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Necessary conditions for strong hyperbolicity of first order systems

by

Waichiro MATSUMOTO and Hideo YAMAHARA

§0. Introduction, definitions and theorems.

On higher order scalar equations, the strong hyperbolicity is well characterized. (See O. A. Oleinik [13], V. Ja. Ivrii and V. M. Petkov [3], V. Ja. Ivrii [2], L. Hörmander [1], N. Iwasaki [4], [5], [6], etc.) On the other hand, on first order systems, if their coefficients are constant, we also have a complete result. (See K. Kasahara and M. Yamaguti [7]). In case of first order systems with variable coefficients, we have some results, but they are not satisfactory. (See, for example, N. D. Koutev and V. M. Petkov [8], T. Nishitani [10], [11], [12], H. Yamahara [14], [15] etc.).

In this note, we give some necessary conditions for the strong hyperbolicity of first order systems with variable coefficients, assuming that coefficients depend only on the time variable. This is a further developed results of H. Yamahara [14] and [15]. On the other hand, these become sufficient under a reasonable supplementary condition.

Let us consider the following Cauchy problem.

$$(1) \quad \begin{cases} P u \equiv (P_p - B)u \equiv \{D_t - \sum_{i=1}^{\ell} A_i(t, x) D_{x_i} - B(t, x)\}u = f(t, x), \\ u(t_0, x) = u_0(x), \end{cases}$$

where $u(t, x)$, $u_0(x)$, $f(t, x)$ are vectors of dimension N and $A_i(t, x)$, $B(t, x)$ are square matrices of order N with elements in $C^\infty(\Omega)$, (Ω is an open set in $\mathbb{R}_{t, x}^{1+\ell}$). We say

that the Cauchy problem (1) is uniformly well-posed in Ω if the following holds :

$$(2) \quad \begin{cases} \forall K = [T_1, T_2] \times K_0, \forall K' \subset \subset K \\ \exists \omega : \text{a lens-shaped neighborhood of the origin,} \\ \forall (t_0, x_0) \in K', \forall u_0 \in C^\infty(K_0), \forall f \in C^\infty(K), \exists ! u \text{ solution of (1) in } (t_0, x_0) + \omega. \end{cases}$$

Proposition 0.1. *If (1) is uniformly well-posed in Ω , the following holds :*

$$(3) \quad \begin{cases} \forall (\hat{t}_0, \hat{x}_0) \in K', \forall M \in \mathbb{N}, \exists M' \in \mathbb{N}, \exists \delta > 0, \exists C > 0 \\ \forall (t_0, x_0) \in K' \text{ s.t. } |t_0 - \hat{t}_0| \leq \delta, \forall U_0 \in C^M(K) \\ \forall f \in C^{M-1}(K), \exists u \text{ solution of (1) in } K'', (K'' = \{|t - t_0| \leq \delta\} \times \{|x - x_0| \leq \delta\}) \end{cases}$$

and u satisfies

$$(4) \quad |u|_{M, K''} \leq C(|u_0|_{M', K_0} + |f|_{M'-1, K}),$$

where $|u|_{M, K} \stackrel{\text{def}}{=} \sum_{|\alpha| \leq M} \max_{(t, x) \in K} |D_{t, x}^\alpha u(t, x)|.$

By the estimate (4), we have the following theorem.

Theorem 0. *(P. D. Lax and S. Mizohata)*

If (1) is uniformly well posed, all characteristic roots of P_p are real in $\Omega \times \mathbb{R}_\xi^\ell \setminus \{0\}$.

From now on, we always suppose the conclusion of the above theorem.

Definition 0.1. (Strong hyperbolicity)

We say that P_p is strongly hyperbolic when the Cauchy problem (1) of $P_p + B$ is uniformly well posed in Ω for arbitrary choice of $B(t, x)$.

Throughout this note, we assume the following :

Assumption. A_i depends only on t , ($1 \leq i \leq \ell$).

Let $\{\lambda_j\}_{j=1}^d$ be the different characteristic roots of P_p at $t = t_0$ and $\xi = \xi_0 \neq 0$.

We set

$$A^{(0)} = \sum_{i=1}^{\ell} A_i(t_0) \xi_{0i},$$

$$A^{(i)} = \sum_{i=1}^{\ell} \frac{\partial}{\partial t} A_i(t_0) \xi_{0i},$$

\mathcal{P}_j : the projection to the generalized eigenspace of λ_j ,

$$A_j^{(i)} = A^{(i)} \mathcal{P}_j, \quad (0 \leq i \leq 1, 1 \leq j \leq d).$$

Theorem 1. *If P_p is strongly hyperbolic in Ω , the following holds*

$$(5) \quad \mathcal{P}_j (A_j^{(0)} - \lambda_j I_N) (A_j^{(i)})^k (A_j^{(0)} - \lambda_j I_N) = 0$$

for $1 \leq j \leq d$ and $k \in \mathbb{Z}_+ = \{0, 1, \dots\}$.

Remark. Let m^j be the multiplicity of λ_j . At least for $k \geq m^j$, Condition (5) becomes trivial.

Corollary 2. *The lengths of Jordan chains of $A^{(0)}$ are at most 2.*

By virtue of Bronshtein–Mandai’s theorem, the characteristic roots $\lambda^{(j)}(t)$ ($1 \leq j \leq N$) of $P_p(t; \xi_0)$ belong to C_t^∞ . (See T. Mandai [17] and M. D. Bronshtein [16]). Let us set

$$\lambda_0^{(j)}(t) = \lambda^{(j)}(t_0) + (t - t_0) \frac{\partial}{\partial t} \lambda^{(j)}(t_0).$$

Theorem 3. *If $\ell = 1$ and $\{\lambda_0^{(j)}(t)\}_{j=1}^N$ are distinct for $0 < |t - t_0| \leq \exists \delta_0$, condition (5) is sufficient for the strong hyperbolicity of P_p near t_0 .*

Remark. $\lambda_0^{(j)}(t)$ is obtained by $\sum_1 (\frac{\partial}{\partial t})^k A_i(t_0) \xi_{0i}$ with $0 \leq k \leq 2$.

In the following sections 1, 2 and 3, we give a proof of Theorem 1 for $k = 0$ and 1. The proof of Theorem 1 for $k \geq 2$ and that of Theorem 3 will be given in the forthcoming paper [19].

§1. Reduction.

We may assume $t_0 = 0$. We take B as constant matrix and $f = 0$. Let us take Fourier image of (1) on the variable x ;

As our consideration becomes independent of the part which has the factor $t^2 n$, from now on, we take out (j, j) block and omit the subscript "j". We may assume $\lambda = 0$. Further, we set $t = n^{-\sigma} s$ ($\sigma > 0$). Thus, we arrive at

$$(1.4) \quad \begin{cases} P_0 v \equiv \{n^\sigma D_s - (n J + n^{1-\sigma} s A_1 + n^{1-2\sigma} s^2 A_2(s) + B)\}v \\ = 0, \\ v(0) = v_0, \end{cases}$$

where v, v_0 are vectors of dimension m , J, A_1, A_2, B are square matrix of order m and

$$J = J(r, 1) \oplus \dots \oplus J(r, m_r) \oplus J(r-1, 1) \oplus \dots \oplus J(r-1, m_{r-1}) \oplus J(r-2, 1) \oplus \dots \oplus J(1, m_1),$$

$$\sum_{j=1}^r j m_j = m.$$

Here, condition (5) for j in \mathbb{S}_0 is equivalent to

$$(1.5) \quad J(A_1)^k J = 0 \text{ for } k \in \mathbb{Z}_+.$$

Proposition 1.1. *We assume that (1) is uniformly well posed in Ω . If, for P_0 in (1.4), there exists an invertible matrix $N(s, n)$ for $0 < |s| \leq \delta$ and $\ell \geq 2$, ($\ell \in \mathbb{N}$) such that*

$$\tilde{P} = N^{-1} L N = n^\sigma D_s - n^\mu (\tilde{J}(s) + \tilde{K}(s)) - n^{\mu'} C(s, n),$$

$\mu > \mu', \mu > \sigma$, $C(s, n)$ is bounded,

$$\tilde{J} = \bigoplus_{\substack{1 \leq k \leq R \\ 1 \leq h \leq M_R}} \tilde{J}(k, h), \quad \tilde{J}(k, h) = \begin{pmatrix} 0 & a_1^{k, h} & & \\ & 0 & & \\ & & & a_{k-1}^{k, h} \\ & & & 0 \end{pmatrix},$$

$a_1^{k, h}$ is not identically zero, and is analytic for $s \neq 0$,

$\tilde{K} = (K(k, h, k', h'))_{\substack{1 \leq k, k' \leq R \\ 1 \leq h \leq M_k \\ 1 \leq h' \leq M_{k'}}}$: block decomposed with respect to \tilde{J} ,

with

$$K(k,h,k',h') = \begin{pmatrix} \alpha_1^{k,h,k',h'} & & 0 \\ \cdot & & 0 \\ \alpha_k^{k,h,k',h'} & & 0 \end{pmatrix}; \quad k \times k'$$

$\alpha_i^{k,h,k',h'} = 0$ for $i \neq 0 \pmod{\ell}$,

then, we have the following ;

- 1) If $\ell \geq 3$, $\tilde{J} + \tilde{K}$ is nilpotent.
- 2) Let $\det(\lambda I - (\tilde{J} + \tilde{K}))$ be $\sum_{i=0}^m C_i(s) \lambda^{m-i}$.
If $\ell = 2$ and $C_{2i}(s)$ ($C_{2i+1}(s)$, resp.) is even function (odd function, resp.), $\tilde{J} + \tilde{K}$ is nilpotent.

Now, we assume (5) does not hold for $k = 0$ or $k = 1$.

In order to make B stronger than $n^{1-2\sigma} s^2 A_2(s)$, we take $1 - 2\sigma < 0$ ie $\sigma > \frac{1}{2}$.

§2. Maximal connection.

Let us consider

$$\tilde{J} = \bigoplus_{\substack{1 \leq k \leq R \\ 1 \leq h \leq M_R}} \tilde{J}(k,h), \quad \tilde{J}(k,h) \text{ is that in Prop. 1.1, } M = \sum_{j=1}^R j M_j.$$

Corresponding to the blocks of \tilde{J} , we decompose $M \times M$ matrix K to $(K(k,h,k',h'))_{\substack{1 \leq k,k' \leq R \\ 1 \leq h \leq M_k \\ 1 \leq h' \leq M_{k'}}$.

Ex.

$$\tilde{J} = \begin{pmatrix} \tilde{J}(2,1) & & \\ & \tilde{J}(2,2) & \\ & & \tilde{J}(1,1) \end{pmatrix}, \quad K = \begin{pmatrix} \tilde{J}(2,1,2,1) & \tilde{J}(2,1,2,2) & \tilde{J}(2,1,1,1) \\ \tilde{J}(2,2,2,1) & \tilde{J}(2,2,2,2) & \tilde{J}(2,2,1,1) \\ \tilde{J}(1,1,2,1) & \tilde{J}(1,1,2,2) & \tilde{J}(1,1,1,1) \end{pmatrix}$$

We call $(K(k,h,k',h'))$ the block decomposition of K with respect to \tilde{J} .

The following notions are important.

Definition 2.1. (Maximal connection of Jordan chain).

Let $(K(k,h,k',h'))$ be the block decomposition of K with respect to \tilde{J} . If

1)

$$(2.1) \quad \begin{cases} K(R,h,k',h') = \begin{pmatrix} 0 & & 0 \\ \cdot & & \\ 0 & & 0 \\ \alpha_{R,h,k',h'} & & 0 \end{pmatrix}, \\ K(k,h,k',h') = 0 \quad (k < R) \end{cases}$$

for arbitrary h, k' and h' ,

$$(2) \quad K \neq 0,$$

$$(3) \quad \mathcal{A} = (\alpha_{R,h,k',h'})_{1 \leq h, h' \leq M_R} \text{ is nilpotent,}$$

$\tilde{J}+K$ is again nilpotent. We say that in $\tilde{J}+K$, the Jordan chains of \tilde{J} are maximally connected by K , or that K brings a maximal connection (of Jordan chain) to \tilde{J} .

Definition 2.2. (Selfsimilar matrix).

Let us take $1 < R_0 < R_1 < \dots < R_p < R_{p+1}$, such that

$$R_{j+1} = k_j R_j + R_j^0, \quad k_j \geq 1, \quad 0 \leq R_j^0 < R_j, \quad k_j, R_j \in \mathbb{N}.$$

We set

$$A_0 = \begin{pmatrix} 0 & 1 & & \\ & 0 & \cdot & \\ & & \cdot & 1 \\ & & & 0 \end{pmatrix}; \quad R_0 \times R_0,$$

$$A_{j+1} = \underbrace{A_j \oplus \dots \oplus A_j}_{k_j} \oplus A_j^0 + K_j; \quad R_{j+1} \times R_{j+1}$$

A_j^0 = the first R_j^0 rows and R_j^0 columns part of A_j ,

$$(2.2) \quad K_j = (K_j(k,k')); \text{ block decomposition w.r.t. } A_{j+1}$$

$$K_j(h,h+1) = \begin{pmatrix} 0 & 0 \\ \cdot & 0 \\ 0 & \\ \vdots & \\ s & 0 \end{pmatrix}, \quad 1 \leq h \leq k_j,$$

$\hat{J}(i) : i \times i$ = the first i rows and i columns part of A_{p+1} .

We call

$$\hat{J} = \bigoplus_{1 \leq i \leq R_{p+1}} (J(i) \otimes \dots \otimes J(i))_{M_i}$$

a self similar matrix of step $p+1$ and A_j and A_j^0 the factors of step j .

Let \hat{J} be $M \times M$ selfsimilar matrix and K is a $M \times M$ matrix block decomposed w.r.t \hat{J} . Let an element of a block of K belong to $q_p^{(r)}$ -th A_p in the direction of row and to $q_p^{(c)}$ -th A_p in the direction of column. ((k_p+1) -th $A_p = A_p^0$). Further, in A_p , let it belong to $q_{p-1}^{(r)}$ -th A_{p-1} in the direction of row and to $q_{p-1}^{(c)}$ -th A_{p-1} in the direction of column. We continue this procedure up to $q_0^{(*)}$. At last, let it be the $(q_{-1}^{(r)}, q_{-1}^{(c)})$ element of A_0 . We set $q_h = q_h^{(r)} - q_h^{(c)} + 1$ ($-1 \leq h \leq p$).

Definition 2.3. (Address)

We call $q = (q_p, q_{p-1}, \dots, q_{-1})$ the address of the element.

To the set of addresses, we give the dictionary order.

Definition 2.4. (Acceptable matrix).

Let us take $1 > v_{-1} > v_0 > \dots > v_p > 0$, $v = \sum_{j=-1}^p v_j$, and $\sigma > 0$ ($v_j, \sigma \in \mathbb{R}_+$). For a block decomposed matrix K w.r.t. a selfsimilar matrix \hat{J} , if the address q of its element has a q_j such that $q_j = k_j + 1$ and $\sum_{h=0}^{j-1} (q_h - 1)R_h + q_{-1} = R_j^0$ (that is, the element is found at the left-down corner of $R_{j+1} \times R$ matrix in step of $j+1$, $R(\leq R_{j+1})$; free), the element has the form $c(s)n^{1-v'}$, $v' = v'(q) = 2\sigma - \sum_{j=-1}^p (q_j - 1)v_j$ and otherwise, it has the form $c n^{1-v'}$, $v' = v'(q) = \sigma - \sum_{j=-1}^p (q_j - 1)v_j$ and c is constant. Further, if all $v(q)$ are greater than v , we say that K is acceptable w.r.t. $n^{1-v}\hat{J}$. We call $\varepsilon(q) = (v' - v) / (\sum_{j=0}^p (q_j - 1)R_j + q_{-1})$ the descent index of the element with the address q .

When the descent index is smaller, we say that it is more effective.

Remark. Corresponding to the above \hat{J} , we take a shearing operator with weight ε

$$W = \bigoplus_{\substack{1 \leq k \leq R_{p+1} \\ 1 \leq h \leq M_k}} W(k, h, \varepsilon),$$

$W(k, h, \varepsilon) = \text{diag}(1, n^\varepsilon, n^{2\varepsilon}, \dots, n^{(k-1)\varepsilon})$, $\varepsilon = v(q)$. Then, the element with the address q

obtain the order $1-\nu-\varepsilon$ of $W^{-1}(n^{1-\nu} \hat{J})W$ by the shearing transformation $W^{-1} K W$.

Now, we return to the equation (1.4). We assume that (1) is uniformly well-posed.

$$\text{Let us set } W = \bigoplus_{\substack{1 \leq k \leq r \\ 1 \leq h \leq m_k}} W(k, h, \frac{\sigma}{r}).$$

setting $w_1 = W^{-1} w$, w_1 satisfies $P_1 w_1 = 0$,

$$(2.3) \quad P_1 = W^{-1} P_0 W = n^\sigma D_s - (n^{1-\sigma/r} J + n^{1-\sigma/r} s K_1 + s A_1^1(n) + s^2 A_2^1(s; n) + B^1(n)),$$

where $s(n^{1-\sigma/r} K_1 + A_1^1(n))$ is brought from $n^{1-\sigma} s A^1$ and the order of $A_1^1(n)$ is less than $1-\sigma/r$. $s A_1^1(n) + s^2 A_2^1(s; n)$ is acceptable w.r.t. $n^{1-\sigma/r} J$.

$$\text{In } K_1 = (K_1(k, h, k', h'))_{\substack{1 \leq k, k' \leq r \\ 1 \leq h \leq m_k \\ 1 \leq h' \leq m_{k'}}}, \quad K_1(k, k', h') = \begin{pmatrix} 0 & & & \\ & \cdot & & \\ & & 0 & 0 \\ \alpha^{h, k', h'} & & & \end{pmatrix}$$

and $K_1(k, h, k', h') = 0$ for $k < r$. By virtue of Proposition 1.1, $(\alpha^{h, R, h'})_{1 \leq h, h' \leq m_r}$ must be

nilpotent, and then, K_1 brings a maximal connection to J if $K_1 \neq 0$. We can take each Jordan chain in $J + K_1$ composed by vectors of $s^\mu v$, v : constant vector. Replacing $s^\mu v$ by vn we can have a constant matrix N which transform J to \hat{J}_1 , a selfsimilar matrix. We have

$$(2.4) \quad \tilde{P}_1 = N^{-1} P_1 N = n^\sigma D_s - (n^{1-\sigma/r} \hat{J}_1^1(s) + s \tilde{A}_1^1(n) + s^2 \tilde{A}_2^1(s; n) + \tilde{B}^1(n)).$$

Let us set the length of the longest Jordan chain of $\hat{J}_1(s)$ as $R_1 = k_0 R_0 + \ell$, $R_0 = r$, $0 \leq \ell < R_0$. In $s \tilde{A}_1^1(n) + s^2 \tilde{A}_2^1(s; n)$, the highest order on n is given only by the elements with the address $(k_0 + 1, r - 1)$ if $\ell \geq R_0 - 1$, and by those with the address $(k_0, r - 1)$ (and also by those with $(k_0 + 1, r - 2)$ in case of $k_0 = 1$) if $\ell < R_0 - 1$. In the former case, if an element with the address $(k_0 + 1, r - 1)$ does not vanish, after the shearing transformation with weight $\frac{\sigma}{R_0 R_1}$, a maximal connection occurs by virtue of Proposition 1.1. In the latter case, no maximal connection occurs. Continuing this procedure, we arrive at the following proposition.

Proposition 2.1. Let us set $R_{-1} = 1$, $R_0 = R$, $R_{j+1} = k_j R_j + R_j - R_{j-1}$ ($0 \leq j \leq p+1$) and $\hat{R} = k_p R_p + \ell$, ($k_j \in \mathbb{N} = \{1, 2, \dots\}$, $0 \leq \ell < R_p$).

- (1) In the above procedure, if p times maximal connections occur, the highest order part must be the selfsimilar matrix of step $p+1$ replacing R_{p+1} by \hat{R} and has the order $1-\nu$ on n , $\nu = \sum_{j=0}^p \nu_j$, $\nu_j = \frac{\sigma}{R_{j-1} R_j}$.
- (2) The operator \tilde{P}_{p+1} has the following form ;
- (2.5) $\tilde{P}_{p+1} = n^\sigma D_s - (n^{1-\nu} \tilde{J}_{p+1}(s) + s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n) + B^{p+1})$,
 where $s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n)$ is acceptable w.r.t. $n^{1-\nu} \tilde{J}_{p+1}$.
- (3) In $s A_1^{p+1}(n) + s^2 A_2^{p+1}(s; n)$, if $\ell \geq R_p - R_{p-1}$, the highest order is given only by the elements with address $(k_p+1, k_{p-1}, \dots, k_0, r-1)$ and if $\ell < R_p - R_{p-1}$, it is given by those with the address $(k_p, k_{p-1}, \dots, k_0, r-1)$ (and also by those with $(1, \dots, 1, 2, k_\ell-1, k_{\ell-1}, \dots, k_0, r-1)$ in case of $k_{\ell+1} = \dots = k_p = 1$ and $k_\ell \geq 2$ and also by those with $(1, \dots, 1, 2, 1, \dots, 1, r-2)$ in case of $k_0 = k_1 = \dots = k_p = 1$).

Proof By the induction on p .

§3 Proof of Theorem 1, case of $k \leq 1$.

The maximal connections can occur at most $\lfloor \frac{m-r}{r-1} \rfloor$ times. Let no maximal connection occur on \tilde{P}_{p+1} , that is, in $\hat{R} = k_p R_p + \ell$, $\ell \neq R_p - R_{p-1}$ or $\ell = R_p - R_{p-1}$ but all elements with the address $(k_p+1, k_{p-1}, \dots, k_0, r-1)$ vanish.

Let W be the shearing operator corresponding to \tilde{J}_{p+1} in (2.5) with weight ε ($\varepsilon = \frac{\sigma}{R_p R_{p+1}}$ in case of $\ell > R_p - R_{p-1}$ and $\varepsilon = \frac{\sigma}{R_p (R_{p+1} - R_p)}$ in case of $\ell \leq R_p - R_{p-1}$).

We set

$$(3.1) \quad \hat{P}_{p+2} \equiv W^{-1} \tilde{P}_{p+1} W = n^\sigma D_s - \{n^{1-\nu-\varepsilon} (\tilde{J}_{p+1}(s) + s K_{p+2}) + s A_1^{p+2}(n) + s^2 A_2^{p+2}(s; n) + B^{p+2}(n)\},$$

where the orders of A_1^{p+2} and A_2^{p+2} are less than $1-\nu-\varepsilon$. Here the highest order in $B^{p+2}(n)$ is σ and it is given by the elements with the address $(k_p+1, k_{p-1}, \dots, k_0, r)$ in

case of $\ell > R_p - R_{p-1}$ and $(k_p, k_{p-1}, \dots, k_0, r)$ in case of $\ell \leq R_p - R_{p-1}$.

By a suitable choice of B in the original operator P, we can take B^{p+2} such that it has only one non-zero element, (g,1)-element $c_0 n^\sigma$ (c_0 is a large constant), where $g = k_p R_p + \sum_{j=0}^{p-1} (k_j - 1) R_j + r$ if $\ell > R_p - R_{p-1}$ and $g = \sum_{j=0}^p (k_j - 1) R_j + r$ if $\ell \leq R_p - R_{p-1}$.

We consider the characteristic polynomial of the full operator \hat{P}_{p+2} :

$$\det(\lambda I - \{n^{1-v-\epsilon} (\tilde{J}_{p+1}(s) + s K_{p+2}) + s A_1^{p+2}(n) + s^2 A_2^{p+2} + B^{p+2}(n)\}) \\ = \sum_{j=0}^m \alpha_j(s; n) \lambda^{m-j}.$$

$\alpha_g(s; n)$ has the form $c_0 n^\delta s^\mu (1 + o(1))$, $\delta = (g-1)(1-v-\epsilon) + \sigma$ and $\exists \mu \in \mathbb{Z}_+$ ($v = \sum_{j=0}^p \frac{R_j}{R_{j-1}}$). Here, we cannot find Jordan chains which are composed the vector of type $s^\mu v$, v : constant vector.

By virtue of Proposition 1.1, $\tilde{J}_{p+1} + K_{p+2}$ is nilpotent. Let us take $N(s)$ which transforms $\tilde{J}_{p+1} + s K_{p+2}$ to Jordan's normal form and set

$$(3.2) \quad \tilde{P}_{p+2} \equiv N^{-1} \cdot \hat{P}_{p+2} \cdot N = n^\sigma D_s - n^{1-v-\epsilon} J_{p+2} - C(s; n).$$

Here, the commutator $n^\sigma N^{-1}(s) D_s N(s)$ has the same order σ as $B^{p+2}(n)$ and it can give an influence on $\alpha_g(s; n)$. That is, setting

$$\det(\lambda I - n^{1-v-\epsilon} J_{p+2} - C(s; n)) = \sum_{j=0}^m \alpha'_j(s; n) \lambda^{m-j},$$

$\alpha'_g(s; n)$ may have the form $(c_0 + c'_0(s)) n^\delta (1 + o(1))$. However, $c'_0(s)$ is decided by the principal part part of the original operator P and independent of $B^{p+2}(n)$. Thus, $\alpha'_g(s, n) \neq 0$ and it has the order σ , if we take c_0 sufficiently large.

Let σ be i_1/i_0 ($i_0, i_1 \in \mathbb{N}$). $1-v-\epsilon$ is also expressed as i_2/i_0 ($i_2 \in \mathbb{N}$). If $1-v-\epsilon > \sigma$, we can find a matrix $N' \sim I + \sum_{h \in \mathbb{N}} n^{h/i_0} N_h(s)$ such that

$$(3.3) \quad Q \equiv N'^{-1} \circ \tilde{P}_{p+2} \circ N' = n^\sigma D_s - n^{1-v-\epsilon} J_{p+2} - C'(s; n),$$

where $C'(s; n) = (C'(k, h, k', h'))(s; n)$; block decomposition w.r.t. J_{p+2} ,

$$C'(k, h, k', h') = \begin{pmatrix} \gamma_1^{khk'h'} & 0 \\ \cdot & 0 \\ \gamma_k^{khk'h'} & 0 \end{pmatrix} \quad \text{and}$$

$$\gamma_j^{khk'h'} = \sum_{\substack{i \in \mathbb{Z} \\ i/i_0 + j(v+\varepsilon) < 1}} n^{i/i_0 + (j-1)(v+\varepsilon)} \gamma_{ji}^{khk'h'}(s).$$

(See, for example, V.M. Petkov [18] or rather its proof).

We say that a matrix which has the form as C' is admissible to $n^{1-v-\varepsilon} J$. By this transformation, the principal part of $\alpha'_g(s; n)$ is preserved. From now on, we assume that $1-v-\varepsilon > \sigma$.

We introduce a notion :

Definition 3.1. (Stable coefficient of characteristic polynomial)

Let \tilde{C} be admissible to $n^v J$. We set

$$\det(\lambda I - n^v J - \tilde{C}(s; n)) = \sum_{j=0}^m \tilde{\alpha}_j(s; n) \lambda^{m-j}.$$

When the principal part of $\tilde{\alpha}_j$ is preserved by any perturbation of order at most σ , we say that α_j is a stable coefficient of the characteristic polynomial of full operator.

On the stable coefficients, the following proposition was obtained by W. Matsumoto [9].

Proposition 3.1. *If the original Cauchy problem is uniformly well-posed, the characteristic polynomial of full operator has no stable coefficient.*

We transform Q by shearing operator W' corresponding to J with weight $\varepsilon_0 > 0$, ε_0 : very small.

$$(3.4) \quad Q' \equiv W'^{-1} Q W' = n^\sigma D_s - n^{1-v-\varepsilon-\varepsilon_0} J - C''(s; n),$$

where the order of $C''(s; n)$ is less than $1-v-\varepsilon-\varepsilon_0$ and $C''(s, n)$ is admissible to $n^{1-v-\varepsilon-\varepsilon_0} J$. Further, the elements which concern $\alpha'_g(s; n)$ has the order $\sigma + (g-1)\varepsilon_0$ in $C''(s, n)$. This implies that $\alpha'_g(s; n)$ is stable in the characteristic polynomial of the full operator Q' , if we can find σ such that $1-v-\varepsilon > \sigma > \frac{1}{2}$. Then, when we can find a σ such that $1-v-\varepsilon > \sigma > \frac{1}{2}$, we arrive at a contradiction. Here,

the existence of such σ is equivalent to " $g \geq 3$ " and further equivalent to " $r \geq 2$ and if $r = 2$,

$$\text{in } A_1 = (A_1(k, h, k', h'))_{\substack{1 \leq k, k' \leq 2 \\ 1 \leq h \leq m_k \\ 1 \leq h' \leq m_{k'}}} \text{ in (1.4) } \left(A_1(2, h, 2, h') = \begin{pmatrix} * & * \\ \alpha(h, h') & * \end{pmatrix} \right),$$

$(\alpha_{hh'})_{1 \leq h, h' \leq M_2}$ vanishes". " $r \leq 2$ " is equivalent to condition (1.5) with $k = 0$ (Corollary

2) and the rest is equivalent to (1.5) with $k = 1$.

Q.E.D.

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