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# Some remarks on the multi-dimensional Borg-Levinson theorem

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**1.1-dim-case.** The Borg-Levinson theorem is a uniqueness theorem in inverse eigenvalue problems. We first recall the 1-dim case. Consider the Dirichlet problem :

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \leq x \leq 1, \\ y(0) = y(1) = 0, \end{cases}$$

$q(x)$  being a real function. Let

$$\lambda_1 < \lambda_2 < \dots$$

be the eigenvalues. Now suppose for two potentials  $q_1, q_2$ ,

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1.$$

One can then easily see that it does not necessarily imply  $q_1 = q_2$ . To derive the uniqueness of the potential, we must add some auxiliary conditions.

Let  $y = y(x, \lambda) = y(x, \lambda, q)$  be the solution of the Cauchy problem :

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \leq x \leq 1, \\ y(0) = 0, y'(0) = 1. \end{cases}$$

Then we have

**THEOREM (Borg-Levinson).** Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geq 1,$$

$$y'(1, \lambda_i, q_1) = y'(1, \lambda_i, q_2) \text{ for all } i \geq 1.$$

Then  $q_1 = q_2$ .

This is a starting point of 1-dim. inverse problems ([1],[2]). The recent

article of Pöschel–Trabowitz [3] gives a deep insight. It is proved that the map

$$q \rightarrow \{\lambda_i\}_{i=1}^{\infty} \times \{\log|y'(1, \lambda_i, q)|\}_{i=1}^{\infty}$$

defines an analytic isomorphism from  $L^2(0,1)$  to a Hilbert space of infinite sequences.

And also, for any fixed potential  $p$ , the set defined by

$$M(p) = \{q ; \lambda_i(q) = \lambda_i(p) \text{ for } \forall i \geq 1\}$$

is a real analytic manifold (isospectral manifold) with the system of coordi- nates

$$\{\log|y'(1, \lambda_i, q)|\}_{i=1}^{\infty}.$$

Since  $y(x, \lambda_i, q)$  is an eigenfunction of  $-\frac{d^2}{dx^2} + q(x)$  associated with the eigenvalue  $\lambda_i$ , one can see that there is a one to one correspondance between the potential and the eigenvalues and the normal derivatives of eigenfunctions.

**2. n-dim. case.** Next we turn to the  $n$ -dim. case ( $n \geq 2$ ). Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $S$ . Consider the Dirichlet problem :

$$\begin{cases} (-\Delta + q)u = \lambda u \text{ in } \Omega, \\ u|_S = 0. \end{cases}$$

Although we treat the Dirichlet problem here, all of the arguments below also hold for the Neumann or Robin boundary conditions by a suitable modification.

Let  $\lambda_1 < \lambda_2 \leq \dots$  be the eigenvalues. To derive the uniqueness theorem corresponding to the 1-dim. case, we consider the normal derivatives of eigenfunctions. However, one must be careful to choose a system of eigenfunctions, since in the multi-dimensional case eigenvalues are not simple in general.

Let  $m$  be the multiplicity of  $\lambda_i$  and  $u_1, \dots, u_m$  be a real-valued orthonormal eigenfunctions system associated with  $\lambda_i$ . We set

$$E_i = \left\{ \left( \frac{\partial u_1}{\partial \nu}, \dots, \frac{\partial u_m}{\partial \nu} \right) \right\}.$$

$\nu$  being the outer unit normal to  $S$ . One can then see that for two such systems of eigenfunctions  $\{u_1, \dots, u_m\}, \{v_1, \dots, v_m\}$ , there exists an orthogonal matrix  $T \in O(m)$  such that

$$\left( \frac{\partial u_1}{\partial v}, \dots, \frac{\partial u_m}{\partial v} \right) = \left( \frac{\partial v_1}{\partial v}, \dots, \frac{\partial v_m}{\partial v} \right) T.$$

Now, this defines an equivalent relation  $\sim$  in the space of functions on the boundary  $S$ . Further, it shows that for the set  $\{E_i\}$ , the totality of  $E_i$ , there corresponds only one equivalence class, which we denote by  $W_i$ :

$$W_i = \{E_i\} / \sim.$$

Then, we have the following theorem due to Nachman - Sylvester - Uhlmann [4].

**THEOREM A.** Let  $q_1, q_2 \in C^\infty(\overline{\Omega})$ . Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for } \forall i \geq 1,$$

$$W_i(q_1) = W_i(q_2) \text{ for } \forall i \geq 1.$$

Then  $q_1 = q_2$ .

This theorem seems to be a direct generalization of the 1-dimensional Borg - Levinson theorem. So, it is natural to ask : does the map  $q \rightarrow \{\lambda_i\} \times \{W_i\}$  define an isomorphism ? Can  $\{W_i\}$  be the coordinates of the isospectral set of potentials ? The answer is always negative. In fact, we have

**THEOREM B.** Let  $q_1, q_2 \in C^\infty(\overline{\Omega})$ . Suppose that there exists an  $N > 0$  such that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for } \forall i > N,$$

$$W_i(q_1) = W_i(q_2) \text{ for } \forall i > N.$$

Then  $q_1 = q_2$ .

In other words, if  $\lambda_i$  and  $W_i$  are equal except for a finite number of indices  $i$ , the potentials are equal. It also shows that the totality of  $\lambda_i$  and  $W_i$  is too much to determine the potential. It is a common belief that, in contrast to the 1-dim. case, the multi-dimensional inverse eigenvalue problem has a sort of rigidity. Here one can find

an exemple.

**3. Proof of Theorem B.** We sketch the proof of theorem B. Let  $N(\lambda)$  be the Neumann operator, mamely,

$$N(\lambda)f = \frac{\partial v}{\partial \nu},$$

where  $v$  satisfies

$$\begin{cases} (-\Delta + q)v = \lambda v, \\ v|_S = f. \end{cases}$$

We introduce the following notation :

$$\begin{aligned} (f, g) &= \int_{\Omega} f(x) \overline{g(x)} dx, \\ \langle f, g \rangle &= \int_S f(x) \overline{g(x)} dS, \\ \varphi_{\lambda, \omega}(x) &= e^{i\sqrt{\lambda} \omega \cdot x}, \quad \omega \in S^{n-1}, \lambda \in \mathbb{C}. \end{aligned}$$

Let  $S(\lambda, \theta, \omega)$  be defined by

$$S(\lambda, \theta, \omega) = \langle N(\lambda)\varphi_{\lambda, \omega}, \overline{\varphi_{\lambda, -\theta}} \rangle$$

The crucial fact is the following lemma.

**LEMMA C.** If  $\lambda \neq$  eigenvalue,

$$\begin{aligned} S(\lambda, \theta, \omega) &= -\frac{\lambda}{2} (\theta - \omega)^2 \int_{\Omega} e^{-i\sqrt{\lambda}(\theta - \omega)x} dx \\ &\quad + \int_{\Omega} e^{-i\sqrt{\lambda}(\theta - \omega)x} q(x) dx \\ &\quad - (R(\lambda) q \varphi_{\lambda, \omega}, q \overline{\varphi_{\lambda, -\theta}}), \end{aligned}$$

where  $R(\lambda) = (-\Delta_D + q - \lambda)^{-1}$ .

Note that the above expression is similar to the S-matrix in scattering theory.

Now, we recall the Born approximation.

Let  $\mathbb{R}^n \ni \xi \neq 0$  be arbitrarily fixed. Take  $\eta \in S^{n-1}$  such that  $n^\perp \xi$ . For a large parameter  $N$ , we define

$$\begin{cases} \theta_N = C_N \eta + \xi/2N, & C_N = (1 - |\xi|^2/4N^2)^{1/2}, \\ \omega_N = C_N \eta - \xi/2N, \\ \sqrt{t_N} = N + i. \end{cases}$$

They have the following properties :

$$\begin{aligned} \theta_N, \omega_N &\in S^{n-1}, \\ \sqrt{t_N}(\theta_N - \omega_N) &\rightarrow \xi, \text{ as } N \rightarrow \infty, \\ \text{Im } t_N &\rightarrow \infty \text{ as } N \rightarrow \infty, \\ \text{Im } \sqrt{t_N} \theta_N, \text{Im } \sqrt{t_N} \omega_N &\text{ are bounded.} \end{aligned}$$

Invoking these properties, one can easily show

**THEOREM D.**

$$\lim_{N \rightarrow \infty} S(t_N, \theta_N, \omega_N) = -\frac{|\xi|^2}{2} \int_{\Omega} e^{-ix\xi} dx + \int_{\Omega} e^{-ix\xi} q(x) dx.$$

So, one can reconstruct the potential from  $S(\lambda, \theta, \omega)$ .

Now, we prove Theorem B.  $N(\lambda)$  has, formally, the integral kernel :

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \left( \frac{\partial \varphi_i}{\partial \mathbf{v}} \right)(\mathbf{x}) \left( \frac{\partial \varphi_i}{\partial \mathbf{v}} \right)(\mathbf{y}),$$

where  $\varphi_i$  is the eigenfunction. In view of this expression one can show that the assumption of Theorem B implies

$$\|N(\lambda, q_1) - N(\lambda, q_2)\|_{B(L^2(S^{n-1}))} \leq C/|\lambda|$$

for large  $|\lambda|$ . Theorem B then follows from this inequality and theorem D.

From the very proof, one can see that the potential is uniquely determined by the asymptotic properties of the eigenvalues and eigenfunctions.

**4. Variable coefficient case.** We briefly mention the variable coefficients case.

Consider the operator

$$H = - \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + q(x).$$

Assume that for  $|\alpha| \leq N$ ,  $N$  is chosen large enough,

$$\sup_{x \in \Omega} |\partial_x^\alpha (a_{ij}(x) - \delta_{ij})| = \delta < 1.$$

Then the above Theorem B, Lemma C, Theorem D also hold in this case. The proof relies on the method of asymptotic solutions and Fourier integral operators. Note that we are fixing  $a_{ij}$  and seeking  $q(x)$ .

The above results may be extended to higher order elliptic operators and elliptic systems.

## References

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