcal{V}_b(\beta^*(C)), \tag{1.10}$$

so that the transpose, with respect to ν, of any $V \in \mathcal{V}_b(\beta^*(C))$ acting as a differential operator has $\mathcal{C}^\infty$ coefficients. Furthermore notice that

$$\beta^* : L^2_{\text{loc}}(\Omega) \longleftrightarrow L^2_{\text{loc},\nu}(\beta^{-1}(\Omega)) \tag{1.11}$$

for any open set $\Omega \subset \mathbb{R}^N$.

We then *define* the space of (L^2-based) conormal functions associated to the cusp C by iterative regularity on X :

$$J_k L^2_{\text{loc}}(\Omega, C) = \left\{u \in L^2_{\text{loc}}(\Omega); \mathcal{V}_b(\beta^* C)^p u \subset L^2_{\text{loc},\nu}(\beta^{-1}(\Omega)) \ \forall \ p \leq k\right\}. \tag{1.12}$$

The most interesting case corresponds to full conormality, when $k = \infty$, but it is convenient to have the finite order spaces to allow inductive proofs.

The definition (1.12) is actually completely coordinate independent in view of the coordinate independence of the blow up procedure (see for example [**5**]), so applies directly to any hypersurface having just local cusp singularities. In any region where the surface does not have singularities this definition reduces to the standard one for finite order conormality, giving the spaces denoted $I_k L^2_{\text{loc}}(X, C)$ in [**8**].

§2: The main result

Let $\Omega \subset \mathbb{R}_t \times \mathbb{R}^n$ be an open domain and suppose that $J_k(\Omega)$ is a decreasing sequence of function spaces on Ω normalized by

$$L^2_{\text{loc}}(\Omega) = J_0(\Omega) \supset J_1(\Omega) \supset \cdots \supset J_k(\Omega) \supset \cdots \supset \mathcal{C}^\infty(\Omega). \tag{2.1}$$

If the $J_k(\Omega)$ are $\mathcal{C}^\infty(\Omega)$-modules:-

$$\phi \in \mathcal{C}^\infty(\Omega), \ u \in J_k(\Omega) \Longleftrightarrow \phi u \in J_k(\Omega) \tag{2.2}$$

then they can be localized to any open subset $\Omega' \subset \Omega$:

$$J_k(\Omega') = \left\{u \in L^2_{\text{loc}}(\Omega'); \ \forall \ \phi \in \mathcal{C}^\infty_c(\Omega'), \ \phi u \in J_k(\Omega)\right\}. \tag{2.3}$$

The spaces defined by (1.12) are obviously $\mathcal{C}^\infty(\Omega)$-modules and it is clearly that this localization to open subsets gives the same space as the direct definition (1.12) for the subset.

If P is t-hyperbolic and Ω is in the t-forward influence domain of

$$\Omega^- = \{(t, x) \in \Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x; t < 0\} \tag{2.4}$$

then we shall say that J_k-regularity propagates for bounded solutions of (*) provided

$$u \in L^\infty_{\text{loc}}(\Omega), \ (*) \text{ holds and } u, Du \in J_k(\Omega^-) \Longleftrightarrow u, Du \in J_k(\Omega). \tag{2.5}$$

(2.6) Theorem. *Suppose $\Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$ is in the influence domain of Ω^- defined by (2.4) for a second order strictly t-hyperbolic operator P and that $H \subset \Omega$ is a closed hypersurface which is characteristic for P and which is $\mathcal{C}^\infty$ except for cusp singularities then $J_k L^2(\cdot, H)$-regularity propagates for bounded solutions of (*).*

Note that there is a difference between this and the true 'interaction' results for two or three hypersurfaces discussed in [**3**] and [**7**]. Namely if $p \in L \subset H$ is a point in the singular locus of H then, assuming that H has only cusp singularities, any open set $D \subset \Omega$ which contains p in its forward domain of influence must also meet L. That is, cusp singularities cannot arise spontaneously but must be produced by some other singularity such as a swallowtail caustic. For this reason the present result should only be considered as a (necessary) step toward the full analysis of the swallowtail.

§3: Propagation theorem

Suppose that $J(\Omega) \subset L^\infty_{\text{loc}}(\Omega)$ is a linear subspace, then $J(\Omega)$ is said to be a $\mathcal{C}^\infty$ algebra if

$$f \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^q),\ u_i \in J(\Omega),\ i = 1, \dots, q \Longrightarrow f(z, u_1(z), \dots, u_q(z)) \in J(\Omega). \tag{3.1}$$

In particular this is true of $L^\infty_{\text{loc}}(\Omega)$ itself.

We shall say that a sequence of spaces $J_k(\Omega)$ propagates under P if

$$u, Du \in J_k(\Omega^-),\ Pu \in J_k(\Omega) \Longrightarrow u \in J_k(\Omega) \quad \forall\, k \geq 1. \tag{3.2}$$

Here of course we assume that Ω is in the forward dependence domain of Ω.

(3.3) Theorem. *Let P be a second order strictly hyperbolic operator and Ω such that for some $\epsilon > 0$, Ω is contained in the dependence domain of*

$$\Omega_s^- = \{z \in \Omega; t < s\} \quad \forall\, |s| < \epsilon, \tag{3.4}$$

then, provided $J_k(\Omega)$ is a sequence of spaces satisfying (2.2), (3.1) *and* (3.2), *$J_k(\Omega)$-regularity propagates for bounded solutions of* (*).

Thus to prove Theorem (2.6) it is only necessary to check that the spaces $J_k L^2_{\text{loc}}$ satisfy (3.1) and (3.2).

§4: Multiplicative property

It is easy to check that $J_k L^2_{\text{loc}}(\Omega, H)$, for H a closed subspace with only cusp singularities, satisfies (3.1). Indeed from (1.12) $J_k L^2_{\text{loc}}(\Omega, H)$ can be identified with the space $I_k L^2_{\text{loc},\nu}(X, \beta^*(H))$ defined by the iterative regularity with respect to the vector fields $\mathcal{V}_b(\beta^*(H))$. The multiplicative property (3.1) now follows from a suitable form of the Gagliardo-Nirenberg estimates (see [**7**]).

(4.1) Theorem. *Let $\mathcal{V} \subset \mathcal{C}^\infty(\Omega, T\Omega)$ be a Lie algebra of $\mathcal{C}^\infty$ vector fields which is locally finitely generated as a $\mathcal{C}^\infty(\Omega)$-module and for which* (1.10) *holds, then the spaces*

$$I_k L^2_{\text{loc},\nu}(\Omega, \mathcal{V}) = \left\{u \in L^2_{\text{loc},\nu}(\Omega); \mathcal{V}^p u \subset L^2_{\text{loc},\nu}(\Omega)\ \forall\, p \leq k\right\} \tag{4.2}$$

satisfy (3.1), *i.e. $L^\infty_{\text{loc},\nu}(\Omega) \cap I_k L^2_{\text{loc},\nu}(\Omega, \mathcal{V})$ is a $\mathcal{C}^\infty$ algebra.*

§5: Marked Lagrangian distributions

Suppose $\Lambda \subset T^*Y\backslash 0$ is a closed embedded conic Lagrangian submanifold. Consider the space of polyhomogeneous (Kohn-Nirenberg) properly supported pseudodifferential operators which are characteristic on Λ :

$$\mathcal{M}(\Lambda) = \left\{A \in \Psi^1_{\text{KN},P}(Y); \sigma_1(A) = 0 \text{ on } \Lambda\right\}. \tag{5.1}$$

This is a Lie algebra which is locally finitely generated as a $\Psi^0_{\mathrm{KN},P}(Y)$-module. If we define, for each $s \in \mathbb{R}$,

$$I_k H^s_{\mathrm{loc}}(Y,\Lambda) = \{u \in H^s_{\mathrm{loc}}(Y); \mathcal{M}(\Lambda)^p u \subset H^s_{\mathrm{loc}}(Y)\ \forall\ p \le k\} \tag{5.2}$$

then we obtain the Lagrangian distributions of Hörmander [**Ho1**]

$$I^*(Y,\Lambda) = \bigcup_{s\in\mathbb{R}} I_\infty H^s_{\mathrm{loc}}(Y,\Lambda). \tag{5.3}$$

In case $s = 0$, for general k, we use the same notation, $I_k L^2_{\mathrm{loc}}(Y,\Lambda)$, as before.

Now suppose that $\Sigma \subset \Lambda$ is a closed $\mathcal{C}^\infty$ conic submanifold of codimension one. Let $\mathcal{L} = \{\Lambda \setminus \Sigma, \Sigma\}$ be the $\mathcal{C}^\infty$ variety (set of disjoint submanifolds) defined by Σ and Λ. In place of (5.1) consider

$$\mathcal{M}(\mathcal{L}) = \left\{A \in \Psi^1_{\mathrm{KN},P}(Y); a = \sigma_1(A) = 0 \text{ on } \Lambda \text{ and } H_a \text{ is tangent to } \Sigma\right\}. \tag{5.4}$$

Again this is a Lie algebra since

$$\sigma_1([A,B]) = -i\{\sigma_1(A),\sigma_1(B)\} \text{ and } H_{\{a,b\}} = [H_1, H_b], \tag{5.5}$$

and is microlocally finitely generated as a $\Psi^0_{\mathrm{KN},P}(Y)$-module. The obvious generalization of (5.2) yields the space of (L^2-based) marked Lagrangian distributions associated to $\mathcal{L}$:

$$I_k L^2_{\mathrm{loc}}(Y,\mathcal{L}) = \left\{u \in L^2_{\mathrm{loc}}(Y); \mathcal{M}(\mathcal{L})^p u \subset L^2_{\mathrm{loc}}(Y)\ \forall\ p \le k\right\}. \tag{5.6}$$

This space is microlocally equal to $I_k L^2_{\mathrm{loc}}(Y,\Lambda)$ except across the 'mark' Σ. In any case

$$I_k L^2_{\mathrm{loc}}(Y,\Lambda) \subset I_k L^2_{\mathrm{loc}}(Y,\mathcal{L}). \tag{5.7}$$

In case $k = \infty$ it is possible to give a full invariant symbolic description of these spaces (see [**6**]). Notice that

$$u \in I_\infty L^2_{\mathrm{loc}}(Y,\mathcal{L}) \Longrightarrow \mathrm{WF}(u) \subset \Lambda. \tag{5.8}$$

In the present setting these spaces of marked Lagrangian distributions give very convenient decompositions of conormal spaces. For the cusp algebra this will be discussed below. Consider first the simple case of conormal distributions associated to a hypersurface M and a submanifold L of codimension one. In suitable local coordinates near a point of L,

$$M = \{x_1 = 0\},\ L = \{x_1 = x_2 = 0\}. \tag{5.9}$$

The two conormal bundles are

$$\begin{cases} \Lambda_1 = N^*M\backslash 0 = \{(0, x_2, x', \xi_1, 0, 0) \in T^*\mathbb{R}^n\} \\ \Lambda_2 = N^*L\backslash 0 = \{(0,0,x',\xi_1,\xi_2,0) \in T^*\mathbb{R}^n\}. \end{cases} \tag{5.10}$$

The intersection of these two Lagrangian is the submanifold

$$\Sigma = \Lambda_1 \cap \Lambda_2 = \{(0,0,x',\xi_1,0,0) \in T^*\mathbb{R}^n\} \tag{5.11}$$

which is a hypersurface in each. Set $\mathcal{L}_i = \{\Lambda_i \setminus \Sigma, \Sigma\}$ for $i = 1,2$. The conormal functions associated to the $\mathcal{C}^\infty$ variety $\mathcal{S} = \{M \setminus L, L\}$ are defined by iterative regularity with respect to the Lie algebra of vector fields

$$\begin{gathered} \mathcal{V}(\mathcal{S}) = \{V \text{ tangent to both } M \text{ and } L\}, \\ I_k L^2_{\text{loc}}(Y,\mathcal{S}) = \left\{u \in L^2_{\text{loc}}; \mathcal{V}(\mathcal{S})^p u \subset L^2_{\text{loc}} \ \forall\, p \le k\right\}. \end{gathered} \tag{5.12}$$

(5.13) PROPOSITION. *If $L \subset M$ is a submanifold of codimension one and $\mathcal{S}$, $\mathcal{L}_i$, $i = 1,2$ are defined as above then for each $k \in \mathbb{N} \cup \infty$*

$$I_k L^2_{\text{loc}}(Y,\mathcal{S}) = I_k L^2_{\text{loc}}(Y,\mathcal{L}_1) + I_k L^2_{\text{loc}}(Y,\mathcal{L}_2). \tag{5.14}$$

This immediately extends to other cases. For example if H_1 and H_2 are two $\mathcal{C}^\infty$ hypersurfaces meeting transversally set $L = H_1 \cap H_2$ and consider the marked Lagrangians in $T^*Y\backslash 0$

$$\begin{cases} \mathcal{L}_i = \{N^*H_i \setminus N^*L, N^*H_i \cap N^*L\} \quad i = 1,2 \\ \mathcal{L}_3 = \{N^*L \setminus (N^*H_1 \cup N^*H_2), N^*L \cap (N^*H_1 \cup N^*H_2)\} \end{cases} \tag{5.15}$$

Here there are two distinct hypersurfaces in N^*L but they do not intersect. If $\mathcal{S} = \{H_1 \setminus L, H_2 \setminus L, L\}$ is the $\mathcal{C}^\infty$ variety and $I_k L^2_{\text{loc}}(Y,\mathcal{S})$ is defined as in (5.12) then

$$I_k L^2_{\text{loc}}(Y,\mathcal{S}) = \sum_{i=1}^{3} I_k L^2_{\text{loc}}(Y,\mathcal{L}_i). \tag{5.16}$$

§6: MICROLOCAL DECOMPOSITION

Consider again the cusp algebra defined in (1.12). The Lagrangian obtained by closing the conormal to the regular part of the cusp is

$$\Lambda_C = \left\{(x,y,z,\xi,\eta,\varsigma); x = (-\frac{2\eta}{3\xi})^3, y = (-\frac{2\eta}{3\xi})^2, \varsigma = 0, \xi \neq 0\right\} \tag{6.1}$$

The singular locus is $L = \{x = y = 0\}$ and the second Lagrangian

$$\Lambda_L = N^*L\backslash 0 = \{(0,0,z,\xi,\eta,0)\} \tag{6.2}$$

meets the first in a hypersurface

$$\Sigma = \Lambda_L \cap \Lambda_C = \{(0,0,z,\xi,0,0)\} \tag{6.3}$$

in each, even though the intersection is *not* clean, i.e.

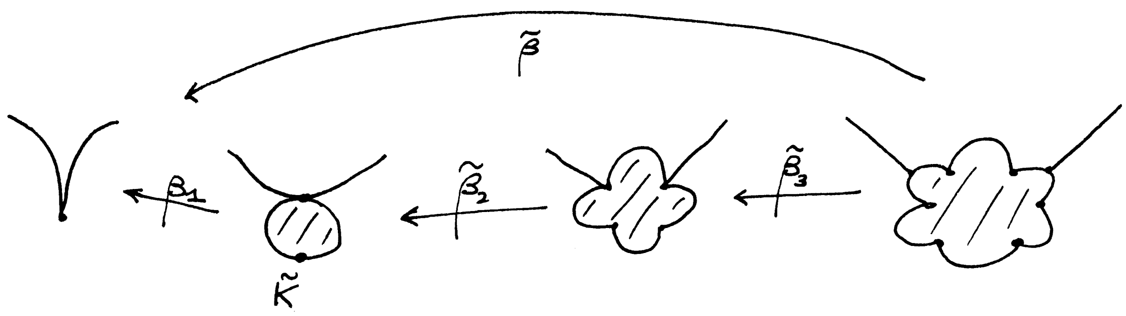
$$T\Lambda_L \cap T\Lambda_C \supsetneq T\Sigma \quad \text{at } \Sigma. \tag{6.4}$$

In any case the marked Lagrangians

$$\mathcal{L}_C = \{\Lambda_C \setminus \Sigma, \Sigma\} \text{ and } \mathcal{L}_L = \{\Lambda_L \setminus \Sigma, \Sigma\} \tag{6.5}$$

are well defined. One might hope that the space in (1.12) decomposes exactly as in (5.14) but this is not the case. The failure of this decomposition is however a relatively minor point arising from the non-projective nature of the blow up in §1 or more accurately the non-projective nature of the spaces of Lagrangian distributions.

Recall that in the definition of the second manifold X_2 in §2 the surface $K = \beta_1^*(H) \cap \partial X_1$ was blown up. In ∂X_1 there is a natural involution, arising from the sign reversal of the defining coordinates in the original manifold (but independent of the choice of coordinates). Let $\tilde{K}$ be the image of K under this involution. Now, $\tilde{K}$ does not meet $\beta_1^*(H)$ so we can freely blow it up at the same time as K giving the manifold $\tilde{X}_2$ and associated blow down map $\tilde{\beta}$. Now proceed as before to define the projective resolution $\tilde{\beta} : \tilde{X} \longrightarrow \mathbb{R}^N$.



Projective resolution of the cusp

The definition leading to (1.12) can now be imitated to define the slightly larger space

$$\begin{gathered}\tilde{J}_k L^2_{\text{loc}}(\Omega, C) = \left\{u \in L^2_{\text{loc}};\ \mathcal{V}_b(\tilde{\beta}^*(H))^p u \subset L^2_{\text{loc},\nu}(\tilde{\beta}^{-1}(\Omega))\ \forall\, p \le k\right\} \\ J_k L^2_{\text{loc}}(\Omega, C) \subsetneqq \tilde{J}_k L^2_{\text{loc}}(\Omega, C)\end{gathered} \tag{6.6}$$

The arguments in §4 show that this space too has the multiplicative property (3.1).

(6.7) PROPOSITION. *If C is the standard cusp in* (1.4) *and $\mathcal{L}_C$, $\mathcal{L}_L$ are the marked Lagrangians* (6.1) *then for any $k \in \mathbb{N} \cup \infty$*

$$\tilde{J}_k L^2_{\text{loc}}(\Omega, C) = I_k L^2_{\text{loc}}(\Omega, \mathcal{L}_C) + I_k L^2_{\text{loc}}(\Omega, \mathcal{L}_L). \tag{6.8}$$

To prove this one can use the fact that both $\mathcal{M}(\mathcal{L}_C)$ and $\mathcal{M}(\mathcal{L}_L)$ are generated by a finite number of differential operators, normalized to have first order by an elliptic factor. These (or the regularity conditions involving them) can be lifted under the blow down maps and the totally characteristic calculus in [**5**] can then be employed to analyse the regularity of the lifts.

§7: LINEAR PROPAGATION

Since the spaces on the right in (6.8) are defined purely microlocally we can use Fourier integral operator methods to check the linear propagation property (3.2) for the sum. Slightly more generally than above consider a closed embedded Lagrangian $\Lambda_1 \subset T^*Y \backslash 0$ and suppose that $Q \in \Psi^1_{\mathrm{KN},P}(Y)$ is of real principal type. Moreover suppose that

$$\Sigma = \Lambda_1 \cap \Sigma(Q) \text{ is a } \mathcal{C}^\infty \text{ hypersurface in } \Lambda_1. \tag{7.1}$$

If this intersection were transversal then the geometry would be that of the microlocal Cauchy problem for Q (see [9]). Instead we shall suppose that

$$H_q \text{ is tangent to } \Lambda_1 \text{ to a fixed finite order along } \Sigma \text{ and is not tangent to } \Sigma. \tag{7.4}$$

In fact for the case of $\Lambda_1 = \Lambda_L$ the conormal to the singular variety of the cusp the tangency is simple.

As a consequence of (7.1) and (7.2) the (bi-directional) flow out of Σ under H_q is a second $\mathcal{C}^\infty$ conic Lagrangian

$$\Lambda_2 = H_q\text{-flow-out of } \Sigma. \tag{7.5}$$

Now set

$$\mathcal{L}_i = \{\Lambda_1 \setminus \Sigma, \Sigma\} \text{ for } i = 1, 2. \tag{7.6}$$

Fixing an orientation of H_q near a point $\lambda \in \Sigma$ we can say that a distribution u is microlocally in $I_k L^2_{\mathrm{loc}}(Y, \mathcal{L}_2)$ in the past of λ if u is microlocally in this space near each point sufficiently near λ on the backward-directed half H_q-bicharacteristic through λ.

(7.7) THEOREM. *Suppose $Q \in \Psi^1_{\mathrm{KN},P}(Y)$ is of real principal type, Λ_1 is a conic Lagrangian satisfying* (7.1) *and* (7.2) *and that u satisfies*

$$Qu \in \sum_{i=1,2} I_k L^2_{\mathrm{loc}}(Y, \mathcal{L}_i) \text{ microlocally near } \lambda \in \Sigma \tag{7.8}$$

then if u is in $I_k L^2_{\mathrm{loc}}(Y, \mathcal{L}_2)$ microlocally in the past of λ it follows that, microlocally near λ, $u \in \sum\limits_{i=1,2} I_k L^2_{\mathrm{loc}}(Y, \mathcal{L}_i)$.

To prove this Q can be reduced to an elliptic multiple of D_x by a Fourier integral operator which also transforms the spaces of marked Lagrangian distributions so that in the image

$$\Lambda_1 = N^*\{y = x^k\},\ \Lambda_2 = \{y = 0\}. \tag{7.9}$$

Explicit computation using oscillatory representations of the marked Lagrangian distributions then yields the result.

It follows directly from Theorem (7.7) that the modified space $\tilde{J}_k L^2_{\mathrm{loc}}(\Omega, C)$ satisfies the linear propagation condition (3.2). Thus Theorem (3.3) holds for this space instead. In fact, using the close relationship between these two sequences of spaces, it is straightforward to show that the smaller space $J_k L^2_{\mathrm{loc}}(\Omega, C)$ also satisfies (3.2), proving Theorem (3.3) as stated.

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