

JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1989), p. 1-9

http://www.numdam.org/item?id=JEDP_1989____A3_0

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SPECTRAL ANALYSIS ON SINGULAR SPACES

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1. In this note we describe some spectral analysis which is naturally attached to arbitrary Riemannian manifolds. These considerations are quite simple but they may prove useful since they give a common structure to rather diverse problems. We illustrate this with some examples, including various very recent results. All facts for which we give no reference will be published elsewhere.

Our model is the analysis of “geometric” operators on compact Riemannian manifolds. On such a manifold, M , of dimension m , consider hermitean vector bundles E_i and differential operators $d_i : C_0^\infty(E_i) \rightarrow C_0^\infty(E_{i+1})$ with the property $d_{i+1} \circ d_i = 0$. Thus we obtain a complex

$$0 \rightarrow C_0^\infty(E_0) \xrightarrow{d_0} C_0^\infty(E_1) \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} C_0^\infty(E_N) \rightarrow 0. \quad (1.1)$$

As is well known, the complex is effectively studied via the operator $d : C_0^\infty(E_{ev}) \rightarrow C_0^\infty(E_{odd})$, where

$$\begin{aligned} E_{ev} &:= \bigoplus_{i \geq 0} E_{2i}, & E_{odd} &:= \bigoplus_{i \geq 0} E_{2i+1}, \\ d(u_0, u_2, \dots) &:= (d_0 u_0 + d_1^t u_2, d_2 u_2 + d_3^t u_4, \dots), \end{aligned} \quad (1.2)$$

and d^t denotes the (formal) adjoint of d_i . From d we obtain a symmetric differential operator $\Delta : C_0^\infty(E) \rightarrow C_0^\infty(E)$, where

$$E := E_{ev} \oplus E_{odd}, \quad \Delta := d^t d \oplus d d^t. \quad (1.3)$$

We will now call the complex *geometric* or a *Dirac complex* if for the principal symbol of Δ we have

$$\sigma(\Delta)(\xi) = |\xi|^2, \quad (1.4)$$

i.e. Δ is a generalized Laplace operator. It follows that $d + d^t$ is a generalized Dirac operator on E in the sense of Gromov and Lawson [Gr+La], and that the splitting $E = E_{ev} \oplus E_{odd}$ is admissible. Conversely, every generalized Dirac operator arises in this way.

Since M is compact, Δ is essentially self-adjoint and the unique self-adjoint extension, also denoted by Δ , is the object of spectral analysis in this case. From ellipticity we see that Δ has a discrete spectrum. This is linked to the geometry of M for example by the classical asymptotic expansion of Minakshisundaram-Pleijel:

$$\text{tr } e^{-t\Delta} \underset{t \searrow 0}{\sim} (4\pi t)^{-m/2} \sum_{j \geq 0} a_j t^j. \quad (1.5)$$

The a_j are recursively defined and locally computable in terms of the geometric data, but the actual computation is possible only for very small values of j . Nevertheless, the method can be used to prove the Atiyah-Singer Index Theorem (cf. [At-Bo-P], [Ge]), via

$$\begin{aligned} \text{ind } d &= \text{tr } e^{-td^*d} - \text{tr } e^{-td d^*} \\ &=: \text{tr}_s e^{-t\Delta}, \end{aligned} \tag{1.6}$$

where tr_s stands for the supertrace.

Another result of importance is the Hodge Theorem which asserts an isomorphism between the cohomology of the complex (1.1) and the finite dimensional vector spaces $\ker \Delta \cap C_0^\infty(E_i)$.

Our general remarks will take these basic results as guideline.

2. From the results mentioned above we abstract the notion of a *Hilbert complex*. By this we mean a family of Hilbert spaces, H_i , and densely defined closed operators $D_i \in \mathcal{C}(H_i, H_{i+1})$, the set of all closed operators which are densely defined in H_i and have values in H_{i+1} . We denote by $\mathcal{D}_i, \mathcal{R}_i$ the domain and the image of D_i , and we require that $D_i(\mathcal{D}_i) \subset \mathcal{D}_{i+1}$ and $D_{i+1}D_i = 0$. Thus we obtain a complex

$$0 \rightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \dots \xrightarrow{D_{N-1}} \mathcal{D}_N \rightarrow 0. \tag{2.1}$$

We denote by

$$\mathcal{H}_i := \ker D_i / \text{im } D_{i-1} \tag{2.2}$$

the cohomology groups of this complex, and we call

$$\beta_i := \dim \mathcal{H}_i \tag{2.3}$$

the i^{th} *geometric Betti number* of the complex (2.1). If the β_i are all finite we call

$$\chi(\mathcal{D}, D) := \sum_{i \geq 0} (-1)^i \beta_i \tag{2.4}$$

the *geometric index*. We get immediately

Lemma 2.1 (Weak Hodge decomposition) Denote by D_i^* the adjoint operator of D_i , with domain \mathcal{D}_i^* and range \mathcal{R}_i^* . With

$$\hat{\mathcal{H}}_i := \ker D_i \cap \ker D_{i-1}^* \tag{2.5}$$

we have the orthogonal decomposition

$$H_i = \hat{\mathcal{H}}_i \oplus \overline{\mathcal{R}_{i-1}} \oplus \overline{\mathcal{R}_i^*}. \tag{2.6}$$

This leads us to define

$$\hat{\beta}_i := \dim \hat{\mathcal{H}}_i \tag{2.7}$$

as the i^{th} *analytic Betti number* of the complex and, in case of finiteness,

$$\hat{\chi}(\mathcal{D}, D) := \sum_{i \geq 0} (-1)^i \hat{\beta}_i \quad (2.8)$$

as the *analytic index*. The case of compact manifolds is mirrored abstractly in the following result. To formulate it we introduce the operators

$$D : \mathcal{D}_{\text{ev}} \rightarrow \mathcal{D}_{\text{odd}} \quad (2.9)$$

and

$$\Delta := D^* D \oplus D D^* \quad (2.10)$$

in analogy with (1.2) and (1.3); note that D is closed and that Δ is self-adjoint.

Theorem 2.2 *The following conditions are equivalent.*

- 1) β_i is finite for all i .
- 2) D is a Fredholm operator.
- 3) $0 \notin \text{spec}_e \Delta :=$ the essential spectrum of Δ .

If one of these conditions is satisfied then each \mathcal{R}_i is closed and $\beta_i = \hat{\beta}_i$.

If one of the above conditions holds then we call (2.1) a *Fredholm complex*. In this case the weak Hodge decomposition becomes a *strong Hodge decomposition* i.e.

$$H_i = \hat{\mathcal{H}}_i \oplus \mathcal{R}_{i-1} \oplus \mathcal{R}_i^* . \quad (2.11)$$

The natural object of spectral analysis with Hilbert complexes is, of course, the self-adjoint operator Δ . In this generality we will, however, not be able to say anything about the nature of the spectrum. But it is useful to note that certain properties of a Hilbert complex are invariant under deformations. To make this precise we introduce the notion of a *complex map*: if we have two Hilbert complexes, (\mathcal{D}, D) and (\mathcal{D}', D') , then a complex map, g , is a family of bounded linear maps $g_i : H_i \rightarrow H'_i$ such that

$$g_i(\mathcal{D}_i) \subset \mathcal{D}'_i, \quad D'_i \circ g_i = g_{i+1} \circ D_i . \quad (2.12)$$

Theorem 2.3 *Let (\mathcal{D}, D) and (\mathcal{D}', D') be isomorphic Hilbert complexes. Then*

- 1) $\beta_i = \beta'_i$ for all i ,
- 2) $\hat{\beta}_i = \hat{\beta}'_i$ for all i ,
- 3) $\text{spec}_e \Delta = \emptyset$ if and only if $\text{spec}_e \Delta' = \emptyset$.

Now let M be an *arbitrary* Riemannian manifold and consider a Dirac complex $(C_0^\infty(E), d)$ as in (1.1); this is well defined since we require compact supports. More generally, consider vector spaces $\Gamma_i \subset C_0^\infty(E_i)$ with

$$d_i(\Gamma_i) \subset \Gamma_{i+1}, \quad (2.13a)$$

$$d_i^t(\Gamma_i) \subset \Gamma_{i-1}, \quad (2.13b)$$

The Riemannian metric and the hermitean structure define the Hilbert space $L^2(E_i)$; we require that Γ_i is large in the sense that its closure, $L^2\Gamma_i$, in $L^2(E_i)$ satisfies

$$C_0^\infty(E_i) \cap L^2\Gamma_i = \Gamma_i . \quad (2.14)$$

We will call the complex (Γ, d) a Dirac complex, too, and refer to $(C_0^\infty(E), d)$ as a *full Dirac complex*.

We ask whether we can associate to the complex (Γ, d) a Hilbert complex (\mathcal{D}, D) . Thus we want to find closed extensions D_i of d_i , with domain \mathcal{D}_i , such that (\mathcal{D}, D) becomes a Hilbert complex. Note that d_i and d_i^t have closed extensions being differential operators. In particular, we have the *closures* or *minimal extensions* $d_{i,\min}$, $d_{i,\min}^t$, and the *maximal extensions*

$$d_{i,\max} := (d_{i,\min}^t)^*, \quad d_{i,\max}^t = d_{i,\min}^*.$$

Any closed extension, D_i , of d_i then satisfies $d_{i,\min} \subset D_i \subset d_{i,\max}$. Following Cheeger [Ch1] we will call any choice of closed extensions for the d_i that produces a Hilbert complex an *ideal boundary condition*. We find without difficulty

Lemma 2.4 *Ideal boundary conditions exist for any Dirac complex (Γ, d) . For example, we may choose $D_i = d_{i,\min}$ or $D_i = d_{i,\max}$.*

Thus we can associate to each Dirac complex the self-adjoint extensions of Δ coming from the various ideal boundary conditions. It is reasonable (cf. [Ch1]) to call the extensions coming from $d_{i,\min}$ and $d_{i,\max}$ the *Dirichlet* and the *Neumann extension*, respectively, to be denoted by Δ_D and Δ_N . It is interesting to remark that the Friedrichs extension, Δ_F , may not belong to an ideal boundary condition (cf. Section 3 for an example). Thus the search for Hilbert complexes leads to distinguished boundary conditions which are not easily detected on the level of the Dirac operator $d : C_0^\infty(E_{\text{ev}}) \rightarrow C_0^\infty(E_{\text{odd}})$ or the Laplace operator $\Delta : C_0^\infty(E) \rightarrow C_0^\infty(E)$. Besides the cohomology theory these “geometric” boundary conditions are a good motivation to study Hilbert complexes.

3. We now explain how various spectral problems on singular spaces fit into this abstract framework.

1) Complete noncompact Riemannian manifolds

By well known results, in this case Δ is essentially self-adjoint for each Dirac complex, hence there is a unique Hilbert complex associated to it. For the de Rham complex, one is interested first of all in the vanishing or nonvanishing of the β_i . In the case of a simply connected manifold, M^m , of negatively pinched curvature, it was conjectured by Singer and Dodziuk [Dod1] that $\beta_i = 0$ for $i \neq m/2$. This was shown to be false in general by Anderson [An]; but recently Gromov [Gr] proved an even stronger result in the Kähler case: Let M be a complete Kähler manifold of complex dimension $m/2$ and with Kähler form $\omega = d\alpha$, where $\sup_{p \in M} \|\alpha(p)\| < \infty$. If M admits a compact quotient

then

$$\hat{\mathcal{H}}^{p,q} = 0 \quad \text{if } p+q \neq m/2, \quad \hat{\mathcal{H}}^{p,q} \neq 0 \quad \text{if } p+q = m/2.$$

Here $\hat{\mathcal{H}}^{p,q}$ denotes the L^2 -harmonic forms of bidegree (p, q) . For more information on nonvanishing of L^2 harmonic forms cf. [Dod2]. For general Dirac complexes, the Γ -index theorem of Atiyah [At] gives a tool to conclude that $\sum_i \beta_i > 0$ if M admits compact quotients.

Another question of interest is the Fredholm property of D which is equivalent to the finiteness of the cohomology, by Theorem 2.2. In this case, one would like to interpret the cohomology topologically. This has been done e.g. in the case of hermitean locally symmetric spaces (cf. [Sa+St], [Lo]) where the cohomology (with local coefficients) coincides with the Goresky-MacPherson intersection homology of the Bailey-Borel compactification.

Even if D is not Fredholm it may happen that the analytic Betti numbers $\hat{\beta}_i$ are finite. In fact, it has been shown by Moscovici [Mo] that this is true for all invariant Dirac operators on locally symmetric spaces of finite volume. Recently, W. Ballmann and the author have shown that this also holds on manifolds with negatively pinched curvature and finite volume. Though it seems very difficult to compute the $\hat{\beta} - i$ individually the analytic index is sometimes more tractable. For results in this direction cf. [At+Don+Si], [Ba+Mo], [Br], [Co+Mo], [Mü1], [Mü2], [R], [St].

2) Algebraic varieties

Let M be the nonsingular locus of an algebraic variety embedded in some projective space $\mathbf{C}P^N$. The Fubini-Study metric on $\mathbf{C}P^N$ induces a Kähler metric on M , and the natural elliptic complex associated to it is the Dolbeault complex. In contrast to the previous example, we now have to expect many ideal boundary conditions. It is very interesting to determine whether any of these leads to a Fredholm complex and, if so, to interpret the cohomology or to calculate the (geometric) index. It was conjectured by MacPherson [McP] that the Neumann extension leads to a Fredholm complex whose index equals the arithmetic genus of any "small" resolution of M . The incorrectness of this conjecture was observed by several people. In the case of a curve it was shown in [Br+Pe+Sch] that all boundary conditions lead to a Fredholm complex and all indices were calculated. In particular, the MacPherson conjecture was shown to hold for the Dirichlet extension; intuitively, the minimal extension does not feel the singularities. For higher dimensions, however, very little is known (cf. e.g. [Ch+Go+McP]), in particular one does not know whether the Dirichlet extension leads to a Fredholm complex.

3. Conic singularities

Now M has a decomposition $M = M_1 \cup U$ where M_1 is a compact manifold with boundary and U is isometric to $(0,1) \times N$ with metric

$$dx^2 + x^2 ds_N(x)^2. \quad (3.1)$$

Here $N = \partial M_1$ is compact, x is the canonical coordinate on $(0,1)$, and $ds_N(x)^2$ is a smooth family of Riemannian metrics on N for $x \in [0,1)$. The spectral analysis on these spaces has been initiated by Cheeger [Ch1] who treated the case of metric cones i.e. $ds_N(x)^2 \equiv ds_N(0)^2$; his analysis was extended to the more general case in [Br+Se2]. There it has been shown that to each Dirac operator d on M one can associate a self-adjoint Dirac operator d^N on N such that the closed extensions of d are parametrized by the subspaces of $\bigoplus_{|s| < 1/2} \ker(d^N - s)$. All these extensions are Fredholm and their

indices can be calculated explicitly. On the other hand, for the de Rham complex there is a unique ideal boundary condition unless $H^{n/2}(H^n) \neq 0$. In the latter case, the ideal boundary conditions are parametrized by the $*_N$ -invariant decompositions of

$H^{n/2}(N^n)$. In particular, it may well happen that $\Delta_D = \Delta_N \neq \Delta_F$ in which case the Friedrichs extension should be regarded as “nongeometric”.

Cheeger [Ch1] has also calculated the cohomology of a suitable ideal boundary condition and identified it with the intersection homology of the closure of M .

The index formula is proved by the heat equation method as in (1.6); if D is closed and Fredholm and if in addition $e^{-t\Delta}$ is trace class then as before

$$\text{ind } D = \text{tr}_s e^{-t\Delta}.$$

One needs, however, a very explicit formula for the coefficients in the asymptotic expansion. This is achieved for the (somewhat stronger) resolvent expansion by means of the Singular Asymptotics Lemma [Br+Se2] which reflects the typical scaling associated with the metric (3.1). The natural contribution of the singularity – i.e. the tip of the cone – turns out to be $\eta(d^N)$, the η -invariant of the operator d^N on N . For the signature operator this was also pointed out by Cheeger.

4) Wedge-like singularities

We now assume that $M = M_1 \cup U$ where M_1 is again compact with boundary and U has now a fibration $(0, 1) \times Z \rightarrow U \xrightarrow{\pi} (0, 1) \times \Sigma$, Z, Σ compact, with metric asymptotically (as $x \rightarrow 0$) equal to

$$d\sigma^2 + dx^2 + x^2 ds_Z(\sigma)^2, \quad (3.2)$$

(x, σ) local coordinates on $(0, 1) \times \Sigma$. In this case the description of boundary conditions is yet rather incomplete. We can associate, however, with each Dirac operator d on M a family $d(\sigma)$, $\sigma \in \Sigma$, of self-adjoint Dirac operators on $Z_\sigma \approx Z$. If

$$\bigoplus_{|s| < 1/2} \ker(d(\sigma) - s) = 0 \quad \text{for all } \sigma \in \Sigma$$

then Δ is essentially self-adjoint hence coincides with the Friedrichs extension. Then again the resolvent expansion can be shown to exist [Br+Se3] implying that D is Fredholm. The typical contribution of the singularity – i.e. Σ – to the index coincides for $\Sigma = S^1$ with a formula given by Witten [W] for the adiabatic limit of η -invariants (cf. [Bi+F], [Ch2] for rigorous derivations). One obtains the quantity

$$\text{const} \int_{S^1} \int_0^\infty t^\alpha \text{tr}_{Z_\sigma} (\beta \dot{d}(\sigma) d(\sigma) e^{-td(s)^2}) dt \Big|_{\alpha=0} d\sigma \quad (3.3)$$

where β is an almost complex structure on Z_σ , $\dot{}$ denotes the σ -derivative, and the t -integral is defined by regular analytic extension. This furnishes a new proof of Witten’s formula in this situation. For arbitrary dimension of Σ one expects a close connection with the recent work of Bismut and Cheeger [Bi+Ch].

5) Orbit spaces of compact group actions

Let M be a compact Riemannian manifold and $G \subset I(M)$ a closed subgroup of the isometry group. If $(C_0^\infty(E), d)$ is a full Dirac complex we get a Dirac complex (Γ, d) setting $\Gamma_i := C_0^\infty(E_i)^G$, the G -invariant sections. It can be shown [Br+H] that in

this case Δ is essentially self-adjoint and that $e^{-t\Delta}$ is trace class. Also, there is an asymptotic expansion

$$\mathrm{tr} e^{-t\Delta} \sim (4\pi t)^{-k/2} \sum_{\substack{i \geq 0 \\ 0 \leq j \leq N}} a_{ij} t^{i/2} \log^j t. \quad (3.4)$$

The coefficients are, however, only very implicitly known. For example, it is an open question whether logarithmic terms do actually occur in (3.4), cf. [Br+Sch].

The spectral analysis can be viewed as analysis of natural elliptic operators on the orbit space M/G .

6) The basic de Rham complex for Riemannian foliations

Let M be a compact Riemannian manifold equipped with a Riemannian foliation \mathcal{F} . The *basic de Rham complex* is the complex given by

$$\begin{aligned} \Gamma_i &:= \{\omega \in \Omega^i(M) \mid X \lrcorner \omega = X \lrcorner d\omega = 0 \text{ for all } X \in C^\infty(T\mathcal{F})\}, \\ d_i &:= \text{exterior differential on } \Omega^i(M), \end{aligned}$$

Then (2.13a) is satisfied but not (2.13b), cf. [K+T]. Nevertheless, taking $D_i =:$ closure of d_i leads to a Hilbert complex which is Fredholm such that the “basic cohomology” is finite. This can be generalized to other “foliated” Dirac operators; also, the heat expansion exists in this case which may lead also to “basic” index theorems (cf. [Br+K]). This analysis can be thought of as spectral analysis on the space of leaves which is highly singular.

7) Domains with singular boundary

Let \tilde{M}^m be a compact Riemannian manifold and M an open subset such that ∂M has finite Minkowski measure in dimension $m - 1 + \delta$, $0 < \delta < 1$. For any Dirac complex $(C_0^\infty(E), d)$ we obtain a corresponding Dirac complex on M by restriction. It has been shown in [Fl+Lap] that the Dirichlet Laplacean on functions, Δ_0 , is discrete and that the counting function

$$N(t) = \#\{\lambda \in \mathrm{spec} \Delta \mid \lambda \leq t\}$$

has the asymptotic behavior

$$N(t) = c_m \mathrm{vol} M t^{m/2} + O(t^{(m-1+\delta)/2}).$$

This implies the asymptotic behavior

$$\mathrm{tr} e^{-t\Delta_0} \sim c'_m \mathrm{vol} M t^{-m/2}.$$

It is therefore interesting to ask, in this setting, whether Δ_F has at least two terms or even a full expansion for $\mathrm{tr} e^{-t\Delta_F}$. If it does not hold in this generality then one would want to know the minimal smoothness requirements on the boundary assuring the existence of an asymptotic expansion.

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