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ON THE SPECTRUM OF SINGULARLY PERTURBED OPERATORS

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Many problems of mathematical physics lead to the study of the behaviour of eigenvalues and eigenvectors of operators singularly depending on a parameter ε. In many cases these operators act in spaces which also depend on the parameter ε.

An abstract theorem is given here on the behaviour of eigenvalues and eigenvectors of a sequence of operators acting in spaces depending on a parameter ε as $\varepsilon \to 0$. This theorem can be applied to study asymptotic properties of eigenvalues and eigenfunctions of boundary value problems for elliptic equations and systems with rapidly oscillating coefficients in perforated domains with a periodic structure, to study spectral properties of a G-convergent sequence of operators, the spectrum of eigenvalue problems in domains with an oscillating boundary and many other problems. We give here the application of this abstract theorem to the Dirichlet problem for elliptic second order equations in a perforated domain and to the problem of the oscillation of systems with concentrated masses.

Let $\mathcal{H}_\varepsilon, \mathcal{H}_o$ be separable Hilbert spaces with the scalar products $(u^\varepsilon, v^\varepsilon)_\varepsilon$, $(u, v)_o$ respectively and $A_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$, $A_o : \mathcal{H}_o \to \mathcal{H}_o$ be linear continuous operators, $Im\ A_o \subset \mathcal{V} \subset \mathcal{H}_o$, where $\mathcal{V}$ is a linear subspace of $\mathcal{H}_o$, $Im\ A_o \equiv \{v : v = A_o u,\ u \in \mathcal{H}_o\}$, $0 < \varepsilon < 1$.

We assume that spaces $\mathcal{H}_\varepsilon$, $\mathcal{H}_o$, v and operators A_ε, A_o satisfy the following conditions $C_1 - C_4$.

C_1 - *There exist linear continuous operators* $R_\varepsilon : \mathcal{H}_o \to \mathcal{H}_\varepsilon$ *such that for any* $f^o \in \mathcal{V}$:

$$(R_\varepsilon f^o\ ,\ R_\varepsilon f^o)_\varepsilon \to \gamma\,(f^o\ ,\ f^o)$$

as $\varepsilon \to 0$, *where* $\gamma = const > 0$ *and* γ *does not depend on* f^o.

C_2 - *Operators* A_ε, A_o *are positive, compact and selfadjoint, moreover norms* $\| A_\varepsilon \|_{\mathcal{L}(\mathcal{H}_\varepsilon)}$ *are bounded by a constant which does not depend on* ε.

C_3 - *For any* $f^o \in \mathcal{V}$

$$\| A_\varepsilon R_\varepsilon f^o - R_\varepsilon A_o f^o \|_\varepsilon \to 0$$

as $\varepsilon \to 0$ *where* $\| u \|_\varepsilon \equiv (u , u)_\varepsilon^{1/2}$.

C_4 - *The family of operators* A_ε *is uniformly compact in the following sense. From any sequence* $f^\varepsilon \in \mathcal{H}_\varepsilon$, *such that* $\sup_\varepsilon \| f^\varepsilon \|_\varepsilon < \infty$, *one can select a subsequence* $f^{\varepsilon'}$ *and a vector* $w^o \in \mathcal{H}^o$ *for which*

$$\| A_{\varepsilon'} f^{\varepsilon'} - R_{\varepsilon'} w^o \|_{\varepsilon'} \to 0 \quad \text{as} \quad \varepsilon' \to 0$$

Let us consider the spectral problems :

(1) $$A_\varepsilon u_\varepsilon^k = \mu_\varepsilon^k u_\varepsilon^k , \quad k = 1, 2, \ldots , \quad u_\varepsilon^k \in \mathcal{H}_\varepsilon , \quad (u_\varepsilon^\ell , u_\varepsilon^m)_\varepsilon = \delta_{\ell m}$$

(2) $$A_o u^k = \mu_o^k u^k, \quad k = 1, 2, \ldots , \quad u^k \in \mathcal{H}_o , \quad (u^\ell , u^m)_o = \delta_{\ell m}$$

where $\delta_{\ell m}$ is Kronecker's symbol, eigenvalues $\mu_\varepsilon^k , \mu_o^k$ are numbered in nonincreasing order and every eigenvalue is counted as many times as its multiplicity.

The following theorem gives estimates for the difference of eigenvalues μ_ε^k and μ_o^k.

THEOREM 1 : *Let for spaces* $\mathcal{H}_\varepsilon , \mathcal{H}_o$ *and operators* A_ε , A_o *conditions* C_1 – C_4 *be fulfilled. Then there exists a sequence* $\beta_\varepsilon^k \to 0$ *as* $\varepsilon \to 0$, $0 < \beta_\varepsilon^k < \mu_o^k$, *such that*:

(3) $$\left| \mu_\varepsilon^k - \mu_o^k \right| \leq \mu_o^k \gamma^{-\frac{1}{2}} (\mu_o^k - \beta_\varepsilon^k)^{-1} \sup_u \| A_\varepsilon R_\varepsilon u - R_\varepsilon A_o u \|_\varepsilon$$

$k = 1, 2, \ldots$, *where the upper bound is taken over all* u *such that*

$$u \in N(\mu_o^k , A_o) \equiv \{ u \in \mathcal{H}_o , A_o u = \mu_o^k u \} , \ \| u \|_o = 1.$$

Let $k \geq O$, $m \geq 1$ *be integers and* $\mu_o^k > \mu_o^{k+1} = \ldots = \mu_o^{k+m} > \mu_o^{k+m+1}$, *i.e. the multiplicity of the eigenvalue* μ_o^{k+1} *of problem* (2) *is equal to* m. *Then for any* $w \in N(\mu_o^{k+1} , A_o)$, $\| w \|_o = 1$, *there exists a linear combination* u^ε *of eigenvectors* $u_\varepsilon^{k+1} , \ldots , u_\varepsilon^{k+m}$ *of problem* (1) *such that*

(4) $$\| u^\varepsilon - R_\varepsilon w \|_\varepsilon \leq M_k \| A_\varepsilon R_\varepsilon w - R_\varepsilon A_o w \|_\varepsilon$$

where the constant M_k *does not depend on* ε.

A sketch of the proof of this theorem is given in [1], the full proof of theorem 1 and many of its applications to problems of mathematical physics are given in [2].

As an application of theorem 1, we consider the asymptotic behaviour of eigenvalues and eigenfunctions of the Dirichlet boundary value problem for a second order elliptic equation in a perforated domain with a periodic structure, which is characterized by a small parameter ε. This problem was posed by J.L.Lions [3] and considered by M. Vanninathan [4].

Let Ω^ε be a perforated domain of the form $\Omega^\varepsilon = \Omega \cap \varepsilon\,\omega$, where Ω is a smooth, bounded domain in $\mathbb{R}^n$, ω is a non bounded domain in $\mathbb{R}^n$, which is invariant with respect to shifts by vectors whose components are integers and where $\varepsilon\,\omega = \{x : \varepsilon^{-1} x \in \omega\}$. We assume that the boundary of ω is smooth and $0 < \varepsilon \leq 1$. Consider the family of elliptic operators

$$\mathcal{L}_\varepsilon(u) \equiv \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\right) - b\left(\frac{x}{\varepsilon}\right)u$$

where $a_{ij}(\xi)$, $b(\xi)$ are periodic smooth functions with period 1 in $\mathbb{R}^n$. We shall call such functions 1-periodic. Assume that the following conditions are satisfied :

$$\kappa_1 |\eta|^2 \leq a_{ij}(\xi)\,\eta_i\,\eta_j \leq \kappa_2\,|\eta|^2 \qquad \kappa_1, \kappa_2 = const > 0\,,$$
$$a_{ij}(\xi) = a_{ji}(\xi) \quad i, j = 1, \dots, r\,, \qquad b(\xi) \geq 0$$

The summation over repeated indexes is assumed. We will characterize the asymptotics of eigenvalues and eigenfunctions as $\varepsilon \to 0$ for the eigenvalue problem :

$$(5)\qquad \begin{cases} \mathcal{L}_\varepsilon(U_\varepsilon^k) + \Lambda_\varepsilon^k\,\rho\left(\frac{x}{\varepsilon}\right) U_\varepsilon^k = 0 & \text{in } \Omega^\varepsilon \\ U_\varepsilon^k \in H_o^1(\Omega^\varepsilon),\ k = 1, 2, \dots, & \text{and } \left(\rho\left(\frac{x}{\varepsilon}\right) U_\varepsilon^k\,,\ U_\varepsilon^j\right)_{L^2(\Omega^\varepsilon)} = \delta_{kj} \end{cases}$$

where $\rho(\xi)$ is a smooth, 1-periodic function in $\mathbb{R}^n$, $\rho(\xi) \geq C_o = const > 0$, eigenvalues are numbered in such a way that $\Lambda_\varepsilon^k \leq \Lambda_\varepsilon^{k+1}$, $k = 1, 2, \dots$, and every eigenvalue is repeated as many times as its multiplicity. Here $H_o^1(\Omega^\varepsilon)$ is the Sobolev space, which consists of functions v, such that :

$$v \in L^2(\Omega^\varepsilon)\,,\ \frac{\partial v}{\partial x_j} \in L^2(\Omega^\varepsilon)\,,\ j = 1, \dots, n\,, \quad \text{and } v = 0 \text{ on } \partial\Omega^\varepsilon.$$

Let $\Phi(\xi)$ be an eigenfunction which corresponds to the first eigenvalue Λ_o of the eigenvalue problem in domain ω with 1-periodic structure

$$(6) \qquad \begin{cases} \dfrac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\dfrac{\partial \Phi}{\partial \xi_j}\right) + \Lambda_o\, \rho(\xi)\Phi(\xi) = 0 \quad \text{in } \omega, \quad \Phi = 0 \text{ on } \partial\omega \\ \Phi(\xi) \text{ is 1-periodic in } \xi, \quad \text{and} \quad \langle \rho(\xi)\Phi^2(\xi) \rangle = 1 \end{cases}$$

where $\langle f \rangle \equiv \int_{Q\cap\omega} f(\xi)\, d\xi$, $Q = \{x: 0 < x_j < 1, j = 1, \dots, n\}$.

As is well known, $\Phi(\xi)$ is a smooth function in ω, $\Phi \neq 0$ in ω and $|\nabla_\xi \Phi| \neq 0$ in a neighborhood of $\partial\omega$.

We represent eigenfunction U_ε^k of problem (5) in the form

$$U_\varepsilon^k(x) = \Phi\left(\frac{x}{\varepsilon}\right) v_\varepsilon^k(x)$$

It is easy to verify that v_ε^k, $k = 1, 2, \dots$, has to be an eigenfunction of the problem :

$$(7) \qquad \begin{cases} M_\varepsilon(v_\varepsilon^k) + \lambda_\varepsilon^k\, \rho\left(\frac{x}{\varepsilon}\right) \Phi^2\left(\frac{x}{\varepsilon}\right) v_\varepsilon^k = 0 \qquad \text{in } \Omega^\varepsilon \\ v_\varepsilon^k = 0 \text{ on } \partial\Omega^\varepsilon \cap \partial\Omega, \text{ and } \left(\rho\left(\frac{x}{\varepsilon}\right) \Phi^2\left(\frac{x}{\varepsilon}\right) v_\varepsilon^k(x),\, v_\varepsilon^j(x)\right)_{L^2(\Omega^\varepsilon)} = \delta_{kj} \end{cases}$$

where $\lambda_\varepsilon^k = \Lambda_\varepsilon^k - \Lambda_o\, \varepsilon^{-2}$,

$$(8) \qquad M_\varepsilon(u) \equiv \frac{\partial}{\partial x_i}\left(\Phi^2\left(\frac{x}{\varepsilon}\right) a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_j}\right) - \Phi^2\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}\right) u$$

Thus, we obtain eigenvalue problem (7) for a second order elliptic equation which degenerates on the part of the boundary of the domain Ω^ε which is given by $\partial\Omega^\varepsilon \cap \Omega$.

By the introduction of weak solutions of the corresponding degenerate boundary value problem one can reduce the eigenvalue problem (7) to the eigenvalue problem for a compact, selfadjoint operator A_ε in the Hilbert space $\mathcal{H}_\varepsilon$. It can be proved that problem (7) has a discrete spectrum which is a non-decreasing sequence of eigenvalues λ_ε^1 , $\lambda_\varepsilon^2, \dots$, where λ_ε^k are numbered with regard for the multiplicity in the same way as for problem (5).

If v_ε^k is an eigenfunction of problem (7) with eigenvalue λ_ε^k, then the function $\Phi(\frac{x}{\varepsilon})\, v_\varepsilon^k(x)$ belongs to the space $H_1^0(\Omega^\varepsilon)$ and it is an eigenfunction of problem (5), corresponding to the eigenvalue

$$\Lambda_\varepsilon^k = \lambda_\varepsilon^k + \varepsilon^{-2}\,\Lambda_o$$

The theory of homogenization can be applied to the operators $M_\varepsilon(u)$. We consider now the boundary value problem for operators $M_\varepsilon(u)$.

$$M_\varepsilon(u^\varepsilon) = -\rho\left(\frac{x}{\varepsilon}\right)\Phi^2\left(\frac{x}{\varepsilon}\right) f^\varepsilon \text{ in } \Omega^\varepsilon, \qquad u^\varepsilon = 0 \text{ on } \partial\Omega^\varepsilon \cap \partial\Omega \tag{9}$$

We note that there are no boundary conditions on the boundary $\partial\Omega^\varepsilon \cap \Omega$. We assume that $f^\varepsilon \in L^2(\Omega^\varepsilon)$ and that u^ε belongs to the space V^ε which consists of functions u such that :

$$\Phi\left(\frac{x}{\varepsilon}\right) u(x) \in L^2(\Omega^\varepsilon), \quad \Phi\left(\frac{x}{\varepsilon}\right)\frac{\partial u(x)}{\partial x_j} \in L^2(\Omega^\varepsilon), \quad j = 1, \dots, n$$

Using the Lax-Milgram theorem, one can prove the existence and the uniqueness of a solution of problem (9).

The homogenized problem for problem (9) has the form :

$$\overline{M}(u) = -\,\rho f^0 \quad \text{in}\,\Omega, \quad u \in H_o^1(\Omega) \tag{10}$$

where

$$\left\{\begin{array}{ll} \overline{M}(u) \equiv \overline{a}^{pq}\dfrac{\partial^2 u}{\partial x_p\,\partial x_q} - \overline{b}\,u\,, & \overline{\rho} = \theta < \Phi^2(\xi)\,\rho(\xi) > \\ \overline{b} = \theta < \Phi^2(\xi)\,b(\xi) >, & \theta = < \Phi^2(\xi) >^{-1} \\ \overline{a}^{pq} = \theta < a_{pj}\,\Phi^2(\xi)\,\dfrac{\partial N_q(\xi)}{\partial \xi_j} + a_{pq}(\xi)\,\Phi^2(\xi) > & \end{array}\right. \tag{11}$$

and the functions $N_q(\xi)$ are solutions of the following boundary value problems :

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial \xi_k}\left(\Phi^2(\xi)\,a_{kj}(\xi)\,\dfrac{\partial N_q(\xi)}{\partial \xi_j}\right) = -\dfrac{\partial}{\partial \xi_k}\left(a_{kq}(\xi)\,\Phi^2(\xi)\right) \quad \text{in } \omega,\ q = 1,\dots,n \\ N_q(\xi) \text{ is 1-periodic in } \xi, \quad < \Phi^2 N_q > = 0 \\ \Phi\,\dfrac{\partial N_q}{\partial \xi_j} \in L^2(Q \cap w),\ j = 1, \dots, n. \end{array}\right. \tag{12}$$

The operator $\overline{M}$ is an elliptic operator with constant coefficients. The corresponding eigenvalue problem has the form

$$\overline{M}(v_o^k) + \lambda_o^k\,\rho\,v_o^k = O \text{ in } \Omega,\ v_o^k \in H_o^1(\Omega),\ (\rho\,v_o^\ell, v_o^m) = \delta_{\ell m} \tag{13}$$

where the eigenvalues λ_o^k are numbered with regard for the multiplicity and $\lambda_o^k \leq \lambda_o^{k+1}$.

THEOREM 2 : *Let* Λ_ε^k, λ_ε^k, λ_o^k *be the* k*–th eigenvalues of the eigenvalue problems* (5), (7), (13) *respectively. Then*

(14) $$\Lambda_\varepsilon^k = \Lambda_o\, \varepsilon^{-2} + \lambda_\varepsilon^k, \qquad |\lambda_\varepsilon^k - \lambda_o^k| \leq C_k\, \varepsilon$$

where the constant C_k *does not depend on* ε.

If the multiplicity of the eigenvalue $\lambda_o = \lambda_o^{\ell+1}$ *is equal to* m, *i.e.*, $\lambda_o^{\ell+1} = \dots = \lambda_o^{\ell+m}$, *and* $v_o(x)$ *is an eigenfunction of problem* (13) *with eigenvalue* λ_o, *then for all* $\varepsilon \in]0, 1]$ *there exists a function* v^ε *such that*

(15) $$\| (v^\varepsilon - v_o)\, \Phi(\frac{x}{\varepsilon}) \|_{L^2(\Omega^\varepsilon)} \leq M_\ell\, \varepsilon$$

where v^ε *is a linear combination of eigenfunctions of the eigenvalue problem* (7), *corresponding to eigenvalues* $\lambda_\varepsilon^{\ell+1}, \dots, \lambda_\varepsilon^{\ell+m}$.

In order to prove theorem 2 one has to introduce suitable spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}_o$ and operators A_ε, A_o such that $A_\varepsilon f^\varepsilon = u^\varepsilon$, $A_o f^o = u$ and u^ε, u are solutions of boundary value problems (9), (10) respectively. One can check that conditions $C_1 - C_4$ are satisfied and therefore theorem 1 is applicable. The eigenvalues of problems (7), (13) and problems (1), (2) are connected by the relations :

$$(\lambda_\varepsilon^k)^{-1} = \mu_\varepsilon^k, \qquad (\lambda_o^k)^{-1} = \mu_o^k$$

Using methods of the theory of homogenization, one can estimate the right-hand side of (3) and prove that it is of order ε. Therefore, estimates (14), (15) are consequences of (3), (4).

The full proof of theorem 2 is given in [2].

Let us consider now another example of the application of theorem 1. We shall study asymptotics of vibrating systems with concentrated masses. These problems were considered in [5] - [11].

Let Ω be a smooth, bounded domain, $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $O \in \Omega$ and O is an origin. Consider the eigenvalue probem :

$$(16) \qquad \begin{cases} \Delta u_\varepsilon^k + \lambda_\varepsilon^k (1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})) u_\varepsilon^k = 0 & \text{in } \Omega, \quad u_\varepsilon^k = 0 \quad \text{on } \partial\Omega \\ (\{1 + \varepsilon^{-m} \chi(\frac{x}{\varepsilon})\} u_\varepsilon^k, u_\varepsilon^j)_{L^2(\Omega)} = \delta_{kj} \end{cases}$$

where $\varepsilon \in]0, 1[$, λ_ε^j are numbered with regard for the multiplicity and $\lambda_\varepsilon^k \leq \lambda_\varepsilon^{k+1}$. Here $\chi(\xi)$ is a bounded measurable function such that $\chi(\xi) > M = const > 0$ for $\xi \in G$ and $\chi(\xi) = 0$ for $\xi \notin G$, where G is an open set of positive measure such that $\overline{G} \subset \Omega$, $O \in G$.

THEOREM 3 : I – *Assume that* $-\infty < m < 2$. *Then*

$$|\lambda_\varepsilon^k - \lambda_o^k| \leq C_k \ \varepsilon^{1-m+n/2}$$

where the constant C_k *does not depend on* ε *and* λ_o^k *is the k–th eigenvalue of the Dirichlet eigenvalue problem in* Ω

$$(17) \qquad \Delta u_o^k + \lambda_o^k u_o^k = 0, \qquad u_o^k = 0 \quad on\ \partial\Omega$$

II – *Assume that* $m > 2$. *Then*

$$\lambda_\varepsilon^k = \Lambda_o^k \varepsilon^{m-2} + \varepsilon^{m-2} \alpha_\varepsilon^k$$

where Λ_o^k *is the k–th eigenvalue of the eigenvalue problem*

$$\Delta_\xi u_o^k + \Lambda_o^k \chi(\xi) u_o^k = 0, \quad \xi \in \mathbb{R}^n, \quad u_o^k \in H$$

Here H *his the completion of* $C_o^\infty(\mathbb{R}^n)$ *– functions with the norm*

$$\| u \|_H = \| \nabla_\xi u \|_{L^2(\mathbb{R}^n)}$$

$$\alpha_\varepsilon^k = C_k (\varepsilon^{n-2} + \varepsilon^{(m-\gamma_n)/2}),$$

$\gamma_3 = 1$, $\gamma_4 = const \in]0, 1]$, $\gamma_n = 0$ *for* $n > 4$ *and the constant* C_k *does not depend on* ε.

III – *Assume that* $m = 2$. *Then*

$$|\lambda_\varepsilon^k - \Lambda^k| \leq C_k \ \varepsilon^{(2-\gamma_n)/2}$$

where the constant C_k *does not depend on* ε. *Here* γ_n *is defined as above and* Λ^k *is the k–th eigenvalue of the eigenvalue problem for the system*

$$\Delta_\xi U_o^k(\xi) + \Lambda^k \chi(\xi) U_o^k(\xi) = 0 \quad in\ \mathbb{R}^n, \quad U_o^k \in H$$

$$\Delta_x u_o^k(x) + \Lambda^k u_o^k(x) = 0, \qquad u_o^k \in H_o^1(\Omega).$$

In order to prove theorem 3 in the case $-\infty < m < 2,\ n \geq 3$, we introduce spaces $\mathcal{H}_\varepsilon$, $\mathcal{H}_o$ and operators A_ε, A_o in such a way that conditions $C_1 - C_4$ of theorem 1 are satisfied. Let $\mathcal{H}_\varepsilon$ and $\mathcal{H}_o$ be $L^2(\Omega)$ spaces with the scalar products :

$$(f^\varepsilon, g^\varepsilon)_{\mathcal{H}_\varepsilon} \equiv \int_\Omega (1+\varepsilon^{-m}\chi(\tfrac{x}{\varepsilon}))f^\varepsilon . g^\varepsilon\, dx \qquad (f^o, g^o)_{\mathcal{H}_o} \equiv \int_\Omega f^o . g^o\, dx$$

respectively. We set :

$$R_\varepsilon f^o = (1 - \mathcal{X}_G(\tfrac{x}{\varepsilon}))f^o \qquad f^o \in \mathcal{H}_o$$

where $\mathcal{X}_G(\xi)$ is the characteristic function of the set G : $\mathcal{X}_G(\xi) = 1$ for $\xi \in G$, $\mathcal{X}_G(\xi) = 0$ for $\xi \notin G$.

We define the operator $A_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ in the following way : for $f^\varepsilon \in \mathcal{H}_\varepsilon$

$$A_\varepsilon f^\varepsilon = u^\varepsilon$$

where u^ε is a solution of the Dirichlet problem

$$\Delta_x u^\varepsilon = -(1+\varepsilon^{-m}\chi(\tfrac{x}{\varepsilon}))f^\varepsilon \quad \text{in } \Omega, \qquad u^\varepsilon \in H_o^1(\Omega)$$

We denote by A_o an operator $A_o : \mathcal{H}_o \to \mathcal{H}_o$ such that for any $f^o \in \mathcal{H}_o$ we have :

$$A_o f^o = u^o$$

where u^o is a solution of the Dirichlet problem

$$\Delta u^o = -f^o(x) \quad \text{in } \Omega, \quad u^o \in H_o^1(\Omega).$$

It is easy to verify that conditions $C_1 - C_4$ are satisfied.

Consider the case $m > 2$, $n \geq 3$. Let u^o be a solution of the problem

(18) $$\Delta_\xi u^o = -\chi(\xi) f^o \quad \text{in } \mathbb{R}^n, \qquad f^o \in L^2_{loc}(\mathbb{R}^n), \qquad u^o \in H$$

We define the space $\mathcal{H}_o$ as $L^2(G)$ with the scalar product

$$(f^o, g^o)_{\mathcal{H}_o} = \int_G \chi(\xi) f^o g^o\, d\xi$$

We set $A_o f^o = \mathcal{X}_G(\xi)\, u^o$, where u^o is a solution of problem (18). It can be proved that A_o is a positive, compact, selfadjoint operator, $A_o : \mathcal{H}_o \to \mathcal{H}_o$.

The space $\mathcal{H}_\varepsilon$ is defined as $L^2(\Omega_\varepsilon)$ with the scalar product

$$(f^\varepsilon, g^\varepsilon)_{\mathcal{H}_\varepsilon} = \int_{\Omega_\varepsilon} (\varepsilon^m + \chi(\xi))\, f^\varepsilon g^\varepsilon\, d\xi$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$ and $\varepsilon^{-1}\Omega = \{x : \varepsilon x \in \Omega\}$. The operator $R_\varepsilon : \mathcal{H}_o \to \mathcal{H}_\varepsilon$ is an extension of $f^o \in L^2(G)$ such that $f^o = 0$ in $\Omega_\varepsilon \setminus G$. We define $A_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ by setting :

$$A_\varepsilon f^\varepsilon = u^\varepsilon$$

where u^ε is a solution of the boundary value problem :

$$\Delta_\xi u^\varepsilon = -(\varepsilon^m + \chi(\xi)) f^\varepsilon \quad \text{in } \Omega_\varepsilon , \qquad u^\varepsilon \in H_o^1(\Omega_\varepsilon).$$

For $\mathcal{H}_\varepsilon, \mathcal{H}_o, A_\varepsilon, A_o$ defined as above, conditions $C_1 - C_4$ of theorem 1 are satisfied and theorem 3 in the cas $m > 2$ is also a consequence of theorem 1.

In a similar way the case $m = 2$ can be considered. The detailed proof of theorem 3 is given in [2].

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