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CR MAPPINGS AND THEIR HOLOLOMORPHIC EXTENSION

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If M is a smooth manifold of real dimension $2n+1$, we say that M is a *CR manifold* of codimension one with *CR bundle* $\mathcal{V}$, if $\mathcal{V}$ is a subbundle of $\mathbb{C}TM$, the complexified tangent bundle of M, satisfying

$$dim_{\mathbb{C}}\mathcal{V} = n, \qquad \mathcal{V} \cap \overline{\mathcal{V}} = 0.$$

Any smooth real hypersurface M in $\mathbb{C}^{n+1}$ is a *CR* manifold of codimension one, where $\mathcal{V}$ is the subbundle of antiholomorphic tangent vectors to M.

Let $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ be two *CR* manifolds of codimension one. A smooth mapping from M into M' is called CR if for all $p \in M$

$$H'(\mathcal{V}_p) \subset \mathcal{V}'_{H(p)}.$$

We recall the following definition introduced in Baouendi-Jacobowitz-Treves [3]. If M is a real analytic hypersurface in $\mathbb{C}^{n+1}$ containing the origin and defined locally by $\rho(z, \overline{z}) = 0$, $d\rho \neq 0$, we say that M is essentially finite at 0 if for any sufficiently small $z \in \mathbb{C}^{n+1} \setminus \{0\}$, there exists an arbitrarily small $\varsigma \in \mathbb{C}^{n+1}$ satisfying: $\rho(z, \varsigma) \neq 0$, $\rho(0, \varsigma) = 0$.

Our main result is the following:

THEOREM 1. *Let M and M' be real analytic hypersurfaces in $\mathbb{C}^{n+1}$ and $H : M \to M'$ a smooth CR mapping, defined near $p_0 \in M$ with $H(p_0) = p'_0$, and satisfying*

$$(1) \qquad H'(\mathbb{C}T_{p_0}M) \not\subset \mathcal{V}'_{p'_0} \oplus \overline{\mathcal{V}}'_{p'_0},$$

where $\mathcal{V}'$ is the CR bundle of M'. If M and M' are essentially finite at p_0 and p'_0 respectively then H extends as a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$.

Theorem 1 was first proved for $n = 1$ by S. Bell and the authors (see [1], [2]). It generalizes the result in the diffeomorphic case proved in [3]. We refer to the references of [2] and [3] for earlier works on holomorphic extendibility of CR mappings under stronger conditions.

The following is a key ingredient in the proof of Theorem 1. If j is a smooth CR function defined on M then there exists a unique formal (holomorphic) power series $J(z) = \sum a_\alpha z^\alpha$, $a_\alpha \in \mathbb{C}$, such that, if $U \ni u \mapsto Z(u) \in \mathbb{C}^{n+1}$ ($U \subset \mathbb{R}^{2n+1}, Z(0) = 0$) is a parametrization of M, then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. On the other hand it is clear that a CR mapping between two hypersurfaces M and M' in $\mathbb{C}^{n+1}$ is given by $(n+1)$ CR functions $(j_1, \ldots, j_{n+1})$. Such a mapping is called of *finite multiplicity* at 0 if

$$dim_{\mathbb{C}} \mathcal{O}[[Z]]/(J(Z)) < \infty,$$

where $\mathcal{O}[[Z]]$ is the ring of formal power series in $(n+1)$ indeterminates and $(J(Z))$ is the ideal generated by $(J_1(Z), \ldots, J_{n+1}(Z))$. Here the dimension is taken in the sense of vector spaces. We have the following:

THEOREM 2. *If M and M' are essentially finite at p_0 and p'_0 respectively then a CR mapping $H : M \to M'$ is of finite multiplicity at p_0 if and only if condition (1) of Theorem 1 holds.*

We may restate Condition (1) in terms of local coordinates. We may assume $p'_0 = H(p_0) = 0$ and M and M' are given locally by

(2) $$Im\ w = \varphi(z, \bar{z}, Re\ w), \qquad Im\ w = \psi(z, \bar{z}, Re\ w)$$

with $\varphi(z, 0, Re\ w) = \psi(z, 0, Re\ w) = 0$; $z \in \mathbb{C}^n$, $w \in \mathbb{C}$. The map H is then given by $n + 1$ CR functions $(f_1, \ldots, f_n, g) = (f, g)$ defined on M. Therefore we have

(3) $$\frac{g - \bar{g}}{2i} = \psi(f, \bar{f}, \frac{g + \bar{g}}{2}).$$

With this notation Condition (1) is equivalent to

$$(4) \qquad \frac{\partial g}{\partial s}(0) \neq 0,$$

with $s = \operatorname{Re} w$. (Here f_j and g are considered as smooth functions of z, $\bar{z}$, s).

Using Theorem 1 as well as Diederich-Fornaess [5], [6], Fornaess [7] and Bell-Catlin [4], we obtain the following

THEOREM 3. *Let D and D' be two bounded pseudoconvex domains in $\mathbb{C}^{n+1}$ with real analytic boundaries and $H : D \to D'$ a proper, holomorphic mapping. Then H extends holomorphically to a neighborhood of $\overline{D}$, the closure of D.*

We give here an outline of the proof of Theorem 1. By solving (3) for $\bar{g}$ we obtain a holomorphic function Q

$$(5) \qquad \bar{g} = Q(f, \overline{f}, g).$$

As in [3] by writing

$$Q(f, \lambda, g) = \sum Q_{\varsigma^\alpha}(f, \overline{f}, g) \frac{(\lambda - \overline{f})^\alpha}{\alpha!}$$

we are reduced to showing that for $z_0 \in \mathbb{C}^n$ fixed, $|z_0| < r$,

$$Q_{\varsigma^\alpha}\left(f(z_0, \bar{z}_0, s), \overline{f}(z_0, \bar{z}_0, s), g(z_0, \bar{z}_0, s)\right)$$

extends as a holomorphic function in $s + it$, $|s| < r$, $-R < t < 0$, for some r, R positive, and satisfies

$$(6) \qquad |Q_{\varsigma^\alpha}| \leq C^{\alpha+1}\alpha!, \qquad C > 0.$$

The main ingredients used in proving the above are the following.

LEMMA 1. *If j is a smooth CR function defined on M then the Taylor series of j in the coordinates (z, s) is given uniquely by*

$$(7) \qquad j \sim \sum a_{\alpha k} z^\alpha w^k \big|_{w=s+i\varphi(s,\bar{s},s)}, \qquad a_{\alpha k} \in \mathbb{C}.$$

A basis for the CR vector fields on M is given by

$$(8) \qquad L_j = \frac{\partial}{\partial \bar{z}_j} - i \frac{\varphi_{\bar{z}_j}}{1 + i\varphi_s} \frac{\partial}{\partial s}, \qquad 1 \le j \le n,$$

LEMMA 2. *If $j(z, \bar{z}, s)$ is a CR function on M, then for all multi-indices α*

$$\overline{L}^\alpha j(0) = \left(\frac{\partial}{\partial z}\right)^\alpha J(0, 0),$$

where $J(z, w) \sim \sum a_{\alpha k} z^\alpha w^k$ is as defined in Lemma 1.

Using the Nullstellensatz we may prove the following.

LEMMA 3. *For $j = 1, \ldots, n$ let $F_j(z, w)$ be the formal power series associated to f_j as in Lemma 1. Let I be the ideal generated by $F_j(z, 0)$, $1 \le j \le n$, the ring $\mathcal{O}[[Z]]$ of formal power series in the indeterminates $z_1, \ldots, z_n$. Then under the assumptions of Theorem 1,*

$$(9) \qquad \dim_{\mathbb{C}} \mathcal{O}[[z]]/I < \infty,$$

and therefore

$$(10) \qquad \det\left(\frac{\partial F_k}{\partial z_j}(z, 0)\right) \not\equiv 0.$$

An immediate consequence of Lemmas 2 and 3 is that there exists a multi-index α such that

$$(11) \qquad \overline{L}^\alpha (\det(\overline{L}_j f_k))(0) \neq 0.$$

LEMMA 4. *For every multi-index α and every z_0, $|z_0| < r$ there exist functions $a(s)$, $b(s)$ holomorphic in the domain $\mathcal{R} = \{s + it; |s| < r, -R < t < 0\}$, smooth in $\overline{\mathcal{R}}$ such that*

$$Q_{\varsigma^\alpha}(f, \overline{f}, g)(z_0, s) = \frac{a(s)}{b(s)}.$$

Lemma 4 is proved by applying successively $\overline{L}^\beta$ to (5) and using (11).

LEMMA 5. *For each* j, $1 \leq j \leq n$, f_j *satisfies a polynomial equation of the form*

$$f_j^{N_j} + a_{N_j-1}^j f_j^{N_j-1} + \cdots + a_0^j = 0,$$

where $a_k^j = a_k^j(L^\gamma \bar{f}, L^\gamma g)$ *is a holomorphic function of the* $L^\gamma \bar{f}$, $L^\gamma g$, *for* $|\gamma| \leq \gamma_0$.

The proof of Lemma 5 uses Lemma 3, as well as repeated applications of the Weierstrass Preparation theorem and the Nullstellensatz.

LEMMA 6. *There exists* N *such that for each multi-index* α, $Q_{\varsigma^\alpha}(f, \bar{f}, g)(z, \bar{z}, s)$ *is a root of a polynomial of the form*

$$(12) \qquad\qquad X^N + b_{N-1}^\alpha X^{N-1} + \cdots + b_0^\alpha = 0$$

where the b_k^α *are holomorphic functions of* $L^\gamma \bar{f}$ *and* $L^\gamma g$, $|\gamma| \leq \gamma_0$, *and satisfies*

$$(13) \qquad\qquad |b_j^\alpha(L^\gamma \bar{f}, L^\gamma g)| \leq (C^{\alpha+1}|\alpha|!)^{N-j}$$

at $(z, \bar{z}, s + it)$ *for* $|z| < r$, $|s| < r$ *and* $-R \leq t \leq 0$.

From Lemmas 4 and 6 it follows, using the Lemma in [2], that each $Q_{\varsigma^\alpha}(f, \bar{f}, g)$ extends holomorphically to $\mathcal{R}$. Finally, the estimate (6) follows from (13).

For higher codimension, a slight modification of the proof of Theorem 1 yields the following.

THEOREM 4. *Let* M *and* M' *be real analytic generic* CR *submanifolds of real codimensional* ℓ *in* $\mathbb{C}^{n+\ell}$ *and* $H : M \to M'$ *a smooth* CR *mapping defined near* $p_0 \in M$, $H(p_0) = p_0'$, *and satisfying*

$$(14) \qquad\qquad \dim_{\mathbb{C}}(H'(\mathbb{C}T_{p_0}M)/\mathcal{V}_{p_0}' \oplus \bar{\mathcal{V}}_{p_0'}') = \ell$$

where $\mathcal{V}'$ *is the* CR *bundle of* M'. *Assume that* M *and* M' *are essentially finite at* p_0, *and that near* p_0, H *extends holomorphically to a wedge of edge* M. *Then* H *extends as a holomorphic mapping from a neighborhood of* p_0 *in* $\mathbb{C}^{n+\ell}$ *to* $\mathbb{C}^{n+\ell}$.

Complete details of the proofs will appear elsewhere.

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