

PIERRE SCHAPIRA

Propagation of analytic singularities up to non smooth boundary

Journées Équations aux dérivées partielles (1987), p. 1-9

http://www.numdam.org/item?id=JEDP_1987____A7_0

© Journées Équations aux dérivées partielles, 1987, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Colloque E.D.P.
 Saint-Jean de Monts
 JUIN 1987

PROPAGATION OF ANALYTIC SINGULARITIES
UP TO NON SMOOTH BOUNDARY

Pierre SCHAPIRA

1.- Propagation for sheaves

We shall follow the notations of [K-S 1]. In particular if X is a real manifold, we denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology, and if $F \in D^b(X)$ we denote by $SS(F)$ its microsupport. Recall that $SS(F)$ is a closed conic involutive subset of T^*X . We shall also make use of the bifunctor μhom , from $D^b(X)^0 \times D^b(X)$ to $D^b(T^*X)$, a slight generalization of the functor of Sato's microlocalization.

Let h be a real C^2 -function defined on an open subset U of T^*X , H_h its hamiltonian vector field. If $(x; \xi)$ is a system of homogeneous symplectic coordinates, with $\omega_X = \sum_j \xi_j dx_j$, then :

$$(1.1) \quad H_h = \sum_j \left(\frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) .$$

If $p \in U$ we denote by b_p^+ the positive half integral curve of H_h issued at p . We define similarly b_p^- and $b_p = b_p^- \cup b_p^+$. We also set for $* = 0, +, -$:

$$(1.2) \quad V_* = \{p \in U ; h(p) \geq 0 \quad (* = +) \quad \text{or} \\ h(p) \leq 0 \quad (* = -) \quad \text{or} \quad h(p) = 0 \quad (* = 0)\}.$$

The following result is easily deduced from [K-S 1, Th. 5.2.1].

Theorem 1.1. Let F and G belong to $D^b(X)$ with

$SS(G) \cap U \subset V_-$, $SS(F) \cap U \subset V_+$. Let $j \in \mathbb{Z}$ and let u be a section of $H^j(\mu\text{hom}(G,F))$ on U . Then $p \in \text{supp}(u)$ implies $b_p^+ \subset \text{supp}(u)$.

(Remark that $\text{supp}(u)$ is contained in V_0).

2.- Wave front sets at the boundary [S 1]

Let M be a real analytic manifold of dimension n , X a complexification of M , Ω an open subset of M . We introduce :

$$(2.1) \quad C_{\Omega|X} = \mu\text{hom}(\mathbb{Z}_{\Omega}, \theta_X) \otimes \omega_{M/X}[n]$$

where $\omega_{M/X}$ is the relative orientation sheaf.

Let π denote the projection $T^*X \longrightarrow X$, and let

$B_M = R\Gamma_M(\theta_X) \otimes \omega_{M/X}[n]$ denote the sheaf of Sato's hyperfunctions on M . There is a natural isomorphism :

$$(2.2) \quad \alpha : \Gamma_{\Omega}(B_M) \xrightarrow{\sim} \pi_* H^0(C_{\Omega|X}) .$$

Hence a hyperfunction u on Ω defines a section $\alpha(u)$ of $H^0(C_{\Omega|X})$ all over T^*X . We set :

$$(2.3) \quad SS_{\Omega}(u) = \text{supp}(\alpha(u)) .$$

Since $H^0(C_{\Omega|X})$ is supported by the conormal bundle T_M^*X , $SS_{\Omega}(u)$ is a closed conic subset of T_M^*X . It coincides with the classical analytical wave front set above Ω , but it may be strictly larger than its closure in T_M^*X (cf. [S 1]).

Now let P be a differential operator defined on X , and assume for simplicity that the principal symbol $\sigma(P)$ never vanishes identically. Let \mathcal{O}_X^P denote the sheaf of holomorphic solutions of the equation $Pf = 0$. Replacing \mathcal{O}_X by \mathcal{O}_X^P in the preceding discussion, we define :

$$(2.4) \quad C_{\Omega|X}^P = \mu_{\text{hom}}(\mathbb{Z}_{\Omega}, \mathcal{O}_X^P) \otimes \omega_{M/X} [n] .$$

Let B_M^P denote the sheaf of hyperfunction solutions of the equation $Pu = 0$. There is a natural isomorphism :

$$(2.5) \quad \alpha : \Gamma_{\Omega}(B_M^P) \xrightarrow{\sim} \pi_* H^0(C_{\Omega|X}^P) .$$

If u is a hyperfunction on Ω solution of the equation $Pu = 0$, we set :

$$(2.6) \quad SS_{\Omega}^P(u) = \text{supp}(\alpha(u)) .$$

Remark that

$$(2.7) \quad SS_{\Omega}^P(u) \subset SS(\mathbb{Z}_{\Omega}) \cap \text{char}(P)$$

(where $\text{char}(P) = \sigma(P)^{-1}(0)$), but in general $SS_{\Omega}^P(u)$ is no more contained in T_M^*X .

I don't know if $SS_{\Omega}^P(u) \cap T_M^*X = SS_{\Omega}(u)$, but this is true when $M \setminus \Omega$ is convex (locally, up to analytic diffeomorphisms).

Of course the preceding discussion extends to solutions of general systems of differential equations (cf. [S 1]).

Now assume $\partial\Omega = N$ is a real analytic hypersurface and let Y be a complexification of N in X . Assume P of order m , Y is non characteristic for P , and a normal vector field to N in M is given, so that the induced system $(D_X/D_X P)_Y$ is isomorphic to D_Y^m ; (as usual, D_X denotes the ring of differential operators).

Let ρ and $\bar{\omega}$ denote the natural maps associated to $Y \longrightarrow X$:

$$(2.8) \quad T^*Y \xleftarrow[\rho]{Y \times T^*X} T^*X \xrightarrow[\bar{\omega}]{} T^*X .$$

Let $u \in \Gamma(\Omega; B_M^P)$ be a hyperfunction on Ω solution of $Pu = 0$, and let $b(u) \in \Gamma(N; B_N^m)$ be its traces. Recall (cf. [S 1], [S 2]) :

Theorem 2.1. In the preceding situation, one has :

$$SS_N(b(u)) = \rho \bar{\omega}^{-1} SS_\Omega^P(u) .$$

In other words, the analytic wave front set of $b(u)$ is exactly the projection of $SS_\Omega^P(u)$. Remark that if $\text{char}(P) \cap SS(\mathbb{Z}_\Omega)$ is contained in T_M^*X , $SS_\Omega^P(u)$ may be replaced by $SS_\Omega(u)$ in Theorem 2.1.

Remark moreover that $b(u)$ does not make sense when $\partial\Omega$ is not smooth, but $SS_\Omega(u)$ always does.

3.- Transversal propagation for non smooth boundaries

Let M be a real analytic manifold, X a complexification of M , Ω an open subset of M .

If $x \in M$, the cone $N_x(\Omega)$ is defined in [K-S 1]. Recall that $N_x(\Omega)$ is an open convex cone of T_x^*M , and $\theta \in N_x(\Omega)$, $\theta \neq 0$ implies that there exists a convex open cone γ (in a system of local coordinates around x) such that $\theta \in \gamma$ and $\Omega + \gamma \subset \Omega$.

We shall have to consider the real underlying structure of T^*X . Recall that if ω_x is the complex canonical 1-form on T^*X , this real symplectic structure is defined by $2\text{Re } \omega_x$.

If h is a real C^2 -function on T^*X , we denote by $H_h^{\mathbb{R}}$ its real Hamiltonian vector field.

If $(z; \zeta)$ is a system of homogeneous holomorphic symplectic coordinates on T^*X , such that $\omega_x = \sum_j \zeta_j dz_j$, and $z = x + iy$, $\zeta = \xi + i\eta$, then

$$(3.1) \quad H_h^{\mathbb{R}} = \sum_j \left(\frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} + \frac{\partial h}{\partial y_j} \frac{\partial}{\partial \eta_j} - \frac{\partial h}{\partial \eta_j} \frac{\partial}{\partial y_j} \right).$$

Now let P be a differential operator on X , u a hyperfunction on Ω , solution of the equation $Pu = 0$. Let $p \in T_M^*X$, $x_0 = \pi(p)$.

Theorem 3.1. Assume :

- a) $\text{Im } \sigma(P) \Big|_{T_M^*X} = 0$
- b) $\pi(H_{\text{Im } \sigma(P)}^{\mathbb{R}}(p)) \in N_{x_0}(\Omega)$.

Let b_p^+ be the positive half integral curve of $H_{\text{Im } \sigma(P)}^{\mathbb{R}}$ issued at p . Then $p \in \text{SS}_\Omega(u)$ implies $b_p^+ \subset \text{SS}_\Omega(u)$.

Proof

We may assume X is open in \mathbb{C}^n and $M = X \cap \mathbb{R}^n$. Then there exists a convex open cone γ such that $\Omega + \gamma \subset \Omega$ (in a neighborhood of x_0) and $\pi(H_{\text{Im } \sigma(P)}^{\mathbb{R}}(p)) \in \gamma$. This last condition implies :

$$\langle d_{\xi} \text{Im } \sigma(P)(x, i\eta), \xi \rangle \geq c|\xi|$$

for some $c > 0$, and all $\xi \in \gamma^0$ (γ^0 is the polar set to γ).

Hence :

$$(3.2) \quad \text{Im } \sigma(P)(x, \xi + i\eta) \leq 0$$

for $(x, \xi + i\eta)$ in a neighborhood of p , $\xi \in \gamma^{\text{oa}}$, where $\gamma^{\text{oa}} = -\gamma^0$.

Since $\Omega + \gamma \subset \Omega$, we have (cf. [K-S 1]) :

$$SS(\mathbb{Z}_{\Omega}) \subset T_M^*X + \gamma^{\text{oa}}.$$

Thus :

$$(3.3) \quad \text{Im } \sigma(P) \leq 0 \quad \text{on } SS(\mathbb{Z}_{\Omega})$$

in a neighborhood of p .

Now let $u \in \Gamma(\Omega; B_M)$ be a solution of the equation $Pu = 0$.

Then u defines a section $\alpha(u) \in \Gamma(T^*X; H^n(\mu\text{hom}(\mathbb{Z}_{\Omega}, \mathcal{O}_X^P)))$ and $p \in SS_{\Omega}(u)$ implies $p \in SS_{\Omega}^P(u)$, that is, $p \in \text{supp}(\alpha(u))$.

Since $SS(\mathcal{O}_X^P) = \text{char}(P) \subset \{\text{Im } \sigma(P) = 0\}$, we may apply Theorem 1.1 and we obtain :

$$b_p^+ \subset SS_{\Omega}^P(u).$$

But $b_p^+ \setminus \{p\}$ is contained in $\pi^{-1}(\Omega)$ and

$SS_{\Omega}^P(u) = SS_{\Omega}(u) = SS_M(u)$ above Ω . Thus $b_p^+ \subset SS_{\Omega}(u)$,

which is the desired result.

4.- Diffraction

We keep the notations of §3, but we assume :

$$(4.1) \quad \Omega = \{x \in M ; x_1 > 0\}$$

$$(4.2) \quad \sigma(P) = \zeta_1^2 - g(z, \zeta')$$

where $z = (z_1, z')$, $\zeta = (\zeta_1, \zeta')$.

Moreover we assume :

$$(4.3) \text{ a) } \frac{\partial}{\partial x_1} g < 0 \quad \text{at } p \quad \text{or} \quad \text{b) } \frac{\partial}{\partial x_1} g \equiv 0 .$$

Theorem 4.1. Under these hypotheses, if $p \in SS_\Omega(u)$ then b_p^+
 or b_p^- is contained in $SS_\Omega(u)$, in a neighborhood of p .

The idea of the proof is the following.

If $\zeta_1 \neq 0$ at p , the result is a particular case of Theorem 3.1 . Otherwise define for $* = 0, 1, -$:

$$\Omega_* = \{z \in X ; x_1 > 0, y' = 0, y_1 \in \mathbb{R} (* = 0) \\ \text{or } y_1 \geq 0 (* = +) \text{ or } y_1 \leq 0 (+ = -)\}$$

Thus $\text{Im } \sigma(P)$ is negative (resp. positive) on $SS(\mathbb{Z}_{\Omega^+})$
 (resp. $SS(\mathbb{Z}_{\Omega^-})$) in a neighborhood of p . Then one can apply Theorem 1.1 to $\mu_{\text{hom}}(\mathbb{Z}_{\Omega_*}, O_X^P)$, $* = +$ or $-$, and one obtain that if $u|_{b_p}$ has compact support, then $u \in H^{n-1}(\mu_{\text{hom}}(\mathbb{Z}_{\Omega_*}, O_X^P))$, and it is not difficult to conclude using the holomorphic parameter z_1 (cf. [S 2]) .

Remark that Theorem 4.1 has been first obtained by Kataoka [Ka] (under hypothesis (4.3) a)) then refined by G. Lebeau [Le] .

An application : Let (x_1, \dots, x_n) be the coordinates on \mathbb{R}^n , and let Ω_1 and Ω_2 be two open half spaces. Set $\Omega = \Omega_1 \cup \Omega_2$ and let u be a hyperfunction on Ω . One can easily prove :

$$(4.4) \quad SS_{\Omega}(u) = SS_{\Omega_1}(u) \cup SS_{\Omega_2}(u) .$$

Now assume $\Omega_i = \mathbb{R} \times \Omega'_i$, ($i = 1, 2$) and u satisfies the wave equation $Pu = 0$, where $P = D_1^2 - \sum_{j=2}^n D_j^2$.

Applying Theorem 4.1 we get that $p \in SS_{\Omega}(u) \implies b_p^+$ or b_p^- is contained in $SS_{\Omega}(u)$, where b_p^+ and b_p^- are the half bicharacteristic curves of $\text{Im } \sigma(P)$.

Problem : to extend this result to the case where

$\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$, and A is any convex closed subset of \mathbb{R}^{n-1} .

Remark that if $\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$, where A is polyedral, and if $p \in SS_{\Omega}(u)$, $b_p^+ \setminus \{p\} \subset \pi^{-1}(\Omega)$ then $b_p^+ \subset SS_{\Omega}(u)$, in view of Theorem 3.1.

BIBLIOGRAPHY

- [K-S 1] KASHIWARA, M. and SCHAPIRA, P. : Microlocal study of sheaves. *Astérisque* 128 (1985).
- [Ka] KATAOKA, K : Microlocal theory of boundary value problems. *J. Fac. Sc. Univ. Tokyo Sect. A. I* : 27, 355-399 (1980), *II* : 28, 31-56 (1981).
- [Le] LEBEAU, G. : Deuxième microlocalisation sur les sous-variétés isotropes. *Ann. Inst. Fourier, Grenoble* 35,2, 145-216 (1985).
- [S 1] SCHAPIRA, P. : Front d'onde analytique au bord II. *Sem. E.D.P. Ecole Polytechnique, Exp. 13* (1986).
- [S 2] SCHAPIRA, P. : Microfunctions for boundary value problems. Volume dedicated to Pr. M. Sato. To appear.