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SINGULARITIES OF THE SCATTERING KERNEL FOR NON CONVEX OBSTACLES

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open domain with C^∞ smooth boundary $\partial\Omega$ and bounded complement $K = \mathbb{R}^3 \setminus \Omega \subset \{x : |x| \leq \rho\}$. The scattering operator S, related to the Dirichlet problem for the wave equation in $R_t \times \Omega$, is an unitary operator from $L^2(R \times S^2)$ into $L^2(R \times S^2)$. The kernel $s(t-t', \theta, \omega)$ of $S-Id$ is called scattering (echo) kernel. For fixed $(\theta, \omega) \in S^2 \times S^2$, $s(t,\theta,\omega) \in \mathcal{S}'(\mathbb{R})$ and

$$(1)\ s(t,\theta,\omega) = (1/8\pi^2)\int_{\partial\Omega} \frac{\partial^2}{\partial t\,\partial n} w(<x,\theta> - t, x, \omega)\, dS_x .$$

Here $w(\tau, x, \omega)$ is the solution to the problem

$$(2)\quad \begin{aligned} &(\partial_\tau^2 - \Delta_x)w = 0 \text{ in } \mathbb{R} \times \Omega,\\ &w = 0 \text{ on } \mathbb{R} \times \partial\Omega,\\ &w|_{\tau < -\rho} = \delta(\tau - <x,\omega>), \end{aligned}$$

n is the interior unit normal to $\partial\Omega$ pointing into Ω and the integral (1) is interpreted in the sense of distributions.

If $\hat{s}(\lambda, \theta, \omega)$ is the Fourier transform of $s(t, \theta, \omega)$, then $a(\lambda, \theta, \omega) = (2\pi/i\lambda)\hat{s}(\lambda, \theta, \omega)$ is called scattering amplitude and its asymptotic as $\lambda \to \infty$ is closed related to the singularities of $s(t, \theta, \omega)$. As was remarked in [3], [8], in general, these singularities are connected with the sojourn times of the so called (ω, θ)-rays defined below. The assumptions in [8] are too difficult for verifications. Nevertheless, some of them are fulfilled for generic obstacles (see [10], [11]).

We expect that for generic directions (ω, θ) the sojourn times of all ordinary (ω, θ)-rays are included in the singular support of $s(t, \theta, \omega)$. In this talk we prove this in the case when

$$K = \cup_{i=1}^{M} K_i, \bar{K}_i \cap \bar{K}_j = \emptyset \; i \neq j, K_i \text{ are strictly convex for} \tag{3}$$
$$i = 1, \dots, M.$$

For a large class of obstacles K of the type (3) the sojourn times of (ω,θ)-rays are not bounded, provided ω and θ suitably chosen. This enables us to study the asymptotics of the sojourn times when the number of reflections goes to infinity and to obtain some scattering invariants. In particular, for two strictly convex obstacles we recover as scattering invariants the distance d between the obstacles and the number c_0 determined by the first sequence of pseudo-poles of the scattering matrix (see [4], [2]).

2. Main results.

Let $\gamma = \cup_{i=0}^{k} l_i$ be a curve in $\mathbb{R}^3$ such that $l_i = [x_i, x_{i+1}]$, $i = 1,\dots,k-1 (k \geq 1)$, are finite segments, $x_i \in \partial\Omega$, while l_0 (l_k) is the infinite segment starting at x_1 (resp. at x_k) and having direction $-\omega$ (resp. θ). Then γ is called (ω, θ)-ray if the following conditions hold :

(i) the open segments l_i^0, $i=0,1,\dots,k$ do not intersect transversally $\partial\Omega$,

(ii) for every $i=0,1,\dots,k-1$ the segments l_i and l_{i+1} satisfy the reflection law at x_{i+1} (see [10], [11]).

A (ω, θ)-ray γ will be called ordinary one if γ has no segments tangent to $\partial\Omega$. For ordinary (ω, θ)-rays γ we can introduce the sojourn time T_γ and the map J_γ (see [3], [8] for the precise definitions). A subset $\mathcal{R} \subset S^2$ is called residual if $\mathcal{R}$ is a countable intersection of open dense sets. Throughout we assume that K has the form (3).

<u>Theorem 1</u>. Let $\omega \in S^2$ be fixed. Then there exists a residual subset $R \subset S^2$ such that for each $\theta \in R$ we have

(4) $\operatorname{singsupp} s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\}$, where $\mathcal{L}_{\omega,\theta}$ is the union of all ordinary (ω, θ)-rays.

Nakamura and Soga [7] established (4) for $\theta = -\omega$ and for two disjoint balls $\mathcal{O}_1, \mathcal{O}_2$, making some restrictions on the distance $(\mathcal{O}_1, \mathcal{O}_2)$ and the diameters of $\mathcal{O}_i$, $i = 1,2$.

The equality (4) is similar to the Poisson relation for generic bounded domains in $\mathbb{R}^2$ connecting the spectrum of the laplacian and the lengths of closed geodesics ([9], [12]). For this reason we will cal $\{T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\}$ scattering length spectrum related to ω, θ.

Under the assumption of Theorem 1 we can describe the leading singularity at $-T_\gamma$, $\gamma \in \mathcal{L}_{\omega,\theta}$. For this purpose denote by x_γ (resp. y_γ) the first (resp. the last) reflection point of γ. Let Z be a plane orthogonal to ω such that $Z \cap \overline{K} = \emptyset$. Denote by $A_\gamma \in Z$ the point where the segment starting at x_γ with direction $-\omega$ hits Z. Therefore, following [8], near $-T_\gamma$ we have

$$(5)\ s(t,\theta,\omega) = (1/2\pi) i^{\sigma_\gamma} (-1)^{m-1} \left| \frac{\det J_\gamma (A_\gamma) \langle n(x_\gamma), \omega \rangle}{\langle n(y_\gamma), \theta \rangle} \right|^{-1/2} \delta'(t+T_\gamma)$$

$$+ a_0 \delta(t+T_\gamma) + \text{smoother terms.}$$

Here $\sigma_\gamma \in \mathbb{N}$ is a Maslov index and m is the number of reflections of γ.

To study the existence of (ω, θ)-rays we are going to introduce the notion of a configuration. By a configuration α with length m $(m \geq 1)$ we mean a symbol $\alpha = (i_1, i_2, \ldots, i_m)$ with $i_j \in \{1,2,\ldots,M\}$ for all j and $i_j \neq i_{j+1}$ for $j = 1,2,\ldots,m-1$.

Definition 1. Let $\omega, \theta \in S_2$ and let γ be a (ω, θ)-ray with successive reflection points $x_1,\ldots,x_m$. We say that γ has type $\alpha = (i_1,\ldots,i_m)$ if $x_j \in \partial K_{i_j}$ for every $j = 1,\ldots,m$.

Definition 2. We say that a configuration $\alpha = (i_1,\ldots,i_m)$ satisfies the condition of visibility with respect to (ω, θ) if the following conditions hold :

(a) for every $x \in \partial K_{i_1}$ (resp. $x \in \partial K_{i_m}$) the ray starting at x with direction $-\omega$ (resp. θ) has no common points with $\cup_{j \neq i_1} \overline{K}_j$ (resp. $\cup_{j \neq i_m} \overline{K}_j$),

(b) for all $j = 1,\ldots,m-1$ the convex hull of $K_{i_j} \cup K_{i_{j+1}}$ do not contain points in $\cup_{r \neq i_j, i_{j+1}} \overline{K}_r$.

Theorem 2. If $\omega \neq \theta$ for every configuration α there exists at most one (ω, θ)-ray of type α. Moreover, if α satisfies the condition of visibility with respect to (ω, θ), then there exists a (ω,θ)-ray of type α.

In the case $M = 2$ the obstacle K satisfies the conditon fo visibility with respect to (ω, θ) if the condition (a) holds for $\{i_1, i_2\} = \{1,2\}$ and $\{i_1, i_2\} = \{2,1\}$.

Corollary 3. Let $\omega \neq \theta$ and let $K = K_1 \cup K_2$ satisfies the condition of visibility with respect to (ω, θ). Then for every $m \geq 1$ there exist exactly two different ordinary (ω, θ)-rays γ_m^i with m reflection points so that the first reflection point of γ_m^i belongs to ∂K_i, $i = 1,2$.

A partial case of Corollary 3 for $\theta = -\omega$ and two disjoint balls has been obtained by Nakamura and Soga [7].

By Theorem 2 we conclude that if we can find a configuration α satisfying the condition of visibility with respect to (ω, θ), then we can construct ordinary (ω, θ)-rays with arbitrary large number of reflections. Thus we get

$$\sup \{T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\} = \infty . \tag{6}$$

It is natural to make the following.

Conjecture. For every obstacle K in the form (3) there exist ω, θ such that (6) holds.

It is not hard to see that for $M = 2,3$ the above Conjecture is fulfilled. Moreover, for a large class of obstacles we can apply Theorem 2. Notice that (6) is a typical property of trapping obstacles since for non-trapping ones the sojourn times of (ω, θ)-rays are uniformly bounded with respect to (ω, θ).

3. Scattering invariants.

In this section we assume $K = K_1 \cup K_2$ and we consider two directions $\omega \neq \theta$ for which the assumption of Corollary 3 holds. Let γ^{ij}_m be the ordinary (ω, θ)-ray having m reflections and such that the first (resp. the last) reflection point of γ^{ij}_m belongs to ∂K_i (resp. ∂K_j). Let T^{ij}_m be the sojourn time of γ^{ij}_m.

<u>Theorem 4</u>. There exist constants $L^{ij}_{\omega,\theta}$ depending on (ω, θ) such that

(7) $$T^{ij}_m = md + L^{ij}_{\omega,\theta} + \varepsilon^{ij}_m$$

with $\varepsilon^{ij}_m \to 0$ as $m \to \infty$ and $d = \mathrm{dist}\,(K_1, K_2)$.

The invariants $L^{ij}_{\omega,\theta}$ are connected with the rays having infinite number relfections ans initial directions ω or $-\theta$. Consider the ray $\gamma^i_\infty(\omega)$, $i = 1,2$ starting at $x^i_\infty \in Z$ with direction ω and having relfection points $\{x^i_k\}^\infty_{k=1}$, $x^i_1 \in \partial K_i$. Set $l^i_m(\omega) = \langle x^i_1, \omega\rangle + \Sigma^{m-1}_{k=1} \|x^i_{k+1} - x^i_k\|$.

Then applying the results of Ikawa [5] (see also [13]), we obtain

(8) $$l^i_m(\omega) = md + L^i_m + O(m^{-N}),\ \forall N.$$

A similar result holds for the ray $l^j_\infty(-\theta)$ with initial direction $-\theta$ and reflection points $\{y^j_k\}^\infty_{k=1}$, $y^j_1 \in \partial K_j$. Thus we obtain the constants L^i_ω, $L^j_{-\theta}$ and $L^{ij}_{\omega,\theta} = L^i_\omega + L^j_{-\theta}$.

We expect that the asymptotic (7) is true with ε^{ij}_m replaced by $O(m^{-N})$, $\forall N$.

From (7) we get

(9) $\lim_{m\to\infty} (T^{ij}_{m+1} - T^{ij}_{m}) = d,$

hence we can recover the distance d from the scattering length spectrum. For two disjoint balls (9) has been obtained in [7].

We may compare (7) with the asymptotics of the lengths of the periodic reflecting rays established in [6] and [1]. In these works the authors consider periodic reflecting rays approximating the boundary [6] or an elliptic periodic ray [1]. In our case we approximate a stable hyperbolic ray related to a hyperbolic fixed point of the billard ball map and this is one of the reasons leading to the asymptotic (8).

Now we turn to the analysis of the asymptotic behavior of the amplitudes $c^{ij}_{m} = 2\pi\, |C^{ij}_{m}|$, C^{ij}_{m} being the coefficient in front of $\delta'(t+T^{ij}_{m})$ in the form (5) of the leading singularity at $-T^{ij}_{m}$. Consider the (linear) Poincaré map P corresponding to the periodic (trapping) ray orthogonal to both ∂K_i, $i = 1,2$. Let μ_i, $i = 1,2$ be the eigenvalues of P greater than 1 and let

$$c_0 = \log((\mu_1\, \mu_2)^{-1/4}).$$

<u>Theorem 5</u>. We have

(10) $\log c^{ij}_{m} = mc_0 + O(1),\ m \to \infty.$

We conjecture that the asymptotic (10) must have a sharp form like (8) with remainder $O(m^{-N})$ for each N.

The result of Theorem 1 tells us that we can determine T^{ij}_{m} and c^{ij}_{m} as the time and the amplitude of the scattering data. Therefore, the asymptotics (7) and (10) imply that we can recover from the scattering data the constants d and c_0, hence we can recover the first sequence of pseudo-poles

$$\lambda_j = -\frac{ic_o}{d} + j\frac{d}{\pi}, j \in \mathbb{Z}$$

of the scattering matrix $S(\lambda)$ (see [2], [4], [5]). On the other hand, the poles of $S(\lambda)$ coincide with their multiplicities with those of the meromorphic continuation of the scattering amplitude $a(-\lambda, \theta, \omega)$ and these poles do not depend on ω, θ. Choosing suitably ω and θ, we could study $a(-\lambda, \theta, \omega)$ instead of $S(\lambda)$. We hope that such approach will be useful for the analysis of the poles of the scattering matrix for trapping obstacles.

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