GIOVANNI M. TROIANIELLO

**A class of weighted function spaces, and intermediate
Caccioppoli-Schauder estimates**

Journées Équations aux dérivées partielles (1988), p. 1-7

http://www.numdam.org/item?id=JEDP_1988____A9_0

© Journées Équations aux dérivées partielles, 1988, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

A CLASS OF WEIGHTED FUNCTION SPACES , AND INTERMEDIATE CACCIOPPOLI-SCHAUDER ESTIMATES

Giovanni M. TROIANIELLO
Dipartimento di Matematica
Università di Roma 1

1 - A THEOREM OF D. GILBARG AND L. HORMANDER

Consider the Dirichlet problem

$$(1) \quad Lu = f \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi,$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $\partial\Omega$ its boundary, and L a linear second order uniformly elliptic differential operator with coefficients defined on $\overline{\Omega}$. The classical Caccioppoli-Schauder approach to (1) provides, under suitable regularity assumptions about $\partial\Omega$ and the coefficients of L , a priori bounds on norms

$$|u|_{C^{k,\delta}(\Omega)}, \quad k = 2, 3, \dots \quad \text{and} \quad \delta \in]0, 1[;$$

this of course requires, to start with, the membership of f in $C^{k-2,\delta}(\overline{\Omega})$ and of φ in $C^{k,\delta}(\partial\Omega)$.

What happens now if we weaken our assumption about φ by requiring that it belong to $C^{k',\delta'}(\partial\Omega)$ for some $k' = 0, 1, \dots$ and some $\delta' \in]0, 1[$ such that $k' + \delta' < k + \delta$? An answer to this question was given by Gilbarg and Hörmander [4] : they provided weighted $C^{k,\delta}$ norm estimates for solutions of (1), the weight consisting of the α -th power of the distance from $\partial\Omega$ with $\alpha \equiv k + \delta - (k' + \delta')$. Note that, for what correspondingly concerns f , the natural regularity requirement is now only that its weighted $C^{k-2,\delta}$ norm be finite.

In order to illustrate the key point of [4] we introduce some notations.
Letting

$$B_r(x^0) \equiv \{x \in \mathbb{R}^N \mid |x - x^0| < r\}$$

$$B_r^+(x^0) \equiv \{x \in B_r(x^0) \mid x_N > x_N^0\}$$

$$S_r^+(x^0) \equiv \{x \in \partial B_r(x^0) \mid x_N > x_N^0\}$$

$$S_r^0(x^0) \equiv \partial B_r^+(x^0) \setminus \overline{S_r^+(x^0)}.$$

(under the convention that the dependence on x^0 , r be depressed if $x^0 = O$, $r = 1$), we define $C_{\alpha}^{k, \delta}(B_R^+)$ as the space of functions $u = u(x)$, $x \in B_R^+$, having finite norms

$$|u|_{C_{\alpha}^{k, \delta}(B_R^+)} \equiv \sup_{S > 0} S^{\alpha} |u|_{C_{\alpha}^{k, \delta}(B_{R[S]}^+)}$$

here, $k = 0, 1, \dots$, $0 < \delta \leq 1$, $\alpha \geq 0$, and $B_{R[S]}^+ \equiv \{x \in B_R^+ \mid x_N > S\}$. (When $\alpha < 0$ the right-hand side in the above definition of norm is finite only for $u = 0$). Through direct investigation of Green's function for the Laplace operator in the upper half space Gilbarg and Hörmander proved the following result (Theorem 3.1 of their paper): let $k = 2, 3, \dots$, $0 < \delta < 1$, $0 \leq \alpha < k + \delta$ and $k + \delta - \alpha \notin \mathbb{N}$; then there exists a constant C such that

$$(2)_k \quad |u|_{C_{\alpha}^{k, \delta}(B^+)} \leq C |f|_{C_{\alpha}^{k-2, \delta}(B^+)}$$

whenever u is a function from $C_{\alpha}^{k, \delta}(B^+)$ which vanishes near S^+ and satisfies (in the pointwise sense)

$$(3) \quad u|_{S^0} = 0, \quad \Delta u = f \text{ in } B^+.$$

What we are going to describe in the present article is an alternative approach to (3), which yields a slightly more general result than the bounds $(2)_k$. Notice that the passage from Δ to more general variable coefficient operators L can be achieved through a perturbation argument as in [4, prop. 4.3]; the case of nonvanishing Dirichlet data φ on S^0 can be handled through suitable extensions of the φ 's to the upper half space [4, lemma 2.3]; finally, partitions of unity and changes of variables near boundary points lead to the general setting of (1) [4, theorem 5.1]. This procedure exhibits rather delicate technical features, if one wants to adopt the "natural" generality for what concerns regularity assumptions about the coefficients of L as well as $\partial \Omega$. The crux of the matter lies, however, within the study of (3).

2 - THE MAIN RESULTS OF THIS ARTICLE

We are going to deal with weak solutions to a problem such as

$$(4) \quad u|_{S^0} = 0, \quad \Delta u = f + f^i_{x_i} \quad \text{in } B^+$$

i.e., for some $p \in]1, \infty[$,

$$u \in H^{1,p}(B^+), \quad u|_{S^0} = 0,$$

$$(5) \quad \int_{B^+} u_{x_i} \varphi_{x_i} dx = \int_{B^+} (-f \varphi + f^i \varphi_{x_i}) dx \quad \forall \varphi \in C_0^\infty(B^+)$$

(summation convention of repeated indices). Here and throughout, $H^{k,p}$ and $H_0^{k,p}$ are the standard notations for Sobolev spaces.

For our study of regularity we find it convenient to introduce new (norms and) function spaces. Namely, for $1 \leq p < \infty$, $\alpha \in \mathbb{R}$ and $0 \leq \lambda \leq N+p$ let

$$[u]_{L_\alpha^{p,\lambda}(B_R^+)} \equiv \sup_{x^0 \in B_R^+, \rho > 0} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{B_R^+ \cap B_\rho(x^0)} x_N^{p\alpha} |u - c|^p dx$$

and denote by $L_\alpha^{p,\lambda}(B_R^+)$ the space of functions $u = u(x)$, $x \in B_R^+$, having finite norms

$$|u|_{L_\alpha^{p,\lambda}(B_R^+)} \equiv \left(\int_{B_R^+} x_N^{p\alpha} |u|^p dx + [u]_{L_\alpha^{p,\lambda}(B_R^+)}^p \right)^{1/p}.$$

It is clear that, for any value of α , $L_\alpha^{p,\lambda}(B_R^+)$ at least contains $C_0^\infty(B_R^+)$.

$L_0^{p,\lambda}(B_R^+)$ is the by now classical campanato space, and $L_0^{p,\lambda}(B_R^+) \sim C^{0,(\lambda-N)/p}(B_R^+)$ if $N < \lambda \leq N+p$ [2]. But we have more :

Lemma 1

For $\alpha \geq 0$ and $N < \lambda \leq N+p$ the spaces $L_\alpha^{p,\lambda}(B_R^+)$ and $C_\alpha^{0,(\lambda-N)/p}(B_R^+)$ are isomorphic.

$L_0^{p,N}(B_R^+)$ is a *BMO* ($\equiv$ Bounded Mean Oscillation) space [6]. The importance of *BMO* spaces as "good substitutes" for C^0 and L^∞ has since long been acknowledged in PDE's (and Harmonic Analysis ...). Take for instance our initial considerations about the classical Caccioppoli-Schauder approach to (1) :

BMO spaces are known to fill the gaps left over by the exclusion of the two values $\delta = 0$ and $\delta = 1$ [3]. But weighted norms lead to another example. Precisely, consider the continuous imbedding

$$(6) \quad C_{\alpha+\beta}^{\alpha, \delta+\beta}(B_R^+) \subset C_{\alpha}^{\alpha, \delta}(B_R^+)$$

which is proven in [4] for $\alpha \geq 0$, $0 \leq \delta < 1$ and $\beta > 0$ with $\delta + \beta \leq 1$, under the restriction $\alpha \neq \delta$. This restriction has far-reaching consequences, such as the above-mentioned requirement $k + \delta - \alpha \in \mathbb{N}$ for the validity of $(2)_k$. But, why cannot $\alpha = \delta$ be allowed? For sure, (6) is false when $\alpha = \delta = 0$, as the one-dimensional example given in [4], that is, $u(x) \equiv \log x$, $0 < x < 1$, clearly shows. But, as it happens, this function u belongs to $L_0^{p,N}([0, 1]) \dots$ We can indeed prove the following result, which contains (6) in all cases except $\alpha \neq 0 = \delta$.

Lemma 2

For $\alpha \geq 0$, $0 \leq \delta < 1$ and $\beta > 0$ with $\delta + \beta \leq 1$, the continuous imbedding

$$L_{\alpha+\beta}^{p, N+p(\delta+\beta)}(B_R^+) \subset L_{\alpha}^{p, N+p\delta}(B_R^+)$$

is valid.

We can now arrive at our results about solutions to (5). Adopting the symbol $L_{\beta}^{\infty}(B^+)$ to denote the space of measurable functions $h = h(x)$, $x \in B^+$, such that

$$|h|_{L_{\beta}^{\infty}(B^+)} \equiv |x_N^{\beta} h|_{L^{\infty}(B^+)}$$

is finite, we begin with first derivatives.

Theorem 1

Let $0 \leq \delta < 1$, $0 \leq \alpha < 1 + \delta$. If, for a suitable value of $p > 1$, u satisfies (5) with $f \in L_{1+\alpha-\delta}^{\infty}(B^+)$ and $f^1, \dots, f^N \in C_{\alpha}^{\alpha, \delta}(B^+)$, then all its first derivatives belong to $L_{\alpha}^{p, N+p\delta}(B_R^+)$, $0 < R < 1$, and satisfy

$$\begin{aligned} \sum_{i=1}^N |u_{x_i}|_{L_{\alpha}^{p, N+p\delta}(B_R^+)} &\leq C(|f|_{L_{1+\alpha-\delta}^{\infty}(B^+)}) \\ &+ \sum_{i=1}^N |f^i|_{C_{\alpha}^{\alpha, \delta}(B^+)} + |u|_{H^{1,p}(B^+)} \end{aligned}$$

with C independent of $u, f, f^1, \dots, f^N$.

The passage to second derivatives is performed, so to speak, through "differentiation" of (5) with respect to $x_1, \dots, x_{N-1}$. Without loss of generality, it can be assumed that $f^1 = \dots = f^N = 0$; as for f , the "natural" requirement becomes

$$f \in C_{\alpha}^{0, \delta}(B^+)$$

for $0 \leq \alpha < 2 + \delta$. It is the range $1 + \delta \leq \alpha < 2 + \delta$, of course, that poses new difficulties: no longer is then f in some $L^p(B^+)$, so that the $H^{2,p}$ regularity theory does apply to (5), and the above results about u are not inherited by u_{x_S} , $S = 1, \dots, N-1$. But $H^{2,p}$ regularity does apply to $x_N u$, and $U \equiv x_N u_{x_S}$ satisfies, in the weak sense,

$$U|_{S_{R_1}^0} = 0, \quad \Delta U = -x_N f_{x_S} + 2 u_{x_S x_N} \quad \text{in } B_{R_1}^+$$

for any $R_1 \in]0, 1[$. We can thus arrive at.

Theorem 2

Let $0 \leq \delta < 1$, $0 \leq \alpha < 2 + \delta$. If, for a suitable value of $p > 1$, u satisfies (5) with $f \in C_{\alpha}^{0, \delta}(B^+)$ and $f^1 = \dots = f^N = 0$, then all its second derivatives belong to $L_{\alpha}^{p, N+p\delta}(B_R^+)$ when restricted to B_R^+ , $0 < R < 1$, and satisfy

$$(7) \quad \sum_{i,j=1}^N |u_{x_i x_j}|_{L_{\alpha}^{p, N+p\delta}(B_R^+)} \leq C(|f|_{C_{\alpha}^{0, \delta}(B^+)} + |u|_{H^{1,p}(B^+)})$$

with C independent of u, f .

(If we want to be more specific in the choice of p , we take $p = 2$ for $0 \leq \alpha < \frac{1}{2} + \delta$ and $1 < p < \frac{1}{\alpha - \delta}$ for $\frac{1}{2} + \delta \leq \alpha < 1 + \delta$ in both Theorems 1 and 2, $p = 2$ for $1 + \delta \leq \alpha < \frac{3}{2} + \delta$ and $1 < p < \frac{1}{\alpha - 1 - \delta}$ for $\frac{3}{2} + \delta \leq \alpha < 2 + \delta$ in Theorem 2).

When $\text{supp } u \cap S^+ = \emptyset$, (7) holds for $R = 1$ without the term $|u|_{H^{1,p}(B^+)}$ on its right hand side. This means that $(2)_2$ holds for all values of α in the range $[0, 2 + \delta[$, $0 < \delta < 1$, that is, without exception for $\alpha = \delta$ and $\alpha = 1 + \delta$. Since the procedure leading to Theorem 2 can be repeated for all higher order derivatives, $(2)_k$ holds whenever $k = 2, 3, \dots$ and $0 \leq \alpha < k + \delta$, $0 < \delta < 1$, no exception being made for $k + \delta - \alpha \in \mathbb{N}$.

As for $\delta = 0$, we simply mention that $C_{\alpha}^{0, 0}(B^+)$ could safely be

replaced by $L^\infty(B^+)$ throughout. The above results can therefore be said to contain "weighted versions of the $L^\infty \rightarrow BMO$ type of regularity".

A few words about our techniques. The main tools are estimates such as

$$(8) \quad \int_{B_\rho(x^0)} |\nabla w|^p dx \leq C(p) \left[\left(\frac{\rho}{r}\right)^N \int_{B_r(x^0)} |\nabla w|^p dx + \sum_{i=1}^N \int_{B_r(x^0)} |h^i|^p dx \right]$$

and

$$(9) \quad \int_{B_\rho(x^0)} |\nabla w - (\nabla w)_{\rho;\alpha}|^p dx \leq C(p,\alpha) \left[\left(\frac{\rho}{r}\right)^{Np} \int_{B_r(x^0)} |\nabla w - (\nabla w)_{r,\alpha}|^p dx \right. \\ \left. + \sum_{i=1}^N \int_{B_r(x^0)} |h^i - (h^i)_{r,\alpha}|^p dx \right],$$

which hold whenever w satisfies

$$w \in H^{1,p}(B_r(x^0)),$$

$$\int_{B_\rho(x^0)} w_{x_i} \varphi_{x_i} dx = \int_{B_r(x^0)} h^i \varphi_{x_i} dx \quad \forall \varphi \in C_0^\infty(B_r(x^0))$$

where $0 < \rho \leq r < \infty$, $x^0 \in \mathbb{R}^N$; in (9), the symbol $(\cdot)_{\rho;\alpha}$ denotes average over $B_\rho(x^0)$ with respect to $x_N^\alpha dx$, $\alpha \geq 0$. We need p from $[1,2]$. For $p = 2$, (8) and (9) are obtained [3] through typical techniques of the Hilbert space theory of elliptic PDE's. The passage to $1 < p < 2$ requires some preliminary results from the corresponding $H^{k,p}$ theory which can be found, for instance, in [7].

If spheres $B_\rho(x^0)$ are replaced throughout by hemispheres $B_\rho^+(x^0)$ - and w is required to vanish on $S_r^0(x^0)$ - the counterpart of (8) is obviously valid for $1 < p \leq 2$, while the counterpart of (9) is only needed here for $p = 2$ as in [3].

Detailed proofs will appear in a forthcoming article.

The results mentioned here could be compared with those of [1], [5], where the perturbing role of the boundary appears through degeneration of operators rather than explosion of some norms of free terms (and boundary data).

BIBLIOGRAPHY

- [1] : P. BOLLEY - J. CAMUS - G. METIVIER
Estimations de Schauder et régularité Höldérienne pour une classe de problèmes aux limites singuliers.
Comm. in P D E 11, 1135 - 1203 (1985).
- [2] : S. CAMPANATO
Proprietà di Hölderianità di alcune classi di funzioni.
Ann. Sc. Norm. Sup. Pisa 17, 173 - 188 (1963).
- [3] : S. CAMPANATO
Equazioni ellittiche del secondo ordine e spazi $L^{2,\lambda}$.
Ann. Mat. Pura Appl. 69, 321 - 382 (1965).
- [4] : D. GILBARG - L. HORMANDER
Intermediate Schauder estimates.
Arch. Rat. Mech. An. 74, 297 - 318 (1980).
- [5] : C. GOULAOUIC - N. SHIMAKURA
Régularité Höldérienne de certains problèmes aux limites elliptiques dégénérés.
Ann. Sc. Norm. Sup. Pisa, 10 (1983).
- [6] : F. JOHN - L. NIRENBERG
On functions of bounded mean oscillation.
Comm. Pure Appl. Math. 14, 415 - 426 (1961).
- [7] : G. M. TROIANIELLO
Elliptic differential equations and obstacle problems.
Plenum Publ. Co., New-York, (1987).