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Remarks on the Klein-Gordon equation

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1. Introduction. From the work of Klainerman [6] and Shatah [7] it is known that for non-linear perturbations of the Klein-Gordon equation in $\mathbf{R}^{1+n}$,

$$\square u + u = F(u, u', u''), \tag{1.1}$$

where $\square = \partial_0^2 - \partial_1^2 - \cdots - \partial_n^2$, F vanishes of second order at 0, and F is linear in u'', the Cauchy problem with small data in C_0^∞ has a global solution if $n \geq 3$.

The main purpose of this paper is to examine the remaining cases $n = 1, 2$. We shall begin by studying in Section 2 the solutions of the unperturbed Klein-Gordon equation $\square u + u = 0$ in considerable detail for arbitrary n. This covers the estimates of von Wahl [8] and gives in addition a much more precise description of the asymptotic properties to serve as a goal in the study of (1.1). In Section 3 we discuss L^2 estimates for the inhomogeneous linear Klein-Gordon equation in the spirit of Klainerman [6]. His estimates for the case $n = 3$ were not sharp but sufficient to establish global existence theorems then. Their analogue for $n = 1$ or $n = 2$ would not give a good estimate for the lifespan of the solutions. We shall therefore reexamine the estimates of [6] for arbitrary dimension, but some of them may not be sharp when $n > 3$. Using these bounds we outline in Section 4 how existence theorems for (1.1) follow when F vanishes of second order or of third order at 0. In the second case we believe that our results are optimal, but it is feasible that the lifespan of the solutions must be of the same order of magnitude in the two cases. Some evidence in favor of that is presented in Section 5. In particular we discuss the case $n = 0$ there, that is, the ordinary differential equation

$$u'' + u = F(u, u').$$

Some new idea seems needed to decide what the optimal results should be when $n = 1$ or $n = 2$.

2. Asymptotic behavior of solutions of the Klein-Gordon equation. In this section we shall discuss the solution of the Cauchy problem

$$\square u + u = 0; \quad u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x), \tag{2.1}$$

where u is a function of $(t, x) \in \mathbf{R}^{1+n}$, $\square = \partial_t^2 - \Delta$, and $u_j \in \mathcal{S}(\mathbf{R}^n)$. It is immediately obtained by Fourier transformation in the x variables,

$$\hat{u}(t, \xi) = \tfrac{1}{2}(\hat{u}_0(\xi) - i\hat{u}_1(\xi)/\langle\xi\rangle)e^{it\langle\xi\rangle} + \tfrac{1}{2}(\hat{u}_0(\xi) + i\hat{u}_1(\xi)/\langle\xi\rangle)e^{-it\langle\xi\rangle}.$$

Here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. This gives a splitting $u = u_+ + u_-$ where

$$\partial_t u_\pm = \pm i\langle D_x\rangle u_\pm, \quad u_\pm(0, x) = \varphi_\pm,\ \varphi_\pm = (u_0 \mp i\langle D_x\rangle^{-1}u_1)/2. \tag{2.2}$$

Since $\varphi_\pm \in \mathcal{S}$, we can as well study the problem

$$\partial_t u = i\langle D_x\rangle u, \quad u(0, x) = \varphi \in \mathcal{S}, \tag{2.2$_+$}$$

The splitting above is Lorentz invariant, for the spectrum of $u_\pm$ as a function in $\mathbf{R}^{1+n}$ is contained in $\{(\pm\langle\xi\rangle, \xi), \xi \in \mathbf{R}^n\}$, and these hyperboloids are disjoint. More explicitly, the Fourier transform of $u_\pm$ in all variables is $2\pi\delta(\tau \mp \langle\xi\rangle)\hat{\varphi}_\pm(\xi)$. The solution of (2.2)$_+$ is

$$u(t, x) = (2\pi)^{-n} \int e^{i(t\langle\xi\rangle + \langle x, \xi\rangle)}\hat{\varphi}(\xi)\, d\xi. \tag{2.3}$$

A *formal* application of the method of stationary phase suggests that one should look for a point ξ where

$$t\xi/\langle\xi\rangle + x = 0,$$

that is,

$$|\xi|^2 = |x|^2/(t^2 - |x|^2), \quad \langle\xi\rangle^2 = t^2/(t^2 - |x|^2).$$

There is a unique solution when $|x| < |t|$ but none otherwise. If $|x| < t$ the critical value of the phase is

$$t\langle\xi\rangle + \langle x, \xi\rangle = t(\langle\xi\rangle - |\xi|^2/\langle\xi\rangle) = t/\langle\xi\rangle = (t^2 - |x|^2)^{\frac{1}{2}},$$

and the Hessian matrix is $t(\delta_{jk}/\langle\xi\rangle - \xi_j\xi_k/\langle\xi\rangle^3)$. The determinant is $t^n\langle\xi\rangle^{-2-n}$ as is immediately seen when $\xi_2 = \cdots = \xi_n = 0$, so we expect that the main contribution to $u(t,x)$ must be

$$e^{i\sqrt{t^2-|x|^2}}(2\pi)^{-n/2}(\langle\xi\rangle^{n+2}|t|^{-n})^{\frac{1}{2}}e^{i\pi n/4}\hat{\varphi}(\xi).$$

This suggests that $u(t,x)e^{-i\sqrt{t^2-|x|^2}}$ behaves when $t > 0$ as a symbol of order $-n/2$ which vanishes for $|x| > t$, hence can be estimated by $(1 + |t| + |x|)^{-n/2}$ times any power of $(1 + |t - |x||)/(1 + t + |x|)$. This is what we shall prove apart from an additional term which lies in the Schwartz space $\mathcal{S}$. In order to be able to deduce some estimates for Cauchy data of finite smoothness we shall first state a weaker result when the Cauchy data just have Fourier transforms in a suitable symbol space.

THEOREM 2.1. *Assume that the Fourier transform of φ is a C^∞ function $\hat{\varphi}$ such that for every multi-index α*

$$|D^\alpha\hat{\varphi}(\xi)| \le C_\alpha(1 + |\xi|)^{N-|\alpha|}, \quad \xi \in \mathbf{R}^n. \tag{2.4}$$

If $N < -(n+1)/2$ it follows that for $|t| + |x| \ge 1$

$$|u(t,x)| \le C(|t| + |x|)^{N+1}(1 + (t^2 - |x|^2)_+)^{M_+/2}(1 + (|x|^2 - t^2)_+)^{M_-}, \tag{2.5}$$

where M_- is arbitrary and $M_+ = \max(0, -\frac{n}{2} - N - 1)$; the constant C depends on M_-.

COROLLARY 2.2. *For any integer $\nu \ge n/2$ the forward fundamental solution E of the Klein-Gordon equation can be written in the form*

$$E = \sum_{|\alpha|\le\nu} D_x^\alpha E_\alpha \tag{2.6}$$

where for any M_-

$$|E_\alpha(t,x)| \le C(|t| + |x|)^{-\nu}(1 + (t^2 - |x|^2)_+)^{(2\nu-n)/4}(1 + (|x|^2 - t^2)_+)^{M_-}, \tag{2.7}$$

if $|t| + |x| \ge 1$. We can also write

$$\partial_t E = \sum_{|\alpha|\le\nu+1} D_x^\alpha E_\alpha$$

for some other E_α satisfying (2.7).

Corollary 2.2 contains the estimates of von Wahl [8], which we shall only state in a special case:

COROLLARY 2.3. *For the solution of (2.1) we have the estimates*

$$|u(t,x)| \le C|t|^{-n/2} \sum_{|\alpha|+j\le(n+3)/2} \int |D^\alpha u_j(y)|\,dy, \quad |t| \ge 1, \tag{2.8}$$

$$|u(t,x)| \le C \sum_{|\alpha|+j\le n} \int |D^\alpha u_j(y)|\,dy, \quad |t| < 1. \tag{2.9}$$

We shall finally give a precise statement of the results on the asymptotic behavior of the solution of (2.1) when $u_j \in \mathcal{S}(\mathbf{R}^n)$, motivated at the beginning of the section.

THEOREM 2.4. *If $\varphi \in S$ then the solution of (2.2)$_+$ can be written in the form*

$$u(t,x) = U_0(t,x) + U_+(t,x)e^{i\varrho} \tag{2.10}$$

where $U_0 \in S(\mathbf{R}^{n+1})$, $\varrho = \operatorname{sgn} t\sqrt{t^2-|x|^2} = t\sqrt{1-|x|^2/t^2}$ and U_+ is a polyhomogeneous symbol of order $-n/2$ with support in the double light cone,

$$U_+(t,x) \sim (+0 - i\varrho)^{-n/2}\sum_0^\infty \varrho^{-j}w_j(t,x)$$

where $w_j(t,x) = w_j(1,x/t)$. The leading term is given by

$$w_0(t,x) = (2\pi)^{-n/2}(t/\varrho)\hat{\varphi}(-x/\varrho),$$

interpreted as 0 when $t^2 \geq |x|^2$.

The decomposition in Therem 2.4 is essentially unique:

THEOREM 2.5. *If $v_0 \in S(\mathbf{R}^{n+1})$ and v_+, v_- are symbols for the standard metric with support in the light cone such that*

$$v_0(t,x) + v_+(t,x)e^{i\sqrt{t^2-|x|^2}} + v_-(t,x)e^{-i\sqrt{t^2-|x|^2}} = 0 \tag{2.11}$$

then $v_\pm \in S$.

Let us now return to the solution of (2.1); what we have done so far concerns only the term u_+ in (2.2). For u_- we have the same result with φ_+ replaced by φ_- and t replaced by $-t$. Hence

THEOREM 2.6. *If $u_0, u_1 \in S$, then the solution of (2.1) can be written in the form*

$$u(t,x) = U_0(t,x) + U_+(t,x)e^{i\varrho} + U_-(t,x)e^{-i\varrho}, \quad \varrho = \operatorname{sgn} t\sqrt{t^2-|x|^2}, \tag{2.12}$$

where $U_0 \in S(\mathbf{R}^{n+1})$, U_+, U_- have their supports in the double light cone and $(+0 \mp i\varrho)^{n/2}$ are polyhomogeneous symbols of order 0 with fully homogeneous terms and leading terms

$$(2\pi)^{-n/2}(t/\varrho)\hat{\varphi}_\pm(\mp x/\varrho),$$

interpreted as 0 outside the double light cone. Here $\varphi_\pm$ are defined by (2.2).

By symbolic calculus we can successively compute the complete symbol from the leading symbol, which was given in Theorem 2.6. This is a much better way of calculating the asymptotic expansion than by using the method of stationary phase, except for the leading term which must be obtained from it.

3. L^2, L^∞ estimates for the Klein-Gordon equation. If u is a solution of the inhomogeneous Klein-Gordon equation

$$\Box u + u = f; \quad u(0,.) = u_0,\ \partial_t u(0,.) = u_1; \tag{3.1}$$

where $u_j \in C_0^\infty$, then it is well known that

$$(\|u'(t,.)\|^2 + \|u(t,.)\|^2)^{\frac12} \leq (\|u_1\|^2 + \|u_0'\|^2 + \|u_0\|^2)^{\frac12} + \int_0^t \|f(s,.)\|\,ds \tag{3.2}$$

where the norms are L^2 norms and u' denotes the derivatives with respect to t and x. This gives control of u and the first derivatives of u in L^2 for fixed t. As emphasized by Klainerman [6], the equation (3.1) implies

$$\Box Z^I u + Z^I u = Z^I f \tag{3.1'}$$

if Z^I is any product of the vector fields

$$\partial/\partial t, \quad \partial/\partial x_j, \quad j = 1, \ldots n, \tag{3.3}$$

$$t\partial/\partial x_j + x_j\partial/\partial t, \; x_j\partial/\partial x_k - x_k\partial/\partial x_j, \quad j,k = 1,\ldots,n; \tag{3.4}$$

generating the Lie algebra of the inhomogeneous Lorentz group. Thus one can hope to get estimates of all $Z^I u$ in L^2, for fixed t, even for perturbations of the equation (3.1) such that $Z^I f$ in turn can be estimates by some $Z^J u$. The purpose of this section is to examine to what extent one can recover the maximum norm estimates of u established in Section 2 if one controls sufficiently many L^2 norms of $Z^I u$ and of $Z^I f$. When $n = 3$ such estimates are the core of Klainerman [6]. His estimates are not sharp but sufficient to establish global existence theorems then. However, their analogue for $n = 1$ or $n = 2$ would not give a good estimate for the lifespan of the solutions. We shall therefore reexamine the estimates of [6] for an arbitrary dimension n.

As in Klainerman [6] we assume that $\operatorname{supp} u_j$ is contained in a fixed ball $\{x; |x| \le B\}$. We translate by the distance $2B$ in the time direction, that is, replace (3.1) by

$$\Box u + u = f, \; t \ge 2B; \quad u(2B,.) = u_0, \; \partial_t u(2B,.) = u_1. \tag{3.1$''$}$$

It follows that

$$|x| \le t - B, \quad (t,x) \in \operatorname{supp} u, \tag{3.5}$$

when $t \ge 2B$, as we always assume. This implies that

$$t^2 - |x|^2 \ge B(t + |x|) \ge 2B^2, \quad (t,x) \in \operatorname{supp} u. \tag{3.6}$$

By a suitable rearrangement of the arguments of Klainerman [6] we obtain with $I_k = (2^{k-1}, 2^{k+1})$:

PROPOSITION 3.1. *If u satisfies (3.1)$''$ and (3.6), then the estimate*

$$\sup t^{\frac{n}{2}} |u(t,x)| \le C\Big(\sum_{|I| \le (n+6)/2} \sum_k 2^k \sup_{2B \le \tau \in I_k} \|Z^I f(\tau,.)\| + \sum_{|\alpha|+j \le (n+8)/2} \|D^\alpha u_j\|\Big). \tag{3.7}$$

is valid where Z^I is a product of $|I|$ vector fields of the form (3.4).

In (3.7) the supremum is taken for all $\tau \ge 2B$ in the right hand side. This is acceptable although not convenient when proving global existence theorems for non-linear perturbations. However, it is not adequate in dimensions 1 and 2 where only a finite lifespan can be proved. In that case we want to be able to take the supremum for $t \le s$, say, in both sides. This can cause additional problems when $|x| < |t|/2$, but we can prove:

PROPOSITION 3.2. *If $n \le 3$ and $s \ge 2B$, then*

$$\sup_{2B \le t \le s} t^{\frac{n}{2}} |u(t,x)| \le C\Big(\sum_{|I| \le n+2} \sum_k 2^k \sup_{I_k \cap (2B,s)} \|Z^I f(t,.)\| + \sum_{|\alpha|+j \le n+3} \|D^\alpha u_j\|\Big), \tag{3.8}$$

provided that u satisfies (3.1)$''$ and (3.6). Here Z^I is a product of $|I|$ vector fields of the form (3.4).

There is a similar but weaker result for arbitrary n.

REMARK: When $n \ge 2$ one can modify the energy integral method so that one gets direct control of L^2 norms over the hyperboloids $\varrho =$ constant; one can then replace Propositions 3.1 and 3.2 by the standard Sobolev inequality.

4. Existence theorems. The energy estimates recalled at the beginning of Section 3 also work if the coefficients of $\Box$ are perturbed:

LEMMA 4.1. *Let u be a solution of the perturbed Klein-Gordon equation*

$$\Box u + u + \sum_{j,k=0}^{n} \gamma^{jk}(x)\partial_j\partial_k u = f, \quad 0 \le x_0 < T, \tag{4.1}$$

where $x_0 = t$. If u vanishes for large $|x|$, and if

$$\sum |\gamma^{jk}| \le \tfrac{1}{2},$$

then

$$(\|u'(t,.)\|^2 + \|u(t,.)\|^2)^{\frac{1}{2}} \le 2\Big((\|u'(0,.)\|^2 + \|u(0,.)\|^2)^{\frac{1}{2}} + \int_0^t \|f(s,.)\|\,ds\Big)\exp\Big(\int_0^t 2\Gamma(s)\,ds\Big) \tag{4.2}$$

where

$$\Gamma(s) = \sum \sup |\partial_i\gamma^{jk}(s,.)|.$$

This well known lemma combined with Proposition 3.2 allows one to prove along the lines of Klainerman [5,6] (see also [3]) that if $u_0, u_1 \in C_0^\infty(\mathbf{R}^n)$ then there is some c such that the equation (1.1) has a solution with Cauchy data

$$u(0,.) = \varepsilon u_0, \quad \partial u(0,.)/\partial t = \varepsilon u_1, \tag{4.3}$$

if ε is small and

$$t \le \begin{cases} c/\varepsilon^2, & \text{if } n = 1 \\ e^{c/\varepsilon}, & \text{if } n = 2. \end{cases}$$

However, one can do better by using an approximate solution of the Cauchy problem as in [4]. In fact, we shall outline a proof that the constant c above can in fact be chosen arbitrarily.

THEOREM 4.2. *Assume that the function F in (1.1) is in C^∞, vanishes of second order at 0, and is affine linear in u''. Let the number n of space variables be 1 or 2. Then the equation (1.1) with Cauchy data (4.3) where $u_0, u_1 \in C_0^\infty$ has a C^∞ solution for $0 \le t \le T_\varepsilon$ where $\varepsilon\sqrt{T_\varepsilon} \to \infty$ if $n = 1$, and $\varepsilon \log T_\varepsilon \to \infty$ if $n = 2$, as $\varepsilon \to 0$.*

When $n = 3$ there is a global solution for small ε (Klainerman [6], Shatah [7]), and this remains true for all larger dimensions.

PROOF: As in Section 3 it is more convenient to put the Cauchy boundary condition at $t = 2B$ instead, where B is an upper bound for $|x|$ in $\operatorname{supp} u_0 \cup \operatorname{supp} u_1$. At first we shall just present the arguments of Klainerman [5, 6] to prove the weaker results stated before the theorem. To separate the terms in F involving second order derivatives we write

$$F(u, u', u'') = f(u, u') - \sum \gamma^{jk}(u, u')\partial_j\partial_k u,$$

where f vanishes of second order and γ^{jk} vanishes of first order at the origin. The equation (1.1) can then be written

$$(\Box + 1 + \sum \gamma^{jk}(u, u')\partial_j\partial_k)u = f(u, u'). \tag{4.4}$$

Choose a positive integer $N \ge 6$, and let A be a positive number. We wish to show that if

$$t^{n/2}(|Z^I u(t,x)| + |Z^I u'(t,x)|) \le M\varepsilon, \quad |I| \le N,\ 2B \le t \le T, \tag{4.5}$$

and if

$$\varepsilon \int_{2B}^{T} t^{-n/2}dt \le A, \tag{4.6}$$

then there is strict inequality in (4.5) for small ε provided that M is large and A is small enough. Here Z^I is any product of $|I|$ operators of the form (3.3) or (3.4). Combined with the local existence theorems this will give the lower bound for the lifespan mentioned before the statement of the theorem. Let $s = 2N$ and apply Z^I to (4.4) for all I with $|I| \le s$. This gives

$$(4.7) \qquad (\Box + 1 + \sum \gamma^{jk}(u,u')\partial_j\partial_k)Z^I u = Z^I f(u,u') + \sum [\gamma^{jk}, Z^I]\partial_j\partial_k u + \sum \gamma^{jk}[\partial_j\partial_k, Z^I]u.$$

Set

$$(4.8) \qquad M_s(t) = \sum_{|I|\le s} (\|Z^I u(t,.)\| + \|\partial Z^I u(t,.)\|);$$

since $[\partial_j, Z] = \pm\partial_k$ for some k or is 0, an equivalent norm is obtained if ∂Z^I is replaced by $Z^I\partial$. We wish to estimate the L^2 norm of the right hand side of (4.7) by means of $M_s(t)$, noting that (u,u') is small by (4.5). We can write $[\gamma^{jk}, Z^I]\partial_j\partial_k u$ as a sum of terms of the form

$$-(Z^J\gamma^{jk}(u,u'))Z^K\partial_j\partial_k u,$$

where $|J| + |K| = |I| \le s$, $|J| \ne 0$. Thus $|K| + 1 \le s$, and $Z^J\gamma^{jk}$ can be estimated by a sum of products of the form

$$Z^{J_1}(u,u')\dots Z^{J_r}(u,u'), \quad |J_1| + \dots + |J_r| = |J| \le s.$$

Since $s + 1 < 2N + 2$ we can apply (4.5) to all factors except one, which we estimate using (4.8). We argue similarly for $Z^I f(u,u')$ and $\gamma^{jk}[\partial_j\partial_k, Z^I]u$, regarding $f(u,u')$ as a quadratic form in (u,u') with coefficients depending on (u,u'). The L^2 norm of the right hand side of (4.7) can therefore be estimated by $CM\varepsilon t^{-n/2}M_s(t)$. By Lemma 4.1 it follows that for $t \le T$

$$M_s(t) \le C(\varepsilon + M\varepsilon \int_{2B}^t M_s(\tau)\tau^{-n/2}\,d\tau),$$

hence by Gronwall's lemma and (4.6)

$$M_s(t) \le C\varepsilon e^{CMA}.$$

For $g = (\Box + 1)u$ we also obtain

$$\|Z^I g(t,.)\| \le CM\varepsilon t^{-n/2}M_s(t), \quad |I| \le s - 1,$$

so it follows from Proposition 3.2 that

$$t^{n/2}|Z^I u(t,x)| \le C'MA\varepsilon e^{CMA} + C''\varepsilon, \quad |I| + 4 \le s - 1.$$

Since $s - 5 = 2N - 5 \ge N + 1$, we confirm (4.5) with strict inequality if $M > C''$ and A is small enough. If $N = 7$ we get a maximum norm estimate for one derivative more than needed, and this can be used successively to get bounds for all $\|Z^I u(t,.)\|$ when $t \le T$, without any further decrease of A. In view of the local existence theorem it follows that (4.6) does not hold for the lifespan T_ε of the C^∞ solution of the Cauchy problem, so we have $2\varepsilon T_\varepsilon^{\frac{1}{2}} > A$ if $n = 1$, and $\varepsilon \log T_\varepsilon > A$ if $n = 2$.

To get the stronger result in the theorem we must first estimate not u itself but the deviation of u from an approximate solution. To construct it, let V be the solution of the equation $(\Box + 1)V = 0$ with Cauchy data u_0, u_1 when $t = 2B$. Then $|x| \le t - B$ in $\operatorname{supp} V$, and by Theorem 2.7 we have a decomposition (2.16) of V. Since $U_\pm$ and all derivatives are rapidly decreasing when $|x| > t - 2B$, we can cut them off so that they vanish where $|x| > t - B$, which implies that U_0 vanishes there too. Since $U_0 e^{i\varrho}$ is also in $\mathcal{S}$ we can include it in U_+ so we have in fact

$$V = V_+ e^{i\varrho} + V_- e^{-i\varrho}; \quad |x| \le t - B \text{ if } (t,x) \in \operatorname{supp} V_\pm,$$

where $V_\pm$ are symbols of order $-n/2$ with principal symbols given in Theorem 2.7.
εV has the required Cauchy data and $(\Box + 1)\varepsilon V = 0$, but we only know that

$$F(\varepsilon V, \varepsilon V', \varepsilon V'') = O(\varepsilon^2 t^{-n}).$$

Let F_2 be the quadratic part of the Taylor expansion of F, so that $F - F_2$ vanishes of third order at 0. We can write

$$F_2(V, V', V'') = V_2 e^{2i\varrho} + V_0 + V_{-2} e^{-2i\varrho}$$

where V_j are symbols of degree $-n$ with $|x| \le t - B$ in the support. (We are using here that the quotient by ϱ of a symbol with such support is again a symbol, of one degree lower. This will be used often in what follows without explicit mention.) If G is such a symbol, of degree μ, and $a \ne \pm 1$, then one can easily find another symbol H with the same degree and support such that

$$(\Box + 1)(He^{ia\varrho}) - Ge^{ia\varrho} \in S, \quad \text{for } t \ge 2B.$$

The leading term of H is $G/(1 - a^2)$, which is all that we need to conclude that for

$$w_\varepsilon = \varepsilon(V_+ e^{i\varrho} + V_- e^{-i\varrho}) - \varepsilon^2(V_2 e^{2i\varrho}/3 - V_0 + V_{-2} e^{-2i\varrho}/3) \tag{4.9}$$

we have

$$(\Box + 1)w_\varepsilon = F(w_\varepsilon, w'_\varepsilon, w''_\varepsilon) + R_\varepsilon,$$

where for all I, if Z^I is a product of the operators (3.3), (3.4),

$$|Z^I w_\varepsilon(t,.)| \le C_I \varepsilon t^{-n/2}, \tag{4.10}$$

$$|Z^I R_\varepsilon(t,.)| \le C_I(\varepsilon^3 t^{-3n/2} + \varepsilon^2 t^{-n-1}). \tag{4.11}$$

Now the measure of the support of $w_\varepsilon(t,.)$ and of $R_\varepsilon(t,.)$ is $O(t^n)$, so it follows with some new constants that

$$\|Z^I w_\varepsilon(t,.)\| \le C_I \varepsilon, \quad \|Z^I R_\varepsilon(t,.)\| \le C_I(\varepsilon^3 t^{-n} + \varepsilon^2 t^{-\frac{n}{2}-1}). \tag{4.12}$$

Also note that for all α

$$|\partial^\alpha(u - w_\varepsilon)| \le C_\alpha \varepsilon^2, \quad t = 2B, \tag{4.13}$$

if u is the solution of the Cauchy problem (1.1), (4.3). This is obvious when $\alpha_0 \le 1$ and follows inductively for larger α_0 if we use the equations satisfied by u and by V.
Write $v = u - w_\varepsilon$ and subtract the equation

$$(\Box + 1 + \sum \gamma^{jk}(w_\varepsilon, w'_\varepsilon)\partial_j\partial_k)w_\varepsilon = f(w_\varepsilon, w'_\varepsilon) + R_\varepsilon$$

from (4.4). This gives

$$\begin{aligned}(\Box + 1 + \sum \gamma^{jk}(u, u')\partial_j\partial_k)v = f(u, u') - f(w_\varepsilon, w'_\varepsilon) - R_\varepsilon \\ + \sum(\gamma^{jk}(w_\varepsilon, w'_\varepsilon) - \gamma^{jk}(u, u'))\partial_j\partial_k w_\varepsilon.\end{aligned} \tag{4.14}$$

Set

$$N_s(t) = \sum_{|I| \le s}(\|Z^I v(t,.)\| + \|\partial Z^I v(t,.)\|) + \sum_{|I| \le s-6} t^{n/2} \sup(|Z^I v(t,.)| + |\partial Z^I v(t,.)|), \tag{4.15}$$

and assume as in the first part of the proof that

$$s + 1 < 2(s - 5), \text{ that is, } s \ge 12. \tag{4.16}$$

When estimating $N_s(t)$ we shall assume that

$$N_s(t) \le \varepsilon, \quad t \le T, \tag{4.17}$$

and confirm afterwards that this is true with ε replaced by $\varepsilon/2$ if ε is small, which makes the hypothesis harmless.

Application of Z^I to (4.14) gives us even more terms than in (4.7),

$$\begin{gathered}(\square + 1 + \sum \gamma^{jk}(u,u')\partial_j\partial_k)Z^I v = \sum_1^5 g_j,\\ g_1 = -Z^I R_\varepsilon, \quad g_2 = Z^I(f(u,u') - f(w_\varepsilon, w_\varepsilon')),\\ g_3 = \sum Z^I(\gamma^{jk}(w_\varepsilon, w_\varepsilon') - \gamma^{jk}(u,u'))\partial_j\partial_k w_\varepsilon,\\ g_4 = \sum[\gamma^{jk}(u,u'), Z^I]\partial_j\partial_k v, \quad g_5 = \sum \gamma^{jk}(u,u')[\partial_j\partial_k, Z^I]v.\end{gathered} \tag{4.18}$$

Using (4.10), (4.12), (4.17) in (4.18) we obtain

$$\|g_1(t,.)\| \le C(\varepsilon^3 t^{-n} + \varepsilon^2 t^{-\frac{n}{2}-1}), \quad \|g_j(t,.)\| \le C\varepsilon t^{-\frac{n}{2}} N_s(t), \text{ if } j > 1. \tag{4.19}$$

In view of (4.13) we conclude that

$$\|Z^I v(t,.)\| \le C(\varepsilon^2 + \varepsilon \int_{2B}^t \tau^{-\frac{n}{2}} N_s(\tau)\, d\tau), \quad |I| \le s.$$

By Gronwall's lemma it follows that

$$N_s(t) \le C\varepsilon^2 \exp(CA), \quad 2B \le t \le T \tag{4.20}$$

if (4.6) holds. The estimate (4.17) with ε replaced by $\varepsilon/2$ is a consequence of (4.20) for small enough ε, no matter how large A is. Starting from $s = 13$ we can now derive estimates for higher derivatives of u when $t \le T$ as in the first part of the proof. By the local existence theorem it follows that (4.6) is not true for the life span T_ε. Since A is arbitrary now, the theorem is proved.

We have in fact also proved that locally uniformly in $t \in (0,\infty)$ we have for the solution u_ε of the Cauchy problem (1.1), (4.3)

$$u_\varepsilon(t/\varepsilon^2, x)/\varepsilon - V(t/\varepsilon^2, x) = O(\varepsilon(t/\varepsilon^2)^{-\frac{n}{2}})$$

as $\varepsilon \to 0$, if $n = 1$. When $n = 2$ we have a similar result with t/ε^2 replaced by $e^{t/\varepsilon}$. Thus the nonlinearity is not felt much during the time for which we have proved the existence of the solution. This suggests strongly that the lifespan is actually much longer than stated in Theorem 4.2. We shall discuss this question further in Section 5 without giving a definite answer though.

If the perturbation F vanishes of third order at the origin then another factor $\varepsilon t^{-\frac{n}{2}}$ appears in the estimates made in the first part of the proof. This gives global existence for small ε when $n = 2$, since t^{-n} is integrable in $(2B, \infty)$ then. When $n = 1$ we get existence for $\varepsilon^2 \log t \le c$ for some $c > 0$.

5. Remarks and questions. We shall begin with discussing the ordinary differential equation which is the analogue of the non-linear Klein-Gordon equation with no space variables present,

$$u'' + u = f(u,u'); \quad u(0) = \varepsilon u_0, \ u'(0) = \varepsilon u_1; \tag{5.1}$$

where $f \in C^\infty$ vanishes of second order at 0 and $u_0^2 + u_1^2 = E_0$ is independent of ε. Since the solution of the unperturbed equation does not decay at infinity one might expect that the life span T_ε should only be of the order $1/\varepsilon$ in general. However, it is much longer than that. Set

$$8\gamma = -(\partial_1^3 + \partial_0^2\partial_1)f(u_0,u_1) - \partial_1\partial_0 f(u_0,u_1)(\partial_0^2 + \partial_1^2)f(u_0,u_1)\Big|_{u_0=u_1=0}. \tag{5.2}$$

THEOREM 5.1. *If T_ε is the life span of the solution u_ε of (5.1), we have*

$$\lim_{\varepsilon\to 0} \varepsilon^2 T_\varepsilon = \infty, \ \text{if}\ \gamma \geq 0, \quad (-\gamma E_0) \varliminf_{\varepsilon\to 0} \varepsilon^2 T_\varepsilon \geq 1 \ \text{if}\ \gamma < 0.$$

Moreover,

$$(u_\varepsilon(t/\varepsilon^2)^2 + u'_\varepsilon(t/\varepsilon^2)^2)/\varepsilon^2 \to E_0/(1+E_0\gamma t) \tag{5.3}$$

uniformly on $[0, t_0]$ *if* $1 + E_0\gamma t_0 > 0$.

For the proof one modifies the energy so that the quadratic terms in f are taken into account. Maybe one should do so also for the genuine Klein-Gordon equation in order to get a correct conclusion on the lifespan of the solution. For the case of several variables this is not as easy. However, if one simplifies the equation by keeping only the radial derivatives in a polar hyperbolic coordinate system, it can be done. This may justify question if in fact the Cauchy problem (1.1), (4.3) has global solutions for small ε when $n = 2$, and if $\varliminf_{\varepsilon\to 0} \varepsilon^2 \log T_\varepsilon$is always positive when $n = 1$. Is there even a global solution when $n = 1$ and $f(u, u') = u^2$?

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