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Linear and nonlinear field equations


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The aim of this lecture is to illustrate how some recent geometric techniques which were usual to derive global existence and long time existence results for non linear wave equations ([1], [2], [3]) can be applied to tensorial field equations. We limitate ourselves here in describing the results which we have obtained in collaboration with D. Christodoulou, to the linear Maxwell and Spin - 2 equations in Minkowski space (see [4]). The latter are a linearised version of the Einstein equations in vacuum and their study important in our attempt to prove the global non linear stability of the Minkowski metric.

Consider the Minkowski space $\mathbb{R}^{3+1}$ with canonical coordinates $(x^\alpha) \alpha = 0,1,2,3$ and metric

$$ds^2 = \eta_{\alpha\beta} \, dx^\alpha \, dx^\beta$$

where $\eta$ is the diagonal matrix with entries $(-1,1,1,1)$. The coordinate $x^0$ is usually denoted by $t$. The following vector fields are conformal killing i.e. vector fields $X$ so that $\ell^X \eta$ is proportional to $\eta$.

(2i) The 4 generators of the translation group

$$T_\mu = \frac{\partial}{\partial x_\mu} \quad \mu = 0,1,2,3$$

(2ii) The 6 generators of the Lorentz group

$$\Omega_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}$$

where $x_\mu = \eta_{\mu\nu} x^\nu$
(2iii) The scaling vector field

\[ S = x^\mu \frac{\partial}{\partial x^\mu} \]

(2iv) The 4 accelerations vector fields

\[ K_\mu = -2x_\mu S + \langle x, x \rangle \frac{\partial}{\partial x^\mu} \]

with \( \langle x, x \rangle = \eta_{\mu \nu} x^\mu x^\nu \).

The Lie algebra \( \Pi \) generated by \( T, \Omega, S \) plays a very important role in what follows. Given a tensor \( U \) in \( \mathbb{R}^{3+1} \), we define the norms

\[ \|U(t)\|_{\Pi, s}^2 = \sum_{i=1}^{3} \int_{\mathbb{R}^3} |L_{X_1} \cdots L_{X_k} U(t, x)|^2 \, dx \]

with the sum taken over all generator \( X_1 \cdots X_k \), \( 0 \leq k \leq s \), of \( \Pi \).

Here, \( L_{X_1} \cdots L_{X_k} U \) denotes the repeated Lie derivatives of \( U \) with respect to \( X_1 \cdots X_k \) and \( \| \cdot \| \) denotes the euclidian norm in \( \mathbb{R}^{3+1} \).

Also, \( x \) refers to \( x^1, x^2, x^3 \), \( dx = dx^1 \, dx^2 \, dx^3 \).

The Maxwell equations in \( \mathbb{R}^{3+1} \) apply to antisymmetric 2-tensors \( F_{\alpha\beta} \) which are required to satisfy the following two pairs of equations

\[ (M_1) \quad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \]

\[ (M_{i1}) \quad F_{\alpha\beta;\gamma} = 0 \]

where \( F_{\alpha\beta;\gamma} \) denotes the covariant differentiation of \( F \) relative to the flat Minkowski metric. The energy momentum tensor of \( (M_1) \) is given by the 2-tensor
\begin{equation}
Q_{\alpha\beta} = F_{\alpha\gamma} F_{\beta}^{\gamma} + \star F_{\alpha\gamma} \star F_{\beta}^{\gamma}
\end{equation}

where \( \star F \) is the Hodge dual of \( F \) i.e. \( \star F_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \) and \( \varepsilon_{\alpha\beta\gamma\delta} \) the components of the volume 1-form of \( \mathbb{R}^{3+1} \). We remark that \( Q \) has the following properties.

- symmetric in \( \alpha, \beta \)
- traceless, i.e. \( \eta^{\alpha\beta} Q_{\alpha\beta} = 0 \)
- satisfies the positive energy condition i.e. given any two time-like vector fields \( X, Y \), \( \langle X, X \rangle < 0 \), \( \langle Y, Y \rangle < 0 \), both future oriented, we have:

\[ Q(X, Y) = Q_{\alpha\beta} X_{\alpha} Y_{\beta} > 0 \]

\[ Q^{;\beta}_{\alpha} = 0 \]

The energy momentum tensor allows one to derive energy estimates for (M). Let \( F \) be a solution of (M), and consider \( X \) a time-like vector field. Let \( P^\alpha = Q^{\alpha\beta} X_\beta \) be the \( X \)-momentum of \( F \). Then,

\begin{equation}
P^{;\alpha}_{\alpha} = \frac{1}{2} Q^{\alpha\beta} (X^{;\alpha}_{\alpha\beta} + X^{;\beta}_{\alpha\alpha})
\end{equation}

The expression \( X^{;\alpha}_{\alpha\beta} + X^{;\beta}_{\alpha\alpha} \) is precisely \( \int_X \eta_{\alpha\beta} \) and thus proportional to \( \eta \), if we choose \( X \) to be conformal killing. On the other hand, since \( Q \) is traceless, we conclude that any choice of a conformal killing vector field leads to a conservation law in (5) i.e.

\begin{equation}
P^{;\alpha}_{\alpha} = 0
\end{equation}

Integrating (5') on slots \([0,t] \times \mathbb{R}^3\) we infer that,

\begin{equation}
\int_{t=\text{const}} Q(T_0, X) dx = \int_{t=0} Q(T_0, X) dx
\end{equation}

where \( T_0 = \frac{\partial}{\partial x^0} = \frac{3}{\partial t} \).
According to the positive energy condition, \( \mathcal{Q}(T, \mathbf{X}) \) is everywhere positive if \( \mathbf{X} \) is time-like. The only two choices of conformal killing time-like vector fields are \( \mathbf{X} = T_0 \) and \( \mathbf{X} = K_0 \). In fact, let \( \overline{K}_0 = K_0 + T_0 \).

Then, according to (6), we have

\[
(6') \quad \int_{t=\text{const}} Q(T_0, \overline{K}_0) \, dx = \int_{t=0} Q(T_0, \overline{K}_0) \, dx
\]

According to (6'), we introduce the norm

\[
(7) \quad \| F(t) \|_{\Pi, s}^\# = \left( \int_{\mathbb{R}^3} Q(T_0, \overline{K}_0) \, dx \right)^{1/2}
\]

where \( Q \) is the energy momentum tensor of \( F \). Also, we define

\[
(7') \quad \| F(t) \|_{\Pi, s}^{\#2} = \sum \| \mathcal{L}_{X_i} \ldots \mathcal{L}_{X_{i_k}} F(t) \|_{\Pi, s}^2
\]

with the sum extended over all choices of vector fields \( X_1, \ldots, X_k \), \( 0 \leq k \leq s \), among the generators of \( \Pi \). Now, due to the conformal equivalence of the equations (M), we can easily check that if \( F \) is a solution, the so is \( \mathcal{L}_{X} F \) for any conformal vector field \( X \). As a consequence, we conclude that

\[
(8) \quad \| F(t) \|_{\Pi, s}^\# = \| F(0) \|_{\Pi, s}^\#
\]

and finite if the right hand side is finite. Since the right hand side depends only on initial conditions for \( F \) at time \( t=0 \), we conclude that \( \| F(t) \|_{\Pi, s}^\# \) can be made globally finite, by requiring appropriate conditions at infinity, for the initial data. Finally, we concisely this feet to derive uniform decay properties for \( F \). To state our theorem, we need to introduce null frames in \( \mathbb{R}^{3+1} \). Thus, let \( e_+ = \partial_t + \partial_r \), \( e_- = \partial_t - \partial_r \) and \( e_1, e_r \) vector fields orthogonal to \( e_+, e_- \), and to each other, and of length one. The vector field \( \frac{\partial}{\partial r} \) is the radial vector field \( \sum_{i=1}^3 \frac{x_i}{|x|} \partial_i \) with \( |x|^2 = \sum_{i=1}^3 (x_i)^2 \).
We decompose $F$ relative to the null frame according to:

$$\alpha_A = F_{A^+}, \quad \alpha_A = F_{A^-}, \quad A = 1, 2$$

$$\rho = F_{+}, \quad \sigma = F_{+}$$

(9)

Here, $\alpha$ and $\alpha_-$ are vectors tangent to the spheres $|x| = \text{const}$ in $\mathbb{R}^3$ while $\rho$ and $\sigma$ are scalars. Clearly they determine the full tensor $F$.

The finiteness of the norm $\|F(t)\|_{\Pi, s}^\#$ can be used to prove the following

**Theorem 1**: Let $F$ be a solution of the Maxwell equations (M) with initial conditions at $t = 0$ for which the norm $I_s = \|F(0)\|_{\Pi, s}^\#$, $s \geq 2$, is finite, then

(i) \[ |F(t,x)| \leq C(1+t)^{-5/2} I_2 \]

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \leq \frac{t}{2} + 1$

(ii) \[ |\alpha(t,x)| \leq C(1 + |t - |x||)^{-3/2} (1 + t + |x|)^{-1} I_2 \]

\[ |(\rho, \sigma)(t,x)| \leq C \left(1 + |t - |x||\right)^{-1/2} (1 + t + |x|)^{-2} I_2 \]

\[ |\alpha(t,x)| \leq C \left(1 + t + |x|\right)^{-5/2} I_2 \]

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \geq \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of $F$ in the interior, $|x| \leq \frac{t}{2} + 1$, or for the derivatives $\alpha, \alpha_-, \rho, \sigma$ relative to the null frame $e_+, e_-, e_1, e_2$ in the exterior $|x| \geq \frac{t}{2} + 1$ (see [4]).
In the second part of this lecture, I will indicate how similar results, based on the same ideas, can be used to derive decay estimates for the Spin-2 equations. These are equations satisfied by 4-tensors \( W_{\alpha\beta\gamma\delta} \) which have all the symmetry properties of the Riemann curvature tensor of metric satisfying the Einstein vacuum equations. Namely,

\[
(w_{\alpha\beta\gamma\delta} = -w_{\beta\alpha\gamma\delta} = -w_{\alpha\beta\delta\gamma})
\]

\[
w_{\alpha\beta\gamma\delta} = w_{\gamma\delta\alpha\beta}
\]

(ii) \[
w_{\alpha\beta\gamma\delta} + w_{\alpha\gamma\delta\beta} + w_{\alpha\delta\gamma\beta} = 0
\]

(iii) \[
w_{\alpha\beta\gamma} = 0
\]

The Spin-2 equations are

\[
(W_{\alpha\beta\gamma\delta} ; e + W_{\alpha\beta\delta\epsilon} ; \gamma + W_{\alpha\beta\epsilon\gamma} ; \delta = 0
\]

As the Maxwell equations, the Spin-2 equations are conformal invariant. In particular, for any solution \( W \) and any conformal vector field \( X \), \( \mathcal{L}_X W \) is also a solution. What corresponds to the energy momentum tensor for the Maxwell equations is now a 4-tensor \( Q \) defined by

\[
Q_{\alpha\beta\gamma\delta} = W_{\alpha\mu\nu} W_{\beta\delta}^{\mu\nu} + *w_{\alpha\mu\nu} *w_{\beta\delta}^{\mu\nu}
\]

with \( *w_{\alpha\beta\gamma} = \epsilon_{\alpha\beta} \ w_{\mu\nu\gamma} \) the Hodge dual of \( W \).

One can prove that \( Q \) satisfied the following properties

- \( Q \) is symmetric and traceless relative to all pair of indices
- \( Q \) satisfied the positive energy condition i.e. given any \( X,Y \) time like and future oriented :

\[
Q(X,X,Y,Y) = Q_{\alpha\beta\gamma\delta} x^\alpha x^\beta y^\gamma y^\delta > 0
\]
whenever $W$ is a solution of (Sp).

One can now proceed as in the derivation of the energy identities for the Maxwell equations to show that

(11) \[ \int_{t=\text{cste}} Q(T_0, T_0, \bar{K}_0, \bar{K}_0) \, dx = \int_{t=0} Q(T_0, T_0, \bar{K}_0, \bar{K}_0) \, dx. \]

Or, introducing the norm

(12) \[ \| W(t) \|^2 = \left( \int_{\mathbb{R}^3} Q(T_0, T_0, \bar{K}_0, \bar{K}_0) \, dx \right)^{1/2} \]

with $Q$ the energy momentum tensor of $W$, and also,

(12') \[ \| W(t) \|^2_{\Pi, s} + \left( \sum_{i_1}^{\Pi} \| \ell X_{i_1} \cdots \ell X_{i_k} W(t) \|^2 \right)^{1/2} \]

for any generators $X_{i_1}, \ldots, X_{i_k}$, $0 \leq k \leq s$ of $\Pi$,

(13) \[ \| W(t) \|^2_{\Pi, s} = \| W(0) \|^2_{\Pi, s} \]

We now decompose $W$ relative to the same null frame introduced above, and introduce

(14) \[ \alpha_{AB} = W_{A+B+} , \quad \beta_A = \frac{1}{2} W_{A+-} , \quad \rho = \frac{1}{4} W_{+-+} \]

Clearly, $\alpha, \alpha, \beta, \beta, \rho, \sigma$ completely determine $W$, and we can prove the following
Theorem 2 : Let \( W \) be a solution of (Sp) with initial conditions at 
\( t = 0 \) for which \( I_s = \| W(0) \|_{\Pi_s} < + \infty \) for some \( s \geq 2 \). Then

(i) \[ |W(t,x)| \leq C (1+t)^{-7/2} I_2 \]

for any \( t > 0, \ x \in \mathbb{R}^3, \ |x| < \frac{t}{2} + 1 \)

(ii) \[ |\alpha(t,x)| \leq C (1 + |t - |x||)^{-5/2} (1 + t + |x|)^{-1} I_2 \]
\[ |\beta(t,x)| \leq C (1 + |t - |x||)^{-3/2} (1 + t + |x|)^{-2} I_2 \]
\[ |(\rho,\sigma)(t,x)| \leq C (1 + |t - |x||)^{-1/2} (1 + t + |x|)^{-3} I_2 \]
\[ |\alpha(t,x)| \leq C (1 + t + |x|)^{-7/2} I_2 \]

for any \( t > 0, \ x \in \mathbb{R}^3, \ |x| > \frac{t}{2} + 1 \).

Similar estimates can be derived for the derivatives of \( W \) (see [4]).

The spirit of these linear estimates can be adjusted to treat the non linear Einstein equations. This, I hope, will be done in a series of papers together with D. Christodoulou.

REFERENCES


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