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Resonance functions of two-body Schrödinger operators


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We consider the Schrödinger operator \(-\Delta + V\) in \(L^2(\mathbb{R}^n)\), \(n \geq 3\), where \(V\) is a short-range, dilation-analytic potential in an angle \(S_\alpha\). A resonance \(\lambda_0\) appears as a discrete eigenvalue of the complex-dilated Hamiltonian [2], a pole of the S-matrix [3] and as a pole of the analytically continued resolvent, acting from an exponentially weighted space to its dual [4,5]. In [2] resonance functions are obtained as square-integrable eigenfunctions of the complex-dilated Hamiltonian, corresponding to the eigenvalue \(\lambda_0\), in [5] they are defined as certain exponentially growing solutions \(f\) of the Schrödinger equation \((-\Delta + V - \lambda_0)f = 0\). In [6] it is proved that for a dilation-analytic multiplicative potential \(V\) with resonance \(\lambda_0\), the resonance functions of [2] and [5] are simply the restrictions of one analytic, \(L^2(S^{n-1})\)-valued function \(f\) on \(S_\alpha\) to rays \(e^{i\varphi \mathbb{R}^+}\) with \(2\varphi > -\text{Arg}\lambda_0\) and to \(\mathbb{R}^+\), respectively.

Moreover, the precise asymptotic behaviour \(f(z) = e^{\frac{ik_0z}{z^{1-n}}} (\tau + o(|z|^{-\varepsilon}))\) with \(\tau \in L^2(S^{n-1})\), where \(k_0^2 = \lambda_0\), is established together with asymptotics for \(f'(z)\). These imply exponential decay in time of resonance states, defined as suitably cut-off resonance functions, as proved in [8].

In this note we shall give a brief account of results on resonance functions, referring for details to [5] and [6].
1. Analytic continuation of resolvent and S-matrix

We introduce the weighted $L^2$-spaces $L^2_{\delta,b} = L^2_{\delta,b}(\mathbb{R}^n)$ for $\delta, b \in \mathbb{R}$ by

$$L^2_{\delta,b} = \{ f \mid \| f \|_{2,\delta,b}^2 = \int_{\mathbb{R}^n} |f(x)|^2 (1+|x|^2)^\delta e^{2br} \, dx < \infty \}$$

where $x \in \mathbb{R}^n, r = |x|$. The weighted Sobolev spaces are defined by

$$H^2_{\delta,b} = \{ f \mid \| f \|_{2,\delta,b}^2 = \sum_{|\alpha| \leq 2} \| D^\alpha f \|_{2,\delta,b}^2 < \infty \}$$

We set $L^2_{\delta,0} = L^2_{\delta}$, $H^2_{\delta,0} = H^2$ and $h = L^2(S^{n-1})$, $S^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$. We assume that the dimension $n \geq 3$

$$\mathfrak{g}^+ = \{ k \in \mathfrak{g} \mid \text{Im} \, k > 0 \} \quad \text{and} \quad \mathfrak{g}^+ = \overline{\mathfrak{g}^+ \setminus \{ 0 \}}.$$ 

$\mathcal{B}(H_1,H_2)$ and $\mathcal{C}(H_1,H_2)$ denote the spaces of bounded and compact operators from $H_1$ into $H_2$, respectively.

The free Hamiltonian $H_0$ in $L^2$ is defined for $u \in \mathcal{D}_{H_0} = H^2$ by $H_0 u = -\Delta u$ with resolvent $R_0(k) = (H_0 - k^2)^{-1} \in \mathcal{B}(L^2)$ for $k \in \mathfrak{g}^+$.

The interaction $Q$ is assumed to be a symmetric, short-range, $S_\alpha$-dilation-analytic operator in $L^2$. Thus, $Q \in \mathcal{C}(H_{-\delta_0}^2,L_{\delta_0}^2)$ for some $\delta_0 \geq \frac{1}{2}$, and if $\{ U(\rho) \}_{\rho \in \mathbb{R}_+}$ is the dilation group on $L^2$ defined by

$$(U(\rho)f)(x) = \rho^nf(\rho x)$$

then the function $Q(\rho) = U(\rho)QU(\rho^{-1})$ on $\mathbb{R}_+$ has an analytic, $\mathcal{C}(H_{-\delta_0}^2,L_{\delta_0}^2)$-valued analytic extension to the angle

$$S_\alpha = \{ \rho e^{i\varphi} \mid \rho > 0, |\varphi| < \alpha \}$$

Moreover, $Q(z) \in \mathcal{C}(H_{-\delta_0}^2,b,L_{\delta_0}^2,b)$ for all $b \in \mathbb{R}$. (This follows from $Q \in \mathcal{C}(H_{-\delta_0}^2,L_{\delta_0}^2)$ if $Q$ is local).
The Hamiltonian $H = H_0 + Q$ is self-adjoint on $\mathcal{D}_H = H^2$, and associated with $H$ is a self-adjoint, analytic family of type $A$, $H(z)$, given by

$$H(z) = z^{-2}H_0 + Q(z),$$

and $H(\rho e^{i\varphi}) = U(\rho)H(e^{i\varphi})U(\rho^{-1})$, so $\sigma(H(z)) = \sigma(H(e^{i\varphi}))$ for $\rho > 0$, $z = \rho e^{i\varphi}$.

We define the operators $H_z$ and their resolvents $R_z(k)$ by

$$H_z = H_0 + z^2Q(z) = z^2H(z), \quad R_z(k) = (H_z - k^2)^{-1}.$$

We note that $R_z(zk) = z^{-2}(H(z) - k^2)^{-1}$.

We have $\sigma_e(H(z)) = e^{-2i\varphi}IR^+$ and $\sigma_d(H(z)) \setminus IR \subset \{ \lambda \mid -2\varphi < \text{Arg} \lambda < 0 \}$.

We define $R(\varphi)$ by $R(\varphi) = \{ k \mid 0 > \text{Arg} k > -\varphi, \quad k^2 \in \sigma_d(H(z)) \}$, $R = \bigcup R(\varphi)$. The points $\lambda = k^2$, $k \in R$, are called resonances.

For our analysis we need the following result, proved in [5]:

**Lemma 1.1.** For $\delta > 0$ let $S_\alpha^\delta = \{ k \in S_\alpha \mid \text{Im}(e^{i(\alpha-\delta)k}) < \varepsilon \}$. There exists $S_\alpha$-dilation-analytic interactions $V_\varepsilon$ and $W_\varepsilon$ with $Q = V_\varepsilon + W_\varepsilon$, such that $H_0 + V_\varepsilon$ has no resonances outside $(S_\alpha^\delta)^2$ and $W_\varepsilon$ decays faster than any exponential. This holds with $W_\varepsilon = g_\varepsilon Qg_\varepsilon$, where $g_\varepsilon(r) = \exp(-\varepsilon r^\beta)$, $\beta = \frac{\pi}{2\alpha}$, for $\varepsilon$ small.

Using Lemma 1.1 one can prove all results for fixed $\delta > 0$ with $S_\alpha^\delta$ replaced by $S_\alpha^\delta$, using the splitting $Q = V_\varepsilon + W_\varepsilon$, and then let $\delta \downarrow 0$. To simplify the presentation, we assume from the outset (although this can strictly speaking not be obtained) that $H_1 = H_0 + V$ has no resonances and fix $\varepsilon$, setting $g = g_\varepsilon$, $W = Qg$, $V = Q - gW$. We denote by $H_{1z}$, $R_{1z}(k)$ etc. the operators obtained by replacing $Q$ by $V$. 

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Basic to our approach is an extended limiting absorption principle proved in [7] and generalized in [5] to non-symmetric, short-range potentials like \( Q_z \). The idea is to consider \(-\Delta\) and \(-\Delta + Q_z\) as operators \( H_{-b}^0 \) and \( H_{-b}^z \) acting in the space \( L^2_{0,-b}, b > 0 \). The spectrum of \( H_{-b}^0 \) and the essential spectrum of \( H_{-b}^z \) coincide with the parabolic region \( p_b = \{ k^2 \mid |\text{Im}k| \leq b \} \), and it is then proved that the resolvents \((H_{-b}^0 - (a + ib + i\epsilon)^2)^{-1}\) and \((H_{-b}^z - (a + ib + i\epsilon)^2)^{-1}\) have boundary values as \( \epsilon \to 0 \) in \( \mathcal{B}(L^2_{\delta,-b}, H^2_{-\delta-b}) \) for \( \frac{1}{2} < \delta \leq \delta_0 \), except at the so-called singular points.

The singular sets \( \Gamma_z^C, \Gamma_z^r \) and \( \Gamma_z \) are defined for \( z = re^{i\varphi}, \varphi > 0 \), by
\[
\Gamma_z^C = \{ k \in \mathbb{C}^+ \mid k^2 = z^2, \lambda \in \sigma_d(H(z)) \},
\]
\[
\Gamma_z^r = z \mathbb{R} \cap \mathbb{R}^+, \quad \Gamma_z = \Gamma_z^C \cup \Gamma_z^r,
\]
and for \( \varphi < 0 \) by \( \Gamma_z^C = -\Gamma_z^C \) and similar for \( \Gamma_z^r \) and \( \Gamma_z \).

For \( \varphi = 0 \), \( \Gamma_z = \Gamma_z^C \cup \Gamma_z^r = \{ k \in \mathbb{C}^+ \mid k^2 \in \sigma_p(H) \} \).

The extended limiting absorption principle for \( H_z \) can then be formulated as follows:

**Theorem 1.2.** For fixed \( z \in S^\alpha_\delta, 0 < \delta \leq \delta_0 \), there exists a meromorphic \( \mathcal{B}(L^2_{\delta}, H^2_{-\delta}) \)-valued function \( R_z^-(k) \) in \( \mathbb{C}^+ \), continuous in \( \mathbb{C}^+ \setminus \Gamma_z \), such that for \( k \in \mathbb{R} \setminus \Gamma_z^r \cup \{ 0 \} \)
\[
R_z^-(k) = e^{ikr} R_z(k + i0) e^{-ikr}
\]
where
\[
R_z(k + i0) = \lim_{\epsilon \to 0} R_z(k + i\epsilon)
\]
in the operator-norm topology of \( \mathcal{B}(L^2_{\delta}, H^2_{-\delta}) \), locally uniformly in \( k \).
For $f \in L^2_{\delta,-b}$, $u = e^{-ikr}R_z(k)e^{ikr}f$ is the unique solution in $L^2_{\delta,-b}$ of the equation $(H_z^{-b} - k^2)u = f$, such that $\mathcal{D}u \in L^2_{\delta-1,-b}$, where $b = \text{Im}k$ and

$$
\mathcal{D}u = (\mathcal{D}u_1,\ldots,\mathcal{D}u_n), \quad \mathcal{D}u_j = \frac{\partial}{\partial x_j} + \frac{n-1}{2r^2} x_j - \frac{ikx_j}{r}
$$

(the radiation condition).

Proof. We refer to [5] for the proof of the Theorem. It utilizes the result of [7] for $H_0$, analytic Fredholm theory and control of the singular points using analyticity in $k$ and $z$.

The trace operators $T_0(k)$, $T_z(k) \in \mathcal{B}(L^2_{\delta},h)$ are defined for $z \in S_\alpha$, by

$$(T_0(\pm k)f)(k,\cdot) = (F_{\pm}f)(k,\cdot), \quad k \in \mathbb{R}^+$$

where

$$(F_{\pm}f)(k,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\pm ik \cdot \omega} f(x) \, dx ,$$

$$T_z(k) = T_0(k)(1 - Q_z R_z(k+10)), \quad k \in \mathbb{R}^+ \setminus \Gamma_z^n.$$  

We set

$$T_0(k) = T_0(k)e^{ikr}, \quad T_z^+(k) = T_z(k)e^{ikr}.$$ 

The following result is proved in [5].

**Theorem 1.3.** For $\frac{1}{2} < \delta < \delta_0$, $z \in S_\alpha$, the $\mathcal{B}(L^2_{\delta},h)$-valued function $T_z^+(k)$ has a continuous extension to $\mathbb{C}^+ \setminus \Gamma_z$ meromorphic in $\mathbb{C}^+$ with poles at $\Gamma_z^C$. The function $T_z^+(k)$ defined for $k \in \mathbb{C}^-(\Gamma_z)$ is analytic in $\mathbb{C}^-(\Gamma_z^C)$ and continuous in $\mathbb{C}^-(\Gamma_z^C)$ as a $\mathcal{B}(h,H^2_{-\delta})$-valued function.

We recall the following formulas from the stationary scattering theory:

$$R(k+i0) = R(-k+i0) + \pi ik^{n-2} T^*(k)T(k), \quad k \in \mathbb{R}^+ \setminus \Gamma_z^n \quad (1.2)$$

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\[ T(k) = S(k)RT(-k) \quad (1.3) \]

where \((RT)(\omega) = \tau(\omega)\) for \(\omega \in \mathbb{R}h\).

Inserting (1.3) in (1.2), we obtain
\[ R(k+i0) = R(-k+i0) + \pi ik^{n-2}T^*(k)S(k)RT(-k) \quad (1.4) \]

The \(S\)-matrix \(S(k)\) of \((H_0,H)\) is given for \(k \in \mathbb{R}^+ \setminus S_1\) by
\[ S(k) = 1 - \pi ik^{n-2}T_0(k)(Q - QR(k+i0)Q)T^*_0(k) \quad (1.5) \]

and the \(S\)-matrix \(S_1(k)\) of \((H_0,H_1)\) by (1.5) with \(Q\) and \(R\) replaced by \(V\) and \(R_1\).

The following result is proved in [3].

**Theorem 1.4.** The \(S\)-matrix \(S(k)\) has a meromorphic extension \(\tilde{S}(k)\) from \(\mathbb{R}^+\) to \(S_\alpha\) with poles at \(R\). The \(S\)-matrix \(S_1(k)\) has an analytic extension \(\tilde{S}_1(k)\) from \(\mathbb{R}^+\) to \(S_\alpha\). Moreover, for \(k > 0\), \(0 < \phi < \alpha\), \(\tilde{S}_1(ke^{-i\phi}) = S_{1,e^{-i\phi}}(k)\), where \(S_{1,e^{-i\phi}}(k)\) is the \(S\)-matrix of \((H_0,H_1)\) at the point \(k\).

From (1.4) and Theorem 1.2 we obtain

**Theorem 1.5.** For any \(b > 0\) the \(\mathcal{B}(L^2_{0,b},H^2_{0,-b})\)-valued function \(R(k)\) has a meromorphic continuation \(\tilde{R}(k)\) from \(\mathbb{C}^+\) across \(\mathbb{R}^+\) to \(S_{\alpha,b} = \{k \in S_\alpha \mid -b < \text{Im}k < 0\}\), given by
\[ \tilde{R}(k) = R(-k) + \pi ik^{n-2}T^*(k)\tilde{S}(k)T(-k) \quad (1.6) \]

The \(\mathcal{B}(L^2_{0,b},H^2_{0,-b})\)-valued function \(R_1(k)\) has an analytic continuation \(\tilde{R}_1(k)\) from \(\mathbb{C}^+ \setminus S_{1c}\) across \(\mathbb{R}^+\) to \(S_{\alpha,b}\), given by (1.6) with \(R, T\) and \(S\) replaced by \(R_1, T_1\) and \(S_1\).

The functions \(\tilde{R}(k)\) and \(\tilde{R}_1(k)\) are connected by the analytically continued symmetrized resolvent equation
\[ \tilde{R}(k) = \tilde{R}_1(k) - \tilde{R}_1(k)g(1 + W\tilde{R}_1(k)g)^{-1}W\tilde{R}_1(k) \quad (1.7) \]
The following result is proved in [5]:

**Theorem 1.6.** \( \tilde{R}(k) \) and \( \tilde{S}(k) \) have the same poles and of
the same order in \( S_{a,b} \).
2. Resonance functions

Let $k_0^2$ be a resonance, and fix $b > -\text{Im}k_0$. Then $k_0$ is a pole of $\tilde{R}(k) \in \mathcal{B}(L^2_{0,b}, H^2_{0,-b})$, defined in Theorem 1.5. Let $C$ be a circle separating $k_0$ from other poles and let

$$p = -\frac{1}{2\pi i} \int_C \tilde{R}_2(k) dk^2$$

be the residue of $\tilde{R}_2(k)$ at $k_0$, $p \in \mathcal{B}(L^2_{0,b}, H^2_{0,-b})$ is of finite rank.

The space $F$ of resonance functions associated with $k_0$ is defined by

$$F = \{ f \in \mathcal{R}_p \mid (-\Delta + Q - k_0^2) f = 0 \}.$$

The following result is proved in [5]:

**Theorem 2.1.** $F$ is the isomorphic image of $N(S^{-1}(k_0))$ and of $N(1 + W\tilde{R}_1(k_0)g)$ via the following maps:

$$N(S^{-1}(k_0)) \ni \tau \mapsto T^*(k_0)\tau = f \in F$$

$$N(1 + W\tilde{R}_1(k_0)g) \ni \phi \mapsto \tilde{R}_1(k_0)g\phi = f \in F$$

**Remark.** From the representation $f = T^*(k_0)\tau$ we conclude by Theorem 1.3 and the uniqueness part of Theorem 1.2 that $f \in H^2_{\delta', -b_0} \setminus L^2_{\delta-1, -b_0}$ for every $\delta > \frac{1}{2}$ and $b_0 = -\text{Im}k_0$. A further analysis yields precise asymptotic estimates. We first establish the analyticity properties, using the second isomorphism.

Applying (1.4) to the operator $H_{1z}$ at a point $zk_0$ with $\text{Arg} zk_0 = 0$ and noting that by Theorem 1.4, $S_{1z}(zk_0) = \tilde{S}_1(k_0)$ we obtain
By Theorems 1.2 and 1.3 we obtain from (1.7)

**Theorem 2.2.** The $\mathcal{B}(L^2_\delta, H^2_\delta)$-valued function $e^{-izk_0r}$ has an analytic extension from $\{z \in \mathbb{C} \mid Re\, z < 0\}$ to $\{z \in \mathbb{C} \mid Arg \, z < 0\}$, given by

$$e^{-izk_0r} = e^{-izk_0r} + \pi i(zk_0)^{n-2} T^*_1(zk_0) S^*_1(k_0) R T_1(-zk_0)^{n-2}$$

Recalling that $W_z = Q_z g(rz)$, where $g(rz) = \exp{-\epsilon(rz)^\beta}$ with $\beta > 1$, we obtain from Theorem 2.2

**Theorem 2.3.** The $C(L^2)$-valued function $W_z R_1(zk_0) g(rz)$ has an analytic continuation from $\{z \in \mathbb{C} \mid Arg \, zk_0 > 0\}$ to $\{z \in \mathbb{C} \mid Arg \, zk_0 \leq 0\}$, given by $W_z R_1(zk_0) g(rz)$.

By standard dilation-analytic arguments $\sigma(W_z R_1(zk_0) g(rz))$ is constant. Let $C$ be a circle separating $-1$ from the rest of $\sigma(W_z R_1(zk_0) g(rz))$ and set

$$P(z) = -\frac{1}{2\pi i} \int_C (-\lambda + W_z R_1(zk_0) g(rz))^{-1} d\lambda.$$ 

Then $P(z)$ is a dilation-analytic $\mathcal{B}(L^2)$-valued function of $z$, and $P(z)$ is a projection on the finite-dimensional algebraic null space of $1 + W_z R_1(zk_0) g(rz)$. Let $\phi \in N(1 + W_1 R_1(k_0) g(rz))$ and pick an $S_\alpha$-dilation-analytic vector $\eta$ in $L^2$ such that
\( \phi = P(1) \eta \). Then \( \phi(z) = P(z) \eta(z) \in N(1 + \tilde{R}_{1z}(z_k \theta) g(rz)) \), and \( \phi(z) \) is dilation-analytic.

We now obtain, using the second isomorphism of Theorem 2.1,

\textbf{Theorem 2.4.} Let \( f \in F \). Then there exists an \( S_\alpha \)-dilation-analytic, \( H^2_{-\delta} \)-valued function \( \chi(z) \), such that \( f = e^{i k_0 r} \chi(1) \) and for \( \text{Arg } z k_0 > 0 \)

\[ f(z) = e^{i k_0 z r} \chi(z) \in N(H(z) - k_0^2) . \]

Moreover, \( \chi(z) \notin L^2_{\delta-1} \) for all \( z \in S_\alpha, \delta > \frac{1}{2} \).

\textbf{Proof.} Define \( f(z) \) by

\[ f(z) = \begin{cases} 
izk_0 r & \text{if } z \in R^+_{1z}(z_k \theta) e^{-izk_0 r} g(rz) \phi(z), \text{Im} z_k \theta > 0 \\
izk_0 r & \text{if } z \in R^+_{1z}(z_k \theta) e^{-izk_0 r} g(rz) \phi(z), \text{Im} z_k \theta \leq 0
\end{cases} \]

where \( R^+_{1z}(z_k \theta) \) is defined similarly to \( R^-_{1z}(z_k \theta) \), replacing \( -b \) by \( b \) and \( e^{i \alpha} \) with \( e^{i \alpha} \) in Theorem 1.2. Clearly, \( f(z) \) is continuous for \( z_k \theta \in IR^+ \). By Theorem 1.2 and 2.2, \( \chi(z) = e^{-izk_0 r} f(z) \) is an analytic \( H^2_{-\delta} \)-valued function in \( S_\alpha \).

It follows from the uniqueness part of Theorem 1.2 that \( \chi(z) \notin L^2_{\delta-1} \) for \( \text{Im} z_k \theta < 0 \). The fact that \( \chi(z) \notin L^2_{\delta-1} \) for \( \text{Im} z_k \theta > 0 \) then follows by the next Lemma, proved in [6]:

\textbf{Lemma 2.5.} Let \( \chi(z) \) be an \( S_\alpha \)-dilation-analytic vector, and define \( h(\phi) \) for \( \phi \in (-\alpha, \alpha) \) by

\[ h(\phi) = \inf \{ s \mid \chi(e^{i \phi}) \notin L^2_{-s} \} . \]

Then either \( h(\phi) = -\infty \) or \( h(\phi) > -\infty \) and \( h \) is convex in \( (-\alpha, \alpha) \).
Using this Lemma together with a recent result of Agmon [1], giving the precise asymptotic behaviour of \( f(z) \) for \( \text{Arg} z k_0 > 0 \), we finally obtain the desired asymptotic estimates of \( f \) and \( f' \). We refer to [6] for the proof.

**Theorem 2.6.** Assume that \( Q \) is an \( S_\alpha \)-dilation-analytic multiplicative potential such that \( |Q(z)(x)| \lesssim C|x|^{-1-\epsilon} \) for \( z \in S_\alpha \) and \( |x| \geq R \). Let \( f \in F \). Then \( f \) is an analytic, \( k \)-valued function \( f(z,\cdot) \) on \( S_\alpha \) of the form

\[
f(z,\cdot) = e^{ik_0 \frac{1-n}{2} \frac{1}{z}} g(z,\cdot)
\]

where

\[
g(z,\cdot) = \tau + O(|z|^{-\epsilon})
\]

\[
g'(z,\cdot) = O(|z|^{-1-\epsilon})
\]

uniformly in any smaller angle \( S'_\alpha \) for some \( \epsilon > 0 \). Moreover, \( \tau \in \mathbb{N}(\overline{S}^{-1}(k_0)) \) and \( f = C T^*(\overline{k}_0) \tau \),

\[
C = k_0 \frac{n-1}{2} \frac{1-n}{2} \frac{1}{(2\pi)^{\frac{d}{2}}}.
\]
REFERENCES:


