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Some problems in inverse scattering theory


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We shall consider the Schrödinger operator $H_v = -\Delta + v(x)$ in $\mathbb{R}^n$, where $n = 3, 5, \ldots$. We assume that $v \in \mathcal{V},$ i.e.

$$\int (1 + |x|)^{\alpha - (n-2)} |v^{(\alpha)}(x)| dx < \infty$$

for any $\alpha$.

Some of the main problems we consider are the following:

(a) Analysis of bound states and poles of the scattering matrix.

(b) Backward scattering.

(c) The characterization problem for scattering matrices.

This talk will be a continuation of the authors lecture at École Polytechnique [6], and we shall mainly give some comments to (a).

We shall study families of intertwining operators $A$ such that

$$H_v A = AH_0$$

or equivalently

$$(\Delta_x - \Delta_y - v(x)) A(x, y) = 0.$$

(We shall always identify operators with their distribution kernels.) Let $\mathcal{M}$ be the set of all $U(x,y) \in L^1_{loc}$ such that

$$\|U\|_{\mathcal{M}} = \max \left\{ \sup_{x} \int |U(x,y)| dy, \sup_{y} \int |U(x,y)| dx \right\} < \infty.$$

Then $\|U\|_{L^1_{loc}} \leq \|U\|_{\mathcal{M}}$ for $1 \leq p \leq \infty$ if $U \in \mathcal{M}$. We let $\mathcal{M}_\theta$ be the subspace of $\mathcal{M}$ consisting of $U$ such that $(y-x,\theta) \geq 0$ in its support. Here $\theta \in S^{n-1}$ and $\mathcal{M}_{\theta,\lambda}$ is the set of $U$ in $\mathcal{M}_\theta$ such that

$$e^{-\lambda(y-x,\theta)} U(x,y) \in \mathcal{M}_\theta.$$

The spaces $\mathcal{M}$, $\mathcal{M}_\theta$ and $\mathcal{M}_{\theta,\lambda}$ are Banach algebras. Finally $\mathcal{M}_{\theta,\lambda}$ is defined by the following conditions:

$$\int |U(x,y)| dy \to 0 \text{ as } |x| \to \infty, \ x/|x| \to \theta$$

and

$$\int |U(x,y)| dx \to 0 \text{ as } |y| \to \infty, \ y/|y| \to -\theta.$$
THEOREM 1. Let $v \in \mathcal{V}$ be real valued and $\theta \in S^{n-1}$. Then there is a unique $A_{\theta} \in \bigcup_{\lambda \geq 0} I + M_{\theta,\lambda}$ such that $H_v A = AH_0$. Moreover, $A_{\ast,\theta} \circ A_{\theta} = I$.

The distribution $A_{\theta}$ is constructed as the infinite sum $\sum_{n=0}^{\infty} U_n$, where $U_0(x,y) = \delta(x - y)$, and

$$U_{n+1} = E_{\theta} * (vU_n),$$

Here $(vU_n)(x,y) = v(x)U_n(x,y)$, and $E_{\theta}$ is the fundamental solution for $\Delta_x - \Delta_y$, which is uniquely determined from the following conditions:

(i) $\langle y - x, \theta \rangle \geq 0$ in the support of $E_{\theta}$,
(ii) $E_{\theta}(x + t\theta, y + t\theta) \to 0$ in $D'(R^n \times R^n)$ as $|t| \to \infty$.
(iii) $E_{\theta} = \sum c_{\alpha,\beta} \partial_\alpha \partial_\beta h_{\alpha,\beta}$, where $\phi(x - y)h_{\alpha,\beta}(x,y) \in M$ for any $\phi \in C_0$.

THEOREM 2. There exists a family of $L^1$ functions $q_{\theta}$ in $R^n$ which depend continuously on $\theta$ and are supported in the set where $(x,\theta) \leq 0$, such that

$$A_{\theta}(I - [q_{\theta}]) \in I + M_{\theta}.$$

COROLLARY 3. Assume that $v \in C^\infty_0$. Then the scattering matrix $S_k(\theta,\theta')$ is analytic in the upper half-plane $\Re k \geq 0$ after multiplication by $1 - \overline{q_{\theta'}}(-k)$.

Sketch of proof. One first constructs $B_{\theta} \in I + M_{\theta,0}$ so that

$$B_{\theta}^{-1} H_v B_{\theta} = H_0 + \sum_{i=1}^{N} f_i \otimes g_i,$$

where $f_i$ and $g_i$ are in $L^1$ together with all their derivatives.

Next one defines the $L^1$ functions $q_{i,k}$ by the formula

$$q_{i,k}(y) = \int (f_i \ast g_k)(x)E_{\theta}(x,y) \, dx.$$

Set $[Q] = [q_{i,k}]$, where the right-hand side is considered as a $N \times N$ matrix of convolution operators, and define the vector valued function $\tilde{h} = (h_1, \ldots, h_N)$ by the equation

$$\tilde{h} = (I - [Q])\tilde{g},$$

where $^\infty (I - [Q])$ denotes the co-factor matrix of $I - [Q]$. We can now define the $L^1$ function $q = q_{\theta}$ by the equation

$$\det(I - [Q]) = I - [q].$$

It is easy to see that $\langle x, \theta \rangle \leq 0$ in the support of $q_{\theta}$. Set

$$C_{\theta} = I - [q_{\theta}] + F_{\theta},$$

where $F_{\theta} = \sum_{i=1}^{N} E_{\theta} \ast (f_i \otimes h_i)$. Then $H_v (B_{\theta} C_{\theta}) = (B_{\theta} C_{\theta}) H_0$. Therefore, if we set

$$R(x,y) = A_{\theta}^{-1} B_{\theta} C_{\theta} - \delta(x - y),$$

then $(\Delta_x - \Delta_y)R = 0$ and $\langle y - x, \theta \rangle \geq 0$ in its support. From a uniqueness result for $\Delta_x - \Delta_y$ one then finds that $R$ is constant in the direction of $(\theta,\theta)$, i.e. $R(x + t\theta, y + t\theta) = R(x,y)$ when $t$ is any real number. Since $R + [q] \in M_{\theta,\lambda}$ we conclude that $R + [q] = 0$. Hence

$$A_{\theta}(I - [q_{\theta}]) = B_{\theta} C_{\theta} \in I + \tilde{M}_{\theta,0},$$

and the proof is complete.
REFERENCES