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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open subset and let $P$ be a second order strictly hyperbolic differential operator in $\Omega$ with smooth coefficients. Let $t \in C^\infty(\Omega)$ be a time function for $P$ and define

$$\Omega^\pm = \Omega \cap \{ \pm t > 0 \}.$$  \hfill (1.1)

Assume that $\Omega$ is a domain of dependence of $\Omega^-$. Let $f$ be a smooth function of its arguments and suppose $u, Du \in L^\infty_{loc}(\Omega)$ satisfies

$$Pu = f(z, u, Du); \quad z \in \Omega.$$  \hfill (1.2)

The general question on propagation of singularities of solutions of (1.1) is how do singularities of $u$ in $\Omega^-$ influence singularities of $u$ in $\Omega$. We shall concentrate in the study of some geometric singularities called conormal and the first example is conormality to a smooth hypersurface. Thus let $S \subset \Omega$ be a smooth hypersurface which is characteristic for $P$, let $\mathcal{V}_S$ be the Lie algebra of smooth vector fields tangent to $S$ and denote

$$I_k L^{2}_{loc}(\Omega, \mathcal{V}_S) = \{ u \in L^{2}_{loc}(\Omega) : \mathcal{V}_S^j u \subset L^{2}_{loc}(\Omega), \ j \leq k \}.$$  \hfill (1.3)

Observe that if $u \in I_\infty L^{2}_{c}(\Omega, \mathcal{V}_S)$, then $u$ is smooth away from $S$. In fact one can easily show that in this case the wavefront set of $u$ is contained in the conormal bundle to $S$.

**Theorem 1.1 (Bony, [4])** Let $u, Du \in H^s_{loc}(\Omega), \ s > \frac{3}{2}$, satisfy (1.2). If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}_S)$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}_S)$.  

This result shows that as long as $S$ is smooth $u$ remains conormal to it, but in general characteristic hypersurfaces of $P$ can have rather complicated singularities. In this talk we shall describe the results of [16] and [17] concerning the propagation of conormal singularities for solutions of (1.2) along a hypersurface $\Sigma$ with either a cusp or a swallowtail singularity. These are in some sense, see [2], the only cases where the singularities are stable under small perturbations. These problems have been also studied by M. Beals [3] and R. Melrose [9], in the case of the cusp and G. Lebeau, [6], [7] and J-M. X...
Delort [5] in the case of the swallowtail with the hypotheses that $P$ has real analytic coefficients and the regular part of $\Sigma$ is real analytic.

Before stating our results we have to introduce some notation. Let $\mathcal{W}$ be a Lie algebra and $C^\infty$ module of smooth vector fields on a manifold with corners $X$ and let $\mu$ be a smooth measure on $X$. The space of iteratively regular distributions with respect to $\mathcal{W}$ is then defined as

$$I_k L^2_{\mu,c}(X, \mathcal{W}) = \{ u \in L^2_{\mu,c}(X); \mathcal{W}^j u \in L^2_{\mu,c}(X), \; j \leq k \}. \quad (1.4)$$

2 The Cusp

Let $G$ be a hypersurface with a cusp singularity at a line $L$, i.e there are local coordinates near $q \in L$ such that

$$G = \{(x, y, z) \in \Omega : y^3 = x^2\}, \quad L = \{(x, y, z) : x = y = 0\}. \quad (2.1)$$

Assume that the smooth part of $G$ is characteristic for $P$. Let $\mathcal{V}_G$ be Lie algebra of smooth vector fields tangent to $G$. It is easy to show that the Lie algebra $\mathcal{V}_G$ is characteristic complete, i.e

$$[P, \mathcal{V}_G] \subset \Psi^0(\Omega) \cdot P + \Psi^1(\Omega) \cdot \mathcal{V}_G + \Psi^1(\Omega). \quad (2.2)$$

Where $\Psi^j(\Omega)$ denotes the space of properly supported pseudodifferential operators of order $j$ in $\Omega$. Then by commutator methods, see [4], one obtains

**Theorem 2.1** Let $u, Du \in H^s_{\text{loc}}(\Omega)$, $s > \frac{3}{2}$, satisfy equation (1.2). If $u, Du \in I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_G)$, then $u, Du \in I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_G)$.

Next we recall the spaces of marked Lagrangian distributions introduced by R. Melrose in [9]. Let $\Lambda_G = \text{clos}[N^*(G \setminus L)]$, $\Lambda_G$ is a smooth conic Lagrangian submanifold of $T^*\mathbb{R}^3$. Let $\Lambda_L = N^*L$ and

$$\mathcal{M}_1(G) = \{ A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \text{ at } \Lambda_G, \quad \text{H.a is tangent to } \Lambda_G \cap \Lambda_L \}. \quad (2.3)$$

$$\mathcal{M}_1(L) = \{ A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \text{ at } \Lambda_L, \quad \text{H.a is tangent to } \Lambda_G \cap \Lambda_L \}. \quad (2.4)$$

Let

$$J^G_{k,m}(\Omega) = I_k L^2_{\text{loc}}(\Omega, \mathcal{M}_1(G)) + I_k L^2_{\text{loc}}(\Omega, \mathcal{M}_1(L)). \quad (2.5)$$

In [9] Melrose proves that

$$J^G_{k,m} \subseteq I_k L^2_{\text{loc}}(\Omega, \mathcal{V}_G) \quad (2.6)$$

and
Theorem 2.2 (Melrose, [9]) Let $u, Du \in H^s_{loc}(\Omega)$, $s > \frac{3}{2}$, satisfy equation (1.2). If $u, Du \in J^G_{k+m}(\Omega^-)$, then $u, Du \in J^G_{k+m}(\Omega)$.

Finally we introduce a third space of distributions associated to the cusp. Observe that in local coordinates where (2.1) holds one finds that $G$ is invariant under the $\mathbb{R}^+$ action

$$m^{3-2}_{y}(x,y) = (s^3x, s^2y).$$

This leads to the definition quasi-homogeneous polar coordinates, thus consider the non-round circle

$$S^{1}_{3-2} = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1^4 + \omega_2^2 = 1\}$$

and the manifold with boundary

$$X_{3-2} = S^{1}_{3-2} \times [0, \infty) \times \mathbb{R}.$$ (2.9)

Then define the blow-down map

$$\beta_{3-2} : X_{3-2} \longrightarrow \mathbb{R}^3, \quad \beta_{3-2}(\omega, r, z) = (r^3 \omega_1, r^2 \omega_2, z).$$ (2.10)

Let $\mathcal{W}_G$ be the Lie algebra of smooth vector fields in $X_{3-2}$ which are tangent to $\partial X_{3-2}$ and to $G^{(1)} = \text{clos}_\beta \beta^{-1}_{3-2}[G \setminus L]$. Let $\mu$ be the pull back of the Lebesgue measure by the map $\beta_{3-2}$. Then one defines

$$J^G_k(\Omega) = \{u \in L^2_{loc}(\Omega) : \beta_{3-2}^* u \in I_k L^2_c(X_{3-2}, \mathcal{W}_G)\}.$$ (2.11)

One can easily show that the space $J^G_k(\Omega)$ does not depend on the choice of coordinates such that (2.1) holds. Then see [16], one can show that if $\mathcal{W}_G^1$ is the Lie algebra of smooth vector fields in $X_{3-2}$ that are tangent to $G^{(1)}$ and to the lines $\{\omega_1 = 0, r = 0\}$, $\{\omega_2 = 0, r = 0\}$, then the blow down map $\beta_{3-2}$ induces an isomorphism

$$\beta_{3-2}^* : J^G_{k+m}(\Omega) \leftrightarrow I_k L^2_c(X_{3-2}, \mathcal{W}_G^1).$$ (2.12)

Similarly if $\mathcal{W}_G^0$ is the Lie algebra of smooth vector fields that are tangent to $G^{(1)}$ and vanish on $\partial X_{3-2}$, then

$$\beta_{3-2}^* : I_k L^2_c(\Omega, \mathcal{V}_G) \leftrightarrow I_k L^2_c(X_{3-2}, \mathcal{W}_G^0).$$ (2.13)

In particular one obtains from (2.12) and (2.13) that

$$J^G_k(\Omega) \subset J^G_{k+m} \subset I_k L^2_{loc}(\Omega, \mathcal{V}_G).$$ (2.14)
Figure 1:

The main difficulty in proving a propagation theorem for $J^0_k(\Omega)$ is that this space is not known to have a microlocal characterization. One of the main results of [16] is the following elliptic regularity type of theorem.

**Theorem 2.3** If $u, Du \in H^p_{loc}(\Omega) \cap I_k L^2_{loc}(\Omega, G)$ satisfies equation (1.2), then $u, Du \in J^0_k(\Omega)$.

Theorem 2.3 illustrates an important idea that will be used in the proof of Theorem 7.1. One first proves a propagation theorem for a bigger space which has a microlocal characterization and then uses the equation to show that the solution is actually in the smaller space.

### 3 The Swallowtail

Since the results we wish to prove are local we shall assume that $\Omega \subset \mathbb{R}^3$ is a sufficiently small neighborhood of $O = (0,0,0)$. Let $\Sigma \subset \Omega$ be a hypersurface with a swallowtail singularity at $O \in \Omega$, i.e. there are smooth coordinates $(x,y,z)$ in $\Omega$ such that

$$\Sigma = \{(x,y,z) : \delta(\lambda) = \lambda^4 + z\lambda^2 + y\lambda + x = 0, \text{ has a double real root}\}. \quad (3.1)$$

$\Sigma$ has a cusp singularity at

$$L = \{(x,y,z) : x = -\frac{z^2}{12}, \ y^2 = \left(-\frac{2}{3}z\right)^3\} \quad (3.2)$$

and a self-intersection at

$$H = \{(x,y,z) : y = 0, \ x = -\frac{z^2}{4}, \ z \leq 0\}. \quad (3.3)$$
The continuation of the line $H$ to values of $z > 0$ corresponds to the set of $(x, y, z)$ such that $\delta(\lambda)$ has two double complex roots and therefore is not included in $\Sigma$. Let $\Sigma_{\text{reg}} = \Sigma \setminus [L \cup H]$ be the regular part of $\Sigma$.

The discriminant of the polynomial $\delta(\lambda)$ is given by

$$\Psi(x, y, z) = 16xz^4 - 4y^2z^3 - 128x^2z^2 + 144xyz^2 + 256x^3 - 27y^4. \quad (3.4)$$

Hence one deduces from (3.2) and (3.3) that

$$\Sigma_{\text{reg}} = \{(x, y, z) : (x, y, z) = 0, y \neq 0, x \neq \frac{z^2}{12}\}. \quad (3.5)$$

Assume that $\Sigma_{\text{reg}}$ is characteristic for $P$, i.e. if $p = \sigma^2(P)$ is its principal symbol,

$$p(d\Psi) = 0 \text{ at } \Sigma_{\text{reg}}. \quad (3.6)$$

Assume that $t(0) = 0$ and that

$$\Sigma^- = \Sigma \cap \Omega^- \quad (3.7)$$

is a smooth hypersurface of $\Omega^-$. 

Let $Q$ be the light cone for $P$ over $O$, then, see Proposition 3.3, $Q \cap \Sigma = E \cup B$, where away from $O$, $\Sigma$ and $Q$ intersect transversally at $E$ and are tangent to third order along $B$. Let $\mathcal{V}(\Sigma)$ and $\mathcal{V}(\Sigma, Q)$ be the Lie algebras of smooth vector fields tangent to $\Sigma$ and to $\Sigma$ and $Q$ respectively.

The following is then a simple consequence of the results of [17].

**Theorem 3.1** Let $u, Du \in H^s_{\text{loc}}(\Omega), s > \frac{3}{2}$, satisfy (1.2). If $u, Du \in I_kL^2_{\text{loc}}(\Omega^-, \mathcal{V}(\Sigma, Q))$, then $u, Du \in I_kL^2_{\text{loc}}(\Omega^-, \mathcal{V}(\Sigma, Q))$.

One deduces from Theorem 3.1

**Theorem 3.2** Let $u, Du \in H^s_{\text{loc}}(\Omega), s > \frac{3}{2}$, satisfy (1.2). If $u, Du \in I_kL^2_{\text{loc}}(\Omega^-, \mathcal{V}(\Sigma))$, then $u, Du \in I_kL^2_{\text{loc}}(\Omega, \mathcal{V}(\Sigma, Q))$. 

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In fact the results of [17] are stronger, we show that under the hypotheses of Theorem 3.1 the solution is strongly conormal in the sense of Melrose and Ritter, [12], along $B$ and in the sense of [16] along the cusp line $L$ of $\Sigma$.

In this note we shall restrict ourselves to the case where $u$ satisfies the weakly semilinear equation

$$Pu = f(z, u), \quad z \in \Omega. \tag{3.8}$$

Since it contains all new ideas involved in the proof of Theorem 3.1

I would like to acknowledge that the main new ideas in [17], originated in joint works (in progress) with R.B. Melrose, [13], and with R.B. Melrose and M. Zworski, [14]. I would like to thank them for sharing their ideas with me, for their interest and encouragement. Possible errors in this manuscript are of course my own fault.

4 Outline Of The Proof

To prove Theorem 3.1 in the case of the weakly semilinear equation (3.6) we shall introduce a family of spaces $J_k(\Omega) \subset I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma)), \; k \in \mathbb{N}_0,$ satisfying the following properties:

1) $J_{k+1}(\Omega) \subset J_k(\Omega) \subset L^2_{loc}(\Omega), \; J_0(\Omega) = I^2_{loc}(\Omega)$.

2) $J_k(\Omega)$ is a $C^\infty(\Omega)$-module.

3) $J_k(\Omega) \cap L^\infty_{loc}(\Omega)$ is a $C^\infty$ algebra.

4) $u, Du \in J_k(\Omega) \implies u \in J_{k+1}(\Omega)$.

5) $Pu = f \in J_k(\Omega), \; u = f = 0$ in $\Omega_T = \Omega \cap \{t < T\}$, then $u, Du \in J_k(\Omega)$.

6) If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}(\Sigma))$ in $\Omega^-$ satisfy (3.8), then $u, Du \in J_k(\Omega^-)$.

Proof of Theorem 3.1 : Suppose that such a family of spaces $J_k(\Omega)$ has been constructed. We then proceed by an induction argument. Let $\chi \in C^\infty(\mathbb{R}), \; \chi(s) = 0, \; s < -\frac{1}{2}, \; \chi(s) = 1, \; s > 0$. We obtain from (1.8)

$$P \chi u = \chi f(z, u) + [P, \chi] u. \tag{4.1}$$

If $u, Du \in J_0(\Omega) \cap J_1(\Omega^-)$, it follows from properties 2, 3 and 4 that the right hand side of (4.1) is in $J_1(\Omega)$. Thus one deduces from property 5 that $u, Du \in J_1(\Omega)$. By the same argument it follows that if $u, Du \in J_k(\Omega) \cap J_{k+1}(\Omega^-), \; \ell < k$, then $u, Du \in J_{k+1}(\Omega)$.

To define the spaces $J_k(\Omega)$ we shall introduce a blow-down map

$$\beta : X \to \mathbb{R}^3 \tag{4.2}$$
from a manifold with corners $X$ to $\mathbb{R}^3$ such that the lifts of $\Sigma$ and $Q$ by $\beta$ intersect each other and the boundary of $X$ transversally. We then define

$$J_k(\Omega) = \{ u \in L^2_{loc}(\Omega) : W^j \beta^* u \in L^2_{\mu}(X), \ j \leq k \}.$$  \hspace{1cm} (4.3)

Where $\mathcal{W}$ is a Lie algebra and $C^\infty(X)$ module of smooth vector fields in $X$ and $\mu$ is the lift of the Lebesgue measure of $\mathbb{R}^3$ under $\beta$. It will be a clear consequence of the definition of $X$ and $\mathcal{W}$ that $J_k(\Omega)$, defined by (4.3), satisfies properties 1, 2 and 4. It is a simple consequence of the Gagliardo-Nirenberg type of estimates of [11] that the spaces defined by (4.3) also satisfy property 3. Property 6 follows from Theorem 2.3 and from the results of [15]. The proof of property 5 is of course the most difficult one. The manifold with corners $X$ and the algebra $\mathcal{W}$ will be constructed in Section 6.

5 Model Case

An easy computation shows that, in coordinates where (3.3) holds, $\Sigma$ is invariant under the $\mathbb{R}^4$ action

$$m^{4-3-2}_s(x, y, z, t) = (s^4 x, s^3 y, s^2 z, t), \ s \in \mathbb{R}^+.$$  \hspace{1cm} (5.1)

Let $\mathcal{M}^{4-3-2}(\Omega) = \{ u \in C^\infty(\Omega) : \partial_x^a \partial_y^b \partial_z^c u(0, 0, 0, t) = 0, \ \forall a, b, c \in \mathbb{N}, \ 4a + 3b + 2c \leq r \}$

be the ideal of smooth functions having Taylor series at

$$O = \{(x, y, z, t) \in \Omega; \ x = y = z = 0 \}$$

consisting of terms of homogeneity $r$ or greater with respect to (5.1). A differential operator $P$ is said to have only terms of homogeneity $r'$ or greater, with respect to (5.1), if

$$P : \mathcal{M}^{4-3-2}_r(\Omega) \rightarrow \mathcal{M}^{4-3-2}_{r+r'}(\Omega), \ r \in \mathbb{N}_0, \ r + r' \geq 0.$$  \hspace{1cm} (5.3)

Simple computations show that if $P_0 = D_y^2 - D_z D_x$, then $\Sigma_{reg}$ is characteristic for $P_0$, in general one can prove, see [17] that

**Proposition 5.1** If $P$ and $\Sigma$ are as above and $(x, y, z, t)$ are smooth coordinates in which (3.3) holds, then

$$P = a(D_y^2 - D_z D_x) + P_{-5}, \ a \in C^\infty(\Omega), \ |a| > 0.$$  \hspace{1cm} (5.4)

where $P_{-5}$ has only terms of homogeneity $-5$ or greater with respect to (5.1).
Let $Q_0$ be the light cone for $P_0$ over $O$, then one easily finds that

$$Q_0 = \{(x,y,z) \in \Omega : y^2 - 4xz = 0\}. \quad (5.5)$$

In this model we find that away from $O$, $Q_0$ and $\Sigma$ are tangent to third order along $B_0$ and intersect transversally along $E_0$, where

$$B_0 = \{(x,y,z) \in \Omega : x = y = 0\}, \quad (5.6)$$
$$E_0 = \{(x,y,z) \in \Omega : x = \frac{3}{16} z^2, \ y^2 = -\frac{27}{32} z^3\}. \quad (5.7)$$

Fig 3:

As an immediate consequence of Proposition 5.1 one obtains

**Proposition 5.2** In the local coordinates of Proposition 5.1 one finds that

$$Q = \{(x,y,z,t) \in \Omega; \ q(x,y,z,t) = 0\}, \quad (5.8)$$

where

$$q = q_0 + q', \ q_0 = y^2 - 4xz, \ q' \in M^4 - 3 - 2(\Omega). \quad (5.9)$$

See [17] for a proof. Now we deduce from it more information about the interaction of $Q$ and $\Sigma$.

**Proposition 5.3** With $P$ and $\Sigma$ as in Proposition 5.1, in a small neighborhood of $O$, there are smooth functions $F_i(z,t), \ 1 \leq i \leq 3$, such that

$$Q \cap \Sigma = B \cup E$$

where

$$B = \{x = x^3 F_1(z,t), \ y = x^2 F_2(z,t)\}, \quad (5.10)$$
$$E = \{x = \frac{3}{16} z^2 + x^3 F_3(z,t), y^2 = -\frac{27}{32} z^3 + z^4 F_4(z,t)\}. \quad (5.11)$$

Away from $O$, $Q$ and $G$ meet transversally at $E$ and are tangent of third order at $B$. 

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6 Geometric Resolution

The hypersurfaces $\Sigma$ and $Q$ will be resolved to normal crossing by iterated quasi-homogeneous blow ups. As a first step we define the 4-3-2 blow up of $\mathbb{R}^n$ along $O = (0,0,0)$.

In $\mathbb{R}^3$ consider the non-round sphere

$$S_{4-3-2}^2 = \{(\omega_1, \omega_2, \omega_3); \omega_1^4 + \omega_2^6 + \omega_3^{12} = 1\}$$

and the map

$$\beta_1 : X_1 = [0, \infty) \times S_{4-3-2}^2 \longrightarrow \mathbb{R}^3, \quad \beta_1(s, \omega) = (s^4 \omega_1, s^3 \omega_2, s^2 \omega_3).$$

This is surjective and restricts to a diffeomorphism of $X_1 \setminus \partial X_1$ onto $\mathbb{R}^n \setminus K$. Moreover the $\mathbb{R}^+$ action (5.1) lifts to the standard multiplicative action on the factor $[0, \infty)$.

From these observations above it follows that the lifts of the hypersurfaces and the bicharacteristic $B$ in the model case are:

$$\Sigma^{(1)} = \text{clos}[\beta_1^{-1}(\Sigma \setminus O)] = \{16\omega_1 \omega_3^4 - 4\omega_2^2 \omega_3^3 - 128\omega_1^2 \omega_3^2 + 144\omega_1 \omega_3 \omega_2^2 + 256\omega_3^3 - 27\omega_2^2 = 0\},$$

$$Q_0^{(1)} = \text{clos}[\beta_1^{-1}(Q_0 \setminus O)] = \{\omega_2^2 - 4\omega_1 \omega_3 = 0\}, \quad (6.2)$$

$$B_0^{(1)} = \text{clos}[\beta_1^{-1}(B \setminus O)] = \{\omega_1 = 0, \omega_2 = 0\}. \quad (6.3)$$

Fig 4:
\( \Sigma^{(1)} \) has a cusp singularity at

\[ L^{(1)} = \text{clos}[\beta_{1}^{-1}(L \setminus O)] = \{ \omega_1 = -\frac{1}{12} \omega_3^2, \omega_2 = (-\frac{2}{3} \omega_3)^3 \} \]  

(6.4)

and a self-intersection at

\[ H^{(1)} = \text{clos}[\beta_{1}^{-1}(L \setminus O)] = \{ \omega_1 = -\frac{1}{4} \omega_3^2, \omega_2 = 0 \}. \]  

(6.5)

For reasons that will become clear later on, there are two "great circles" on \( S_{3-2-1}^2 \) that will have to be taken into consideration. We define

\[ C_1 = \{ \omega_1 = 0, r = 0 \}, \]  

(6.6)

\[ C_2 = \{ \omega_3 = 0, r = 0 \}. \]  

(6.7)

More generally we find, see [17]

**Proposition 6.1** In local coordinates in which (3.1) and (5.8) hold the lifts \( \Sigma^{(1)}, Q^{(1)} \) and \( B^{(1)} \) of the hypersurfaces and the bicharacteristic to \( X_1 \) are diffeomorphic, on \( X_1 \), to the model \( \Sigma_0^{(1)}, Q_0^{(1)} \) and \( B^{(1)} \) under a diffeomorphism fixing \( \partial X_1 \) pointwise. Conversely any diffeomorphism preserving (3.1), (5.8) and \( O \), lifts to a diffeomorphism of \( X_1 \) near \( \partial X_1 \) preserving \( \Sigma^{(1)} \) and \( Q^{(1)} \).

The full resolution of the geometry is obtained by blow ups of the three (really six) submanifolds \( L^{(1)}, D_0^{(1)} = Q^{(1)} \cap C_2 \) and \( B^{(1)} \). There are local coordinates \( (s, X, Y, T) \) near \( L^{(1)} \) with

\[ \Sigma^{(1)} = \{ Y^3 = X^2 \}, \]  

(6.8)

near \( D_0^{(1)} \) with

\[ Q^{(1)} = \{ X = Y^2 \}, C_2 = \{ X = 0, r = 0 \}. \]  

(6.9)

near \( B^{(1)} \) with

\[ Q^{(2)} = \{ X = 0 \}, \quad \Sigma^{(1)} = \{ X = Y^4 \}, \quad C_1 = \{ X = Y^2, r = 0 \}. \]  

(6.10)

Thus \( \Sigma^{(1)} \) can be resolved to normal crossing by a \( 3 - 2 \) blow-up of \( L^{(1)} \), thus set

\[ S_{3-2}^1 = \{ (\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1^4 + \theta_2^5 = 1 \} \]  

(6.11)
and in local coordinates (6.8) we construct the map

\[ \beta_{3-2} : [0, \infty), x [0, \infty), x^2 \to \mathbb{R}^{n-3} \]

\[ \beta_{3-2}(s, r, \theta) = (r, s^2 \theta_1, s^2 \theta_2). \]  

(6.12)  

(6.13)

Fig 5:

It will also be necessary to blow-up \( D^{(1)} \) with homogeneity 2-1-1, thus let

\[ S^2_{2-1-1} = \{ (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^2 ; \theta_1^2 + \theta_2^2 + \theta_3^2 = 1 \} \]  

(6.14)

and in local coordinates (6.9) construct the map

\[ \beta_{2-1-1} : [0, \infty)_R \times S^1_{2-1} \times \mathbb{R}^{n-3} \to X_1 \]

\[ \beta_{2-1-1}(s, R, \omega, t) = (R, s^2 \theta_1, s \theta_2, s \theta_3, t). \]  

(6.15)  

(6.16)

Fig 6:

To resolve \( Q^{(1)}, \Sigma^{(1)} \) and \( C_1 \) to normal crossing it will be more convenient to use four normal blow-ups as in [12]. Since \( Q^{(1)} \) and \( \Sigma^{(1)} \) are tangent to third order at \( B^{(1)} \), if \( C_1 \) did not have to be taken into consideration, one could use a 4-1 nonhomogeneous blow-up to resolve \( Q^{(1)} \) and \( \Sigma^{(1)} \) to normal.

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crossing, but $C_1$ destroys the 4-1 homogeneity.

Fig 7:

Since $D^1, L^1$ and $B^1$ are disjoint we can use these maps to replace small neighborhoods of $D_0^{(1)}$, $L^{(1)}$, $B^{(1)}$ by their respective blow ups and so define the manifold with corners $X$ and a blow down map $\beta : X \to X_1$. Let

$$\beta = \beta_2 \circ \beta_1 : X \to \mathbb{R}^n$$

(6.17)

Denote

$$Q^{(2)} = \text{clos}[\beta_2^{-1}(Q^{(1)} \setminus (B^{(1)} \cup D_0^{(1)}))]$$
$$\Sigma^{(2)} = \text{clos}[\beta_2^{-1}(\Sigma^{(1)} \setminus (L^{(1)} \cup B^{(1)}))]$$
$$L^{(2)} = \text{clos}[\beta_2^{-1}(L^{(1)})]$$
$$B^{(2)} = \text{clos}[\beta_2^{-1}(B^{(1)})]$$
$$C_1^{(2)} = \text{clos}[\beta_2^{-1}(C_1 \setminus B^{(1)})]$$
$$C_2^{(2)} = \text{clos}[C_2 \setminus D_0^{(1)}].$$

The circle $C_2^{(2)}$ does not continue into the boundary face introduced by the 2-1-1 blow-up.

The manifold with corners $X$ has twelve boundary hypersurfaces which meet transversally pairs or triples. Let $\rho_L, \rho_B^j, 1 \leq j \leq 8, \rho_D$ and $\rho_K$ be respectively the defining functions of $\beta^{-1}(L)$, each of the eight hypersurfaces of $\beta^{-1}(B), \beta^{-1}(D)$ and $\beta^{-1}(K)$ (These functions are assumed to be extended smoothly past the surfaces they define).
Proposition 6.2 Under the \( C^\infty \) map \( \beta : X \to \mathbb{R}^n \) the lifts
\[
\beta^*(M) = \text{clos}[^{-1}(M \setminus [K \cup L \cup B])], \quad (6.18)
\]
for \( M = Q, \Sigma \) are smooth hypersurfaces that intersect the boundaries of \( X \) transversally. Any \( C^\infty \) diffeomorphism of \( X \) preserving \( \Sigma, Q(1), D_0^{(1)} \) and \( \partial X \) lifts to a \( C^\infty \) diffeomorphism of \( X \) preserving all boundaries and \( \partial X \).

Let \( L^2_c(X) \) be the space of compactly supported square integrable functions in \( X \) with respect to the measure \( \mu = \beta^*(dx dy dz) \). Then the blow down map \( \beta \) gives an isomorphism
\[
\beta^* : L_c(\mathbb{R}^n) \leftrightarrow L^2_c(X). \quad (6.19)
\]
Let \( \mathcal{W} \) be the Lie algebra and smooth vector fields \( W \) on \( X \) satisfying the following properties:
1) \( W \) is tangent to all boundary hypersurfaces.
2) \( W \) is tangent to \( \beta^*(\Sigma) \) and to \( \beta^*(Q) \).
3) \( W \) is tangent to \( C_2^{(2)} \).
4) In local coordinates \((r, s, X)\) in which \( \rho_K = r \) and \( C_1^{(2)} = \{r = X = 0\} \),
\( \mathcal{W} \) is spanned by \( r \partial_r, s \partial_s, X \partial_X, r^2 \partial_X \).

We then define for any integer \( k \)
\[
J_k(\Omega) = \{u \in L^2_c(\Omega) : \beta^* u \in I_k L^2_c(X, \mathcal{W})\} \quad (6.20)
\]
As a consequence of Propositions 6.1 and 6.2 it follows that the spaces \( J_k(\Omega) \) are independent on the choices of coordinates. Moreover from the Gagliardo-Nirenberg type inequalities of [15] one obtains

Proposition 6.3 For any \( k \in \mathbb{N} \), \( J_k(\Omega) \cap L^\infty_\text{loc}(\Omega) \) is a \( C^\infty \) algebra, i.e for any \( f \in C^\infty(\mathbb{R}^n) \) and \( u_i \in J_k(\Omega) \cap L^\infty(\Omega), 1 \leq i \leq m, \)
\[
f(u_1, ..., u_m) \in J_k(\Omega) \cap L^\infty_\text{loc}(\Omega). \quad (6.21)
\]

By writing the generators of \( \mathcal{V}(\Sigma, Q) \) and their lift under the map \( \beta \) it is not hard to see that
\[
J_k(\Omega) \subseteq I_k L^2_\text{loc}(\Omega, \mathcal{V}(\Sigma, Q)) \quad (6.22)
\]
7 The Linear Propagation Theorem

In this section we sketch the proof that the spaces \( J_k(\Omega) \) satisfy

**Theorem 7.1** Let \( f \in J_k(\Omega) \), \( f = 0 \) in \( \Omega^- \). Let \( u \in H^1_{loc}(\Omega) \), \( u = 0 \) in \( \Omega^- \), satisfy

\[
Pu = f.
\]

Then \( u, Du \in J_k(\Omega) \).

**Lemma 7.1** Let \( \phi \in C_0^\infty(X_1) \), \( \phi = 1 \) in sufficiently small neighborhoods of \( L^{(1)}, E^{(1)} \) and \( H^{(1)} \), \( \phi = 0 \) outside slightly bigger neighborhoods. There exist \( v_1, Dv_1 \in J_k(\Omega) \) such that

\[
\beta_1^*(Pv_1) - \phi \beta_1^* f \in I_kL^2_{loc}(X_1, \partial X_1)
\]

The proof of Lemma 7.1 is based on the fact that the lift of the operator \( P \) by the map \( \beta_1 \) is of real principal type in the totally characteristic sense, see [10], in some directions near \( L^{(1)}, E^{(1)} \) and \( H^{(1)} \). One can then use the calculus of totally characteristic Fourier Integral Operators of [10] to transform the operator, the characteristic surfaces and their intersections into model cases. Lemma 7.1 is then a consequence of the mapping properties of these operators.

**Lemma 7.2** Let \( g \in L^2_{loc}(\Omega) \) be such that

\[
\beta^* g \in I_kL^2_{loc}(X, \partial X_1).
\]

Then there exists \( v_2, Dv_2 \in J_k(\Omega) \) such that \( Pv_2 = g \).

The proof of Lemma 7.2 is considerably simpler than the one of Lemma 7.1, it is based on a commutator argument.

7.1 Marked Lagrangian Distributions

Let \( \Lambda \subset T^*\Omega \) be a smooth conic closed Lagrangian and let \( S_2 \subset S_1 \subset \Lambda \) be conic smooth hypersurfaces. Denote

\[
\mathcal{M}(\Lambda)_1 = \{ A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \text{ at } \Lambda, \ H_a \text{ tangent to } S_1 \text{ and to } S_2 \}
\]

and define

\[
I_kL^2_c(\Omega, \mathcal{M}(\Lambda)_1) = \{ u \in L^2_c(\Omega) : \mathcal{M}(\Lambda)^j u \subset L^2_{loc}(\Omega), \ j \leq k \}.
\]
A detailed study of these distributions can be found in [8]. As mentioned in Section 2, the marked Lagrangian Distributions were first introduced by Melrose in [9] to study the cusp case.

Let $\Lambda \Sigma = \text{clos} N^* (\Sigma_{reg})$, $\Lambda Q = \text{clos} N^* (Q \setminus O)$. It is well known that $\Lambda \Sigma$ and $\Lambda Q$ are smooth conic Lagrangian submanifolds of $T^* \mathbb{R}^3$. Let $\Lambda _B = N^* B$ and let $\Lambda _O = T^*_O \mathbb{R}^3$, denote $S_1 = \Lambda \Sigma \cap \Lambda _B = \Lambda Q \cap \Lambda _B = \Lambda \Sigma \cap \Lambda _O$ and $S_2 = \Lambda \Sigma \cap \Lambda _O$. Let $S_3 = \Lambda \Sigma \cap \Lambda Q$ and let $I_k L^2_{ \text{loc}} (\Omega, \mathcal{M}(\Lambda _O)_3)$ be the space of marked Lagrangian distributions to $\Lambda _O$ marked by $S_3$ and $S_2$.

In coordinates where (3.1) holds one obtains that $\mathcal{M}(\Sigma)_1$ is the $\Psi^0(\Omega)$ span of

$$V_1 = 4x \partial_x + 3y \partial_y + 2z \partial_z, \quad V_2 = (2xz - \frac{3}{4} y^2) \partial_x - \frac{1}{2} yz \partial_y + 4xz \partial_z, \quad (7.7)$$

$$P_1 = z(\partial_y^2 - \partial_x \partial_z), \quad P_2 = y(\partial_y^2 - \partial_x \partial_z), \quad (7.8)$$

$$P_3 = 4\partial_x^2 + 2z \partial_y^2 + y \partial_y \partial_z, \quad P_4 = (\partial_y^2 - \partial_x \partial_z) \partial_z, \quad (7.9)$$

$$P_5 = (\partial_y^2 - \partial_x \partial_z) \partial_y. \quad (7.10)$$

Times elliptic factors of the appropriate orders. The space of marked Lagrangian distributions to the swallowtail marked by $S$ and $S_1$ is however too small for our purposes, we shall need a slightly bigger one. Let $P'_S = (3\partial_y^2 - 8\partial_x \partial_z - 12z \partial_x^2) \partial_y^2$ and define the space of “supermarked” Lagrangian distributions to $\Lambda \Sigma \ S$ and $S_1$ as

$$I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda \Sigma)_1)s = \{ u \in L^2_\mathcal{L}(\Omega) : V_1^{\ell_1} V_2^{\ell_2} P_1^{\ell_3} P_2^{\ell_4} P_3^{\ell_5} u \in H^{-\ell}(\Omega), \quad \ell = \ell_1 + \ell_2 + \ell_3 + \ell_4 + 6\ell_5 \leq 3k \}. \quad (7.11)$$

Where the superscript $s$ is for “supermarked”. The spaces of supermarked Lagrangians was introduced by M. Zworski in [18] where a more detailed description of those spaces is given. One defines the space $I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Sigma)_1)s$ for all integers $k$ by complex interpolation. One can easily show that

$$I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda \Sigma)_1)s \subset I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda \Sigma)_1)s. \quad (7.12)$$

Let

$$M_k(\Omega) = I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda \Sigma)_1)s + I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda Q)_1)s + I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda B)_1)s + I_k L^2_\mathcal{L}(\Omega, \mathcal{M}(\Lambda O)_3)s \quad (7.13)$$

be the space of marked Lagrangian distributions to $\Sigma, Q$ and $B$. 

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Lemma 7.3 Let \( g \in J_k(\Omega) \) be such that \( \beta^* g \) is supported away from \( E^{(1)} \), \( H^{(1)} \) and \( L^{(2)} \), then \( g \in M_k(\Omega) \).

The proof of Lemma 7.3 is quite long and consists basically of lifting the generators of each of the components of \( M_k \) under the map \( \beta \). Now we are going to use the same idea as in the case of the cusp, first we prove a propagation theorem for \( M_k(\Omega) \) and then use again the equation to show that the solution is in fact in the smaller space \( J_k(\Omega) \). By commutator methods one can prove

Lemma 7.4 Let \( f \in M_k(\Omega) \), there exist \( v_3, Dv_3 \in M_k(\Omega) \) such that
\[
P v_3 = f.
\]

Then one proves an elliptic regularity type of Theorem which states that

Lemma 7.5 Let \( v, Dv \in M_k(\Omega) \) be such that \( P v \in J_k(\Omega) \). Then
\[
v, Dv \in J_k(\Omega).
\]

When one lifts \( v \in M_k(\Omega) \) under the map \( \beta \) one finds that it may be singular at some circles at the boundary of \( X \), but it turns out that the lift of operator \( P \) under the map \( \beta \) is elliptic in some directions of \( bT^*X \) over those circles and therefore one concludes that if \( v \) satisfies the inclusion \( P v \in J_k(\Omega) \), then \( v \in J_k(\Omega) \). This is the reason why one has to include the great circles in the definition of the spaces, since the hypersurfaces \( \{ z = 0 \} \) and \( \{ z = 0 \} \) are characteristic for \( P_0 \) the lift of the operator \( P \) could not be elliptic on circles \( C_1^{(2)} \) and \( C_2^{(2)} \).

Conclusion of the proof of Theorem 7.1:

Let \( v_1, v_2 \) and \( v_3 \) be as in Lemmas 7.1, 7.2 and 7.3 and \( w = u - v_1 - v_2 - v_3 \).

Then
\[
P w = 0, \ w \in J_k(\Omega) \ \text{in} \ t < 0. \quad (7.14)
\]

Let
\[
\mathcal{M}(\Lambda Q \cup \Lambda \Sigma) = \{ A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \ \text{on} \ \Lambda Q \cup \Lambda \Sigma \} \quad (7.15)
\]

Equation (7.14) implies that
\[
w, Dw \in L^2_{loc}(\Omega^-, \mathcal{M}(\Lambda Q \cup \Lambda \Sigma)). \quad (7.16)
\]

By commutator methods one can easily show that
\[
w, Dw \in L^2_{loc}(\Omega, \mathcal{M}(\Lambda Q \cup \Lambda \Sigma)). \quad (7.17)
\]
By the arguments used in the proof of Lemma 7.3 one can show that

\[ I_k L^2_{loc}(\Omega, \mathcal{M}(\Lambda Q \cup \Lambda \Sigma)) \subset J_k(\Omega). \quad (7.18) \]

This concludes the proof of Theorem 7.1.

References


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