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LINEAR AND NON LINEAR FIELD EQUATIONS

par S. KLAINERMAN

The aim of this lecture is to illustrate how some recent geometric techniques which were usual to derive global existence and long time existence results for non linear wave equations ([1], [2], [3]) can be applied to tensorial field equations. We limitate ourselves here in describing the results which we have obtained in collaboration with D. Christodoulou, to the linear Maxwell and Spin - 2 equations in Minkowski space (see [4]). The latter are a linearised version of the Einstein equations in vacuum and their study important in our attempt to prove the global non linear stability of the Minkowski metric.

Consider the Minkowski space $\mathbb{R}^{3+1}$ with canonical coordinates (x^α) $\alpha = 0,1,2,3$ and metric

(1) $$ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

where η is the diagonal matrix with entries $(-1,1,1,1)$. The coordinate x^o is usually denoted by t . The following vector fields are conformal killing i.e. vector fields X so that $\mathcal{L}_X \eta$ is proportional to η .

(2i) The 4 generators of the translation group

$$T_\mu = \frac{\partial}{\partial x_\mu} \qquad \mu = 0,1,2,3$$

(2ii) The 6 generators of the Lorentz group

$$\Omega_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}$$

where $x_\mu = \eta_{\mu\nu}\, x^\nu$

(2iii) The scaling vector field

$$S = x^{\mu} \frac{\partial}{\partial x^{\mu}}$$

(2iv) The 4 accelerations vector fields

$$K_{\mu} = -2x_{\mu} S + \langle x,x\rangle \frac{\partial}{\partial x^{\mu}}$$

with $\langle x,x\rangle = \eta_{\mu\nu} x^{\mu} x^{\nu}$.

The Lie algebra Π generated by T, Ω, S plays a very important role in what follows. Given a tensor U in $\mathbb{R}^{3+1}$, we define the norms

(3) $$\|U(t)\|^2_{\Pi,s} = \Sigma \int_{\mathbb{R}^3} |\mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_k}} U(t,x)|^2 \, dx$$

with the sum taken over all generator $X_{i_1} \ldots X_{i_k}$, $0 \leqslant k \leqslant s$, of Π . Here, $\mathcal{L}_{X_1} \ldots \mathcal{L}_{X_{i_k}} U$ denotes the repeated Lie derivatives of U with respect to $X_1 \ldots X_k$ and $|.|$ denotes the euclidian norm in $\mathbb{R}^{3+1}$. Also, x refers to x^1, x^2, x^3 , $dx = dx^1\, dx^2\, dx^3$.

The Maxwell equations in $\mathbb{R}^{3+1}$ apply to antisymetric 2 - tensors $F_{\alpha\beta}$ which are required to satisfy the following two pairs of equations

(M_i) $$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$$

(M_{ii}) $$F^{;\beta}_{\alpha\beta} = 0$$

where $F_{\alpha\beta;\gamma}$ denotes the covariant differentiation of F relative to the flat Minkowski metric. The energy momentum tensor of (M_i) is given by the 2 - tensor

(4) $$Q_{\alpha\beta} = F_{\alpha\gamma}\, F^{\gamma}_{\beta} + {}^*F_{\alpha\gamma}\, {}^*F^{\gamma}_{\beta}$$

where ${}^*F$ is the Hodge dual of F i.e. ${}^*F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta}\, F^{\gamma\delta}$ and $\varepsilon_{\alpha\beta\gamma\delta}$ the components of the volume 1-form of $\mathbb{R}^{3+1}$. We remark that Q has the following properties.

- symetric in α, β
- traceless, i.e. $\eta^{\alpha\beta}\, Q_{\alpha\beta} = 0$
- satisfies the positive energy condition i.e. given any two time-like vector fields X, Y, $\langle X,X\rangle < 0$, $\langle Y,Y\rangle < 0$, both future oriented, we have :

$$Q(X,Y) = Q_{\alpha\beta}\, X^{\alpha} Y^{\beta} > 0$$

- $Q^{;\beta}_{\alpha\beta} = 0$

The energy momentum tensor allows one to derive energy estimates for (M). Let F be a solution of (M), and consider X a time-like vector field. Let $P^{\alpha} = Q^{\alpha\beta}\, X_{\beta}$ be the X-momentum of F. Then,

(5) $$P^{;\alpha}_{\alpha} = \frac{1}{2}\, Q^{\alpha\beta}(X_{\alpha;\beta} + X_{\beta;\alpha})$$

The expression $X_{\alpha;\beta} + X_{\beta;\alpha}$ is precisely $\mathcal{L}_X\, \eta_{\alpha\beta}$ and thus proportional to η, if we choose X to be conformal killing. On the other hand, since Q is traceless, we conclude that any choice of a conformal killing vector field leads to a conservation law in (5) i.e.

(5') $$P^{;\alpha}_{\alpha} = 0$$

Integrating (5') on slots $[0,t] \times \mathbb{R}^3$ we infer that ,

(6) $$\int_{t=\text{const}} Q(T_o, X)\, dx = \int_{t=o} Q(T_o, X)\, dx$$

where $T_o = \dfrac{\partial}{\partial x^o} = \dfrac{\partial}{\partial t}$.

According the positive energy condition, $Q(T_o, X)$ is everywhere positive if X is time-like. The only two choices of conformal killing time-like vector fields are $X = T_o$ and $X = K_o$. In fact, let $\overline{K}_o = K_o + T_o$. Then, according to (6), we have

$$(6') \qquad \int_{t=\text{const}} Q(T_o, \overline{K}_o)\,dx = \int_{t=o} Q(T_o, \overline{K}_o)\,dx$$

According to (6'), we introduce the norm

$$(7) \qquad \| F(t) \|^{\#} = \left(\int_{\mathbb{R}^3} Q(T_o, \overline{K}_o)\,dx \right)^{1/2}$$

where Q is the energy momentum tensor of F . Also, we define

$$(7') \qquad \| F(t) \|^{\#2}_{\Pi, s} = \Sigma \, \| \mathcal{L}_{X_{i_1}} \ldots \mathcal{L}_{X_{i_k}} F(t) \|^{\#2}$$

with the sum extended over all choices of vector fields $X_1, \ldots X_k$, $0 \leqslant k \leqslant s$, among the generators of Π . Now, due to the conformal equivalence of the equations (M), we can easily check that if F is a solution, the so is $\mathcal{L}_X F$ for any confomal vector field X . As a consequence, we conclude that

$$(8) \qquad \| F(t) \|^{\#}_{\Pi, s} = \| F(0) \|^{\#}_{\Pi, s}$$

and finite if the right hand side is finite. Since the right hand side depends only on initial conditions for F at time $t = 0$, we conclude that $\| F(t) \|^{\#}_{\Pi, s}$ can be made globally finite, by requiring appropriate conditions at infinity, for the initial data. Finally, we concise this feet to derive uniform decay properties for F . To state our theorem, we need to introduce null frames in $\mathbb{R}^{3+1}$. Thus, let $e_+ = \partial_t + \partial_2$, $e_- = \partial_t - \partial_r$ and e_1, e_r vector fields orthogonal to e_+, e_- , and to each other, and of length one. The vector field $\frac{\partial}{\partial r}$ is the radial vector field $\frac{x^i}{|x|} \partial_i$ with $|x|^2 = \sum_{i=1}^{3} (x^i)^2$.

We decompose F relative to the null frame according to :

(9)
$$\alpha_A = F_{A+} \quad , \quad \underline{\alpha}_A = F_{A-} \quad , \quad A = 1,2$$
$$\rho = F_{+-} \qquad \sigma = {}^*F_{+-}$$

Here, α and $\underline{\alpha}$ are vectors tangent to the spheres $|x| = \text{const}$ in $\mathbb{R}^3$ while ρ and σ are scalars. Clearly they determine the full tensor F .

The finiteness of the norm $\|F(t)\|^{\#}_{\Pi,s}$ can be used to prove the following

<u>Theorem 1</u> : Let F be a solution of the Maxwell equations (M) with initial conditions at $t=0$ for which the norm $I_s = \|F(0)\|^{\#}_{\Pi,s}$, $s \geq 2$, is finite, then

(i) $$|F(t,x)| \leq C(1+t)^{-5/2}\, I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \leq \frac{t}{2} + 1$

(ii) $$|\underline{\alpha}(t,x)| \leq C(1 + |t - |x||)^{-3/2}\,(1 + t + |x|)^{-1}\, I_2$$

$$|(\rho,\sigma)(t,x)| \leq C\,(1 + |t - |x||)^{-1/2}\,(1 + t + |x|)^{-2}\, I_2$$

$$|\alpha(t,x)| \leq C\,(1 + t + |x|)^{-5/2}\, I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \geq \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of F in the interior, $|x| \leq \frac{t}{2} + 1$, or for the derivatives α , $\underline{\alpha}$, ρ , σ relative to the null frame e_+ , e_- , e_1 , e_2 in the exterior $|x| \geq \frac{t}{2} + 1$ (see [4]).

In the second part of this lecture, I will indicate how similar results, based on the same ideas, can be used to derive decay estimates for the Spin - 2 equations. These are equations satisfied by 4 - tensors $W_{\alpha\beta\gamma\delta}$ which have all the symetry properties of the Riemann curvature tensor of metric satisfying the Einstein vacuum equations. Namely,

(i) $$W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma}$$

$$W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta}$$

(ii) $$W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0$$

(iii) $$W^{\beta}_{\alpha\beta\gamma} = 0$$

The Spin - 2 equations are

(Sp) $$W_{\alpha\beta\gamma\delta;\varepsilon} + W_{\alpha\beta\delta\varepsilon;\gamma} + W_{\alpha\beta\varepsilon\gamma;\delta} = 0$$

As the Maxwell equations, the Spin - 2 equations are conformal invariant. In particular, for any solution W and any conformal vector field X, $\mathcal{L}_X W$ is also a solution. What corresponds to the energy momentum tensor for the Maxwell equations is now a 4 - tensor Q defined by

(10) $$Q_{\alpha\beta\gamma\delta} = W_{\alpha\mu\gamma\nu}\, W^{\mu\nu}_{\beta\delta} + {}^{*}W_{\alpha\mu\gamma\nu}\, {}^{*}W^{\mu\nu}_{\beta\delta}$$

with ${}^{*}W_{\alpha\beta\gamma\delta} = \varepsilon^{\mu\nu}_{\alpha\beta}\, W_{\mu\nu\gamma\delta}$ the Hodge dual of W.

One can prove that Q satisfied the following properties

- Q is symetric and traceless relative to all pair of indices
- Q satisfied the positive energy condition i.e. given any X,Y time like and future oriented :

$$Q(X,X,Y,Y) = Q_{\alpha\beta\gamma\delta}\, X^{\alpha}X^{\beta}Y^{\gamma}Y^{\delta} > 0$$

- $Q^{;\delta}_{\alpha\beta\gamma\delta} = 0$

whenever W is a solution of (Sp).

One can now proceed as in the derivation of the energy identities for the Maxwell equations to show that

$$(11) \qquad \int_{t=cste} Q(T_o, T_o, \overline{K}_o, \overline{K}_o)\,dx = \int_{t=o} Q(T_o, T_o, \overline{K}_o, \overline{K}_o)\,dx \quad .$$

Or, introducing the norm

$$(12) \qquad \| W(t) \|^{\#} = \left(\int_{\mathbb{R}^3} Q(T_o, T_o, \overline{K}_o, \overline{K}_o)\,dx \right)^{1/2}$$

With Q the energy momentum tensor of W, and also,

$$(12') \qquad \| W(t) \|^{\#}_{\Pi,s} + \left(\Sigma \, \| \mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_k}} W(t) \|^{\#2} \right)^{1/2}$$

for any generators $X_{i_1}, \ldots, X_{i_k}$, $0 \leq k \leq s$ of Π,

$$(13) \qquad \| W(t) \|^{\#}_{\Pi,s} = \| W(0) \|^{\#}_{\Pi,s}$$

We now decompose W relative to the same null frame introduced above, and introduce

$$(14) \qquad \begin{aligned} \alpha_{AB} &= W_{A+B+} \quad , & \underline{\alpha}_{AB} &= W_{A-B-} \\ \beta_A &= \tfrac{1}{2} W_{A+-+} & \underline{\beta}_A &= \tfrac{1}{2} W_{A-+-} \\ \rho &= \tfrac{1}{4} W_{+-+-} & \sigma &= \tfrac{1}{4} {}^{*}W_{+-+-} \quad . \end{aligned}$$

Clearly, $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \sigma$ completely determine W, and we can prove the following

Theorem 2 : Let W ve a solution of (Sp) with initial conditions at $t=0$ for which $I_s = \| W(0) \|^{\#}_{\Pi,s} < +\infty$ for some $s \geqslant 2$. Then

(i) $$|W(t,x)| \leqslant C\ (1+t)^{-7/2}\ I_2$$

for any $t \geqslant 0$, $x \in \mathbb{R}^3$, $|x| \leqslant \frac{t}{2} + 1$

(ii) $$|\underline{\alpha}(t,x)| \leqslant C\ (1+|t-|x||)^{-5/2}\ (1+t+|x|)^{-1}\ I_2$$

$$|\underline{\beta}(t,x)| \leqslant C\ (1+|t-|x||)^{-3/2}\ (1+t+|x|)^{-2}\ I_2$$

$$|(\rho,\sigma)(t,x)| \leqslant C\ (1+|t-|x||)^{-1/2}\ (1+t+|x|)^{-3}\ I_2$$

$$|\alpha(t,x)| \leqslant C\ (1+t+|x|)^{-7/2}\ I_2$$

for any $t \geqslant 0$, $x \in \mathbb{R}^3$, $|x| \geqslant \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of W (see [4]).

The spirit of these linear estimates can be adjusted to treat the non linear Einstein equations. This, I hope, will be done in a series of papers together with D. Christodoulou.

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