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# Global existence for the nonlinear equations of crystal optics

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1. In the present report we describe a result on global existence for classical solutions of the system of crystal optics in the nonlinear case. According to Courant-Hilbert [1], Born-Wolf [1] or Sommerfeld [1], the system of crystal optics is just Maxwell's system in homogeneous anisotropic media. To desribe what we mean by this, let us consider an electromagnetic field and denote by

- E the electric field density,
- D the electric induction,
- H the magnetic field intensity,
- B the magnetic induction.

All these quantities are of course vectors with 3 components.
Maxwell's equations are then a synthesis of the physical laws which relate $E, D, H, B$, known at Maxwell's time. In differential form and for nonconductive media, they can be written as

(1) $\partial_t B + \mathrm{curl} E = 0, -$ which is Farady's law,

(2) $\partial_t D - \mathrm{curl} H = 0, -$which is Ampere's law -as improved by Faraday,

(3) $\mathrm{div} D = 0, -$which is the law of conservation of the electric induction,

(4) $\mathrm{div} B = 0, -$which is the law of conservation of the electric induction.

The relations (1),(2),(3),(4) do not suffice alone to determine the evolution of $E, D, H, B$ from some given initial state and, possibly, some given boundary conditions. In fact while these equations are valid for any medium, we will also have additional relations between the $E, D, H, B$ which depend on the material under consideration. It is common practice to assume these relations of form

(5) $D = \varepsilon E, B = \mu H,$

where $\varepsilon$ and $\mu$ are $3 \times 3$ matrices which are called dielectric tensor and tensor of magnetic permeability. In general $\varepsilon$ and $\mu$ may depend on a number of parameters, like position, time, temperature, $E$ and $H$. Here we shall assume that they do not depend in an explicit way on the position, the time or the temperature. The first of these assumptions corresponds to the fact that we assume that our medium is homogeneous. For phenomena involving visible light, this is a reasonable assumption for optical crystals. The same is true also for the second assumption, if we assume that our crystal is not changed much in time, and in order for the third assumption to be reasonable, it suffices to assume that the temperature does not change much. In fact, to simplify the situation, we shall assume henceforth that $\varepsilon$ is a function of $E$ alone and that $\mu$ is a constant multiple of the identity:

$$\mu = \begin{pmatrix} \nu & & 0 \\ & \nu & \\ 0 & & \nu \end{pmatrix}, \quad \nu \in R.$$

Physically this means in particular that we assume that the medium is magnetically isotropic. Both Born-Wolff [1] and Sommerfeld [1] agree on the fact that this is a reasonable assumption in crystal optics. We also note that the previous assumptions make of (1),(2),(3),(4),(5) a system of partial differential equations which refers essentially to $E$ and $H$. This system will be linear or not according to wether we assume that the $\varepsilon$ effectively depends on $E$ or not. Since our main result is trivial in the linear case, we shall assume henceforth that $\varepsilon$ effectively depends on $E$. As for the regularity of the function $\varepsilon$, we shall assume that it is $C^\infty$.

Henceforth we shall regard (1),(2),(3),(4) as a system for $E, H$ in which $D$ and $B$ have been replaced by (5).

2. To the system (1),(2),(3),(4) we shall now associate the Cauchy conditions

(6) $E_{|t=0} = E^o, H_{|t=0} = H^o,$

where $E^o$ and $H^o$ are initial conditions which we suppose given on all of $R^3$. Of course, in order that (6) be compatible, we must assume that

(7) $\operatorname{div}(\varepsilon_{|t=0} E^0) = \operatorname{div}(\mu_{|t=0} H^0) = 0,$

and it is standard to observe that (3),(4) follow from (1),(2),(7). The solution of (1),(2),(3),(4),(6) is assumed to be classical, i.e. we assume that it is at least a $C^1$

function defined on $R^3_x \times [0,T)$ where $T > 0$. We must assume then that $E^0$ and $H^0$ are at least $C^1$ but of course we shall measure regularity later on in terms of Sobolev spaces. Note that thus we shall work with a problem which is global in the space variable $x$, which, since we work for a homogeneous medium is not justifiable in physical terms. While we have until now, and shall also later on, motivated our assumptions on $\varepsilon$ and $\mu$ by physical considerations, we should therefore say here, that we are well aware of the fact that the physically relevant problem associated with the equations of crystal optics would be an appropriate initial- boundary value problem. Unfortunately, the results which we shall describe below do not say anything on such problems. (The situation in this respect is however not much different to that from a number of other papers in long-time existence for nonlinear wave equations.)

3. The assumptions on $\varepsilon$ and $\mu$ which we make later on and which essentially all have physical motivations, will show that if $E^0$ and $H^0$ are suitably small then we can find $T > 0$ so that (1),(2),(3),(4),(6) has a solution on $(0,T)$. The main problem is to measure the maximal existence time $T$ of $E, H$ in terms of the magnitudes of $E^0$ and $H^0$ in suitable norms. To do so we shall follow to a large extend the literature on long-time existence for nonlinear wave equations initiated by F.John and S.Klainerman. (Cf. e.g. John [1],[2],[3], Klainerman [1],[2], Klainerman-Ponce [1],Shatah [1].) In particular the general line of argument is here close to the one from Klainerman-Ponce [1].(For an equivalent approach see Shatah [1].) However, it should be said that the system of crystal optics has characteristics of variable multiplicity, which makes one of the key estimates,namely the one concerning the decay properties of the fundamental solution of the linear system of crystal optics (cf. theorem 7 below) much more difficult to obtain .Finally we mention that these decay estimates for are similar to the decay estimates for the fundamental solution of the two-dimensional scalar wave equation. It is therefore no surprize that our results are similar to the results from Klainerman-Ponce [1],Shatah [1] for that equation.

4. Before we continue we must now describe two more assumptions on $\varepsilon$ and $\mu$ which both have physical motivations. The first of these is that $\mu$ is positive definite and that $\varepsilon$ is positive definite at $E = 0$. (Cf. Born-Wolff [1] and Sommerfeld [1,vol.IV].) It follows from this that $\varepsilon$ is positive definite for small $E$, but we do not assume anything on the definitness of $\varepsilon$ for large $E$. The reason is that definitness of $\varepsilon$ for large $E$ would correspond to another choice of energy function than that used later on.

Our second, and last, assumption on $\varepsilon$ is now that

(8) $\quad < E \ , d\,(\varepsilon(E)E) >$ is an exact differential form.

(Here "d" is "exterior differentiation", and $< , >$ is the scalar product in $R^3$.)

Condition (8) is taken from Sommerfeld [1,vol.3,§5.6] where it is motivated by physical considerations : only under this assumption it seems possible to define a reasonable energy function. But we should say that both Sommerfeld [1] and Born-Wolff [1] argue mainly for the linear case, in which $\varepsilon$ is constant. In that case (8) simply means that $\varepsilon$ is symmetric. For a nonlinear situation cf. Yariv [1].) From a practical point of view,(8) brings an enormous technical advantage: together with the positivity of $\varepsilon$ and $\mu$, it transforms (1),(2) in a symmetric hyperbolic system for small solutions. To see that this is so, let us in fact write $\partial_t(\varepsilon(E)E)$ in the form

(9) $\partial_t(\varepsilon(E)E) = A(E)\partial_t E.$

Thus $A(E) = (A_{ij}(E))_{i,j=1}^3$ is simply determined from the condition

(10) $A(E)dE = d(\varepsilon(E)E).$

Since the operator curl has a skew symmetric symbol and $\mu$ is diagonal and positive definite, it suffices to check that $A(E)$ is symmetric and positive definite.

To check the symmetry of $A$ let us then observe that $A(E)dE$ and $< E, A(E)dE >$ are both exact forms (in view of (10) and (8) respectively ) so that we must have

(11) $\sum_{i,j} A_{ij}(E)dE_i \wedge dE_j = 0.$

The symmetry of $A$ can now be read off from (11). Moreover, an explicit computation shows that $A(E) = \varepsilon(0) + O(|E|)$ if $E$ is small. Since $\varepsilon(0)$ is positive definite it follows that indeed $A(E)$ is positive definite for small $E$.

5. We have now seen that (1),(2) is in fact a symmetric hyperbolic system. It is then a result of T.Kato [1] and P.Lax (also cf. Klainerman [1] or Taylor [1] ) that (1),(2),(6) has a solution on $R^3 \times (0,T)$ if $T$ is small. Let us state for later reference a quantitatively precise version of this result in which we denote by $|| \ ||_s$ the norm in the Sobolev spaces $H^s$ :

**Theorem 1** *Assume $s \geq 2$ is fixed and let $\eta > 0$ be chosen so that $F_i \in H^s$, $||F_i||_s \leq \eta$ implies that $A(F_1, F_2, F_3)$ is positive definite. Then there is $\delta^0$ such that for any $(E^0, H^0)$ with $||E^0|| + ||H^0|| < \delta^0$ there is $T > 0$ and a unique solution $(E, H)$ in*

$$C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

*of (1),(2),(6) and such that $||E(t)||_s + ||H(t)||_s \leq \eta$, for all $t$ in $[0,T]$.*

Here $C^0([0,T];H^s)$, respectively $C^1([0,T];H^{s-1})$, is the space of continuous,respectively $C^1$, functions on $[0,T]$ with values in $H^s$, respectively $H^{s-1}$. (This gives sense in particular to the notation $E(t), H(t)$.)

**Remark 2** *Theorem 1 is taken from Kato [1]. A slightly different version of the theorem is stated in Klainerman [1] where it is also observed that if $\delta^0$ is sufficiently small, then one may always assume that*

(12) $T > C(||E^0||_s + ||H^0||_s)^{-1}$

*for some $C$ which does not depend on $E^0, H^0$. Let us also observe that when $E^0, H^0$ are in $\cap_s H^s$ , then it will follow that $E, H$ are in $C^\infty$ on $[0,T] \times R^3$. (This is useful to note if one wants to study the existence of glocal $C^\infty-$ solutions, and follows either directly from Klainerman [1] or from Bony's theory (cf. Bony [1]) applied to the solution given by theorem 1.)*

The relation (12) gives an inferior bound for the lifespan of the solutions $E, H$ of (1),(2),(6). Here we call "lifespan" ( for given $E^0, H^0$) the supremum of all $T > 0$ so that we can find a solution $E, H$ in

$$C^0([0,T];H^s) \cap C^1([0,T];H^{s-1})$$

of (1),(2),(6).

When (3),(4) is also satisfied the lifespan,which we shall denote by $T_{max}(E^0, H^0)$, will actually be much bigger than what we obtain from (12). The reason is ,( as in the related case of nonlinear wave equations,) that nonlinear effects are partially compensated by the decay properties of the fundamental solutions of the linear system of crystal optics. Before we state our main result,we introduce for $q \geq 1$ the notation

$$|u|_{q,s} = \sum_{|\alpha| \leq s} |\partial^\alpha u|_q$$

for the norm in the Sobolev space $L_{q,s}$ .Here $|v|_q$ is the spatial $L_q(R^3)$ -norm of $v$ and "$\partial$", when no variable is specified, always means derivation in the $x-$ variables. Note that for the case $q = 2$ we have $||\ ||_s = |\ |_{2,s}$. (For convenience we shall assume here and later on that $s$ is a natural number. Although this is not completely natural in all cases, it will greatly simplify the situation in the proofs -when we refer to mapping properties of the $L_{q,s}$ spaces.)

**Theorem 3** *Assume that $\varepsilon$ and $\mu$ satisfy all the assumptions made in the above and assume,moreover, that*

(13) $\varepsilon(E) = \varepsilon(0) + O(|E|^3)$.

*Then we can find $s$ and $\delta$ such that if*

(14) $||E^0||_s + ||H^0||_s \leq \delta,\ |E^0|_{8/7,s} + |H^0|_{8/7,s} \leq \delta,$

*and if $E^0, H^0$ satisfies (5), then $T(E^0, H^0) = \infty$, i.e. there is a solution $C^0([0,\infty]; H^s) \cap C^1([0,\infty]; H^{s-1})$ of (1),(2),(3),(4).*

6. We shall not give the proof of theorem 3 in this report, but we shall briefly describe the general line of argument. In fact, as in Klainerman-Ponce [1] (and in a number of related papers ) the idea is the following : at first we apply theorem 1 to find some (possibly small) $T$ and a solution of (1),(2),(3),(4),(6) on $R^3 \times [0, T]$. The main step in the proof is now to show that if $E^0$ and $F^0$ are sufficiently small, then we can estimate $E(t), H(t)$ for any $\tau < T_{max}(E^0, H^0)$ in such a way as to be able to solve the Cauchy problem

(15) $\partial_t(\varepsilon E') - \text{curl } H' = 0,$ for $t > \tau$

(16) $\partial_t(\mu H') + \text{curl } E' = 0,$ for $t > \tau$

(17) $\text{div } (\varepsilon E') = 0, \text{div } (\mu H') = 0,$ for $t > \tau$

(18) $E'_{|t=\tau} = E(\tau),\ \ H'_{|t=\tau} = H(\tau),$

with the aid of theorem 1. Under favorable conditions and if $T_{max}(E^0, H^0) < \infty$ this will lead to an extension of our initial solution $E, H$ beyond $t = T_{max}(E^0, H^0)$. For technical reasons we replace this argument by the slightly more complicated:

**Remark 4** *Let $p = (2k+2)/(2k+1)$ and $q = 2k+2$ for some natural number $k \geq 1$ so that $1/p + 1/q = 1$. (Actually we need later on only the case $k = 3$. We work here and in a number of other results with a general $k$ to show how the degree of the nonlinearity affects the estimates .) Also fix $s, s', s''$ with*

(19) $s \geq 2,\ \ s' + (3/2)(q-2)/q\ + 1 < s,$

*and denote by*

$$M_{s'}(\tau) = \sup_{0 \leq t \leq \tau} (1+t)^{+1/2-1/q}(|E(t)|_{q,s'} + |H(t)|_{q,s'}).$$

*Let $\delta^0$ be the one from theorem 1 and assume that we can find $\eta$, $\delta$ and $T(\delta) > 0$ with the following property:*

*whenever $(E,H) \in C([0,T];H^s) \cap C^1([0,T];H^{s-1})$ is a solution of (1),(2),(3),(4),(6) for which*

(20) $E^0$and $H^0$have compact support,

(21) $||E^0||_s + ||H^0||_s \leq \delta,\ |E^0|_{p,s''} + |H^0|_{p,s''} \leq \delta,$

(22) $\text{div}\,(\varepsilon(E^0)E^0) = \text{div}\,(\mu H^0) = 0, T < T(\delta), M_{s'}(t) \leq \eta, \forall t \in [0,T],$

*it follows that in fact $M_{s'}(t) \leq \eta/2$ and $||E(t)||_s + ||H(t)||_s \leq \delta^0$.*

*Then we can conclude that $T_{max}(E^0,H^0) > T(\delta)$.*

Indeed the argument is almost as before. Denote by $T'_{max}(E^0,H^0)$ (for $E^0, H^0$ as in the remark,) the supremum of all $T$ so that we can find a solution $E, H$ in $C([0,T];H^s) \cap C^1([0,T];H^{s-1})$ of (1),(2),(3),(4),(6) for which $M_{s'}(t) \leq \eta/2$ if $t < T$. The assumptions imply then that

(23) $T'_{max}(E^0,H^0) = \min\,(T_{max}(E^0,H^0), T(\delta)).$

To see this we observe first that $M_{s'}(t)$ is a continuous function of $t$. (Note that Sobolev's imbedding theorem shows that if $(E,H)$ is in $C([0,T];H^s)$ then $(E,H)$ is in $C([0,T];L_{q,s'})$ . The relation between $s'$ and $s$ is precisely the one which makes Sobolev's theorem applicable.) The domain of definition of $M_{s'}$ is $[0, T_{max}(E^0,H^0))$, so the continuity and the assumptions on $\delta$ and $T(\delta)$ imply that $M_{s'}(t)$ cannot surpass the value $\eta/2$ as long as $t \in [0, T_{max}(E^0,H^0)]$ also satisfies $t < T(\delta)$. This proves (23).

Assume now that $T'_{max}(E^0,H^0) = T_{max}(E^0,H^0) < T(\delta)$. From the assumptions it follows in particular that $||E(t)||_s + ||H(t)||_s < \delta^0$ for any $t < T_{max}(E^0,H^0)$. We can therefore apply theorem 1 as before for some $t$ close to $T_{max}(E^0,H^0)$ and extend the solution beyond $t = T_{max}(E^0,H^0)$, a contradiction.

7. According to the preceding remark, we must then try to estimate the $H^s$ norms $||E(t)||_s, ||H(t)||_s$, for solutions of (1),(2),(3),(4),(6). Since (1),(2) is assumed to be hyperbolic, estimates for $||E(t)||_s, ||H(t)||_s$ can indeed be obtained by the energy integral method, if the solution is small. It is in fact not difficult to prove the following result:

**Proposition 5** *Assume $\varepsilon(E) = \varepsilon(0) + O(|E|^k)$ for some $k$ and let $s$ be fixed. Then there are constants $c, \delta^0$ with the following property :*

*if $(E,H)$ is a $C^1$ solution on $R^3 \times [0,T]$ (for some $T$ ) of (1),(2) with*

$$(24)\quad |E(x,t)| \le \delta^0, \quad \sum_{|\alpha|\le s/2+1} |\partial^\gamma E(x,t)| + |\partial^\gamma H(x,t)| \le 1, \quad \forall x,t$$

*and which has compact support in $x$ for every $t$, then*

$$(25)\quad ||E(t)||_s + ||H(t)||_s \le c(||E(0)||_s + ||H(0)||_s)\exp c[\int_0^t |(E,H)(\tau)|^k_{\infty,s/2+1} d\tau].$$

Note that c and $\delta^0$ do not depend here on $T$.

8. The estimate (25) does not suffice to keep $||E(t)||_s + ||H(t)||_s$ small, unless we succeed in giving suitable estimates for $|(E,H)(\tau)|_{\infty,s/2+1}$ . On the other hand,in remark 4 we needed also estimates for $|(E,H)(t)|_{q,s'}$ for some finite $q$. We see therefore that we have to control simultaneously $|\ |_2$, $|\ |_q$, and $|\ |_\infty$ norms of $(E,H)$. (The situation is similar to the one from Klainerman-Ponce or Shatah [1].) What we shall do then is to

a) estimate $|(E,H)(\tau)|_{\infty,s/2+1}$ by $|(E,H)(t)|_{q,s'}$ using Sobolev's imbedding theorem, and

b) estimate $|(E,H)(t)|_{q,s'}$ in terms of $||E(t)||_s + ||H(t)||_s$.

The proof of theorem 3 will then be concluded with a bootstrap argument.

The estimate to which we refered in b) is given in

**Proposition 6** *Assume $\varepsilon(E) = \varepsilon(0) + O(|E|^k)$ for some natural number $k \ge 1$. Let $p = (2k+2)/(2k+1)$ and $q = 2k+2$ and fix $s' \ge 0$.Then there are constants $c, \delta^0, s'', \sigma$ with the following property:*

*assume $(E,H)$ is a solution of (1),(2),(3),(4) on $R^3 \times [0,T]$ which satisfies*

$$(26)\quad |E(\tau)|_\infty \le \delta^0, \forall \tau \in [0,T],$$

$$(27)\quad |E(t)|_{\infty,s'/2+\sigma} + |H(t)|_{\infty,s'/2+\sigma} \le 1,$$

*and which has compact support in $x$ for every $t$.*

*Then it follows that*

$$(28)\quad |E(t)|_{q,s'} + |H(t)|_{q,s'} \le ct^{-1/2+1/q}[|E(0)|_{p,s''} + |H(0)|_{p,s''}] + c\int_0^t (1+t-\tau)^{-1/2+1/q} ||E(\tau)||_{s'+\sigma} |E(\tau)|^k_{q,s'/2+\sigma} d\tau,$$

*for all $t \in [0,T]$. Moreover,$\sigma$ does not depend here on $s'$.*

9. We shall not prove proposition 6 in this report, but we shall mention the main estimate on which the proof is based. This is a decay estimate for the system (1),(2),(3),(4)

linearized around the functions $E = 0, H = 0$. The linear system which we obtain is then actually

$$P(D)u = \begin{pmatrix} -\epsilon(0)\partial_t & curl \\ -curl & -\mu\partial_t \end{pmatrix} u = 0$$

together with the two constraint equations

$$div\ \epsilon(0)u' = 0, div\ \mu u'' = 0.$$

Here $u', u''$ are the electric and magnetic components of $u$. Then we can prove

**Theorem 7** *If $u$ has compact support in $x$ for each fixed $t$, then we must have*

$$|u(x,t)| \leq ct^{-1/2} \sum_{|\alpha|\leq k} |D_x^\alpha u(x,0)|_{L^1}.$$

Note that this is in analogy with classical decay estimates for solutions of the linear constant coefficient wave equation (in space dimension two. Cf.e.g. John [2], Klainerman [1], Segal [1],Strauss [1],[2], Strichartz [1],[2], v.Wahl [1].) However, the situation seems here considerably more complicated, in that for anisotropic media, the system of crystal optics is a system with characteristics of vairable multiplicity and the points at which the characteristic variety vanishes of variable multiplicity come in in an essential way. In fact, the determinant of the symbol of $P(D)$ is

$$p(\tau,\xi) = \frac{\tau^2}{d_1 d_2 d_3}(\tau^4 - \psi(\xi)\tau^2 + \varphi(\xi)|\xi|^2)$$

where

$$d_j = 1/(\nu\varepsilon_j),$$

$$\psi(\xi) = (d_2 + d_3)\xi_1^2 + (d_3 + d_1)\xi_2^3 + (d_1 + d_2)\xi_3^2,$$

$$\varphi(\xi) = d_2 d_3 \xi_1^2 + d_3 d_1 \xi_2^3 + d_1 d_2 \xi_3^2,$$

and where $\varepsilon$ are the eigenvalues of $\varepsilon(0)$. (Here $\tau$ and $\xi$ are the variables Fourier-dual to $t$ and $x$.) The geometric properties of the surface $p(\tau,\xi) = 0$ in $R^4$ can be read off from corresponding properties of the intersection of this surface with $\tau = 1$. The latter, or more precisely $S = \{\xi \in R^3; p(1,\xi) = 0\}$, is called Fresnel's surface and has been studied extensively by a number of mathematicians in the last century. It is easy to see that $S$ consists of two sheets, which are tied together at 4 singular points. Now while $p(1,\xi) = 0$

vanishes of second order at these singular points, it vanishes of first order at all other points from $S$, so we have an instance of variable multiplicity here. (Cf. Courant-Hilbert [1]. ) Apart from these singular points, also the points from $S$ at which the Gaussian curvature vanishes will leed to difficulties. It is a remarkable fact that these points all lie on 4 circles, which circles have the additional property that at all points from a fixed circle the tangent planes to $S$ coincide. (It is through this property that these circles were discovered by R.W.Hamilton in his celebrated work predicting conical refraction. )

Details of the proofs of the results announced in this report will appear elsewhere.

**REFERENCES**


J.M.Bony [1] : Calcul symbolic et propagation des singularites pour les equations aux derivees partielles non lineares. Ann. Sc. E.N.S. 14 (1981), 209-246.

M.Born-E.Wolf [1]: Principles of optics,3rd ed.,Pergamon Press,1964.

R.Courant-D.Hilbert [1]: Methoden der mathematischen Physik. vol.II, Springer Verlag,1937, and revised English version in Interscience Publ.,1962.

F.John [1]: Delayed singularity formation in solutions of nonllinear wave equations in higher dimension. Comm. Pure Appl. Math.,29 (1976),649-681.

[2] : Lower bounds for the life span of solutions of nonlinear wave equations in three dimensions. Comm. Pure Appl. Math.,36 (1983),1-35.

[3] :Almost global existence of elastic waves of finite amplitude arising from small initial disturbances. Comm.Pure Appl.Math.,41:3 (1988),615-667.

T.Kato [1]: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Rat. Mech. Anal. 58 (1975),181-205.

S.Klainerman [1]: Global existence for nonlinear wave equations. Comm. Pure Appl. Math. 33 (1980),43-101.

[2] Long time behaviour of solutions to nonlinear wave equations. Proc. Int. Congress Math. at Warsaw 1983.1209-1215.

S.Klainerman-G.Ponce [1] : Global small amplitude solutions to nonlinear evolution equations. Comm. Pure Appl. Math.,36,(1983),133-141.

O.Liess [1] : Decay estimates for soltions of the system of crystal optics. To appear.

I.E.Segal [1]: Dispersion for nonlinear relativistic equations.Ann. E.N.S., 4$^e$ serie (1968), 459-497.

[2]: Space time decay for solutions of wave equations. Advances in Math.,22 (1976),305-311.

J.Shatah [1]: Global existence of small solutions to nonlinear evolution equations.J. diff. equations,46 (1982),409-425.

A.Sommerfeld [1]: Vorlesungen über theoretische Physik,Bd.III u.IV, Akademische