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COUNTEREXAMPLES TO LOCAL EXISTENCE FOR NONLINEAR WAVE EQUATIONS

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Introduction and the main results. In this paper we study how much regularity of data that is needed to ensure local existence of a solution to quasilinear wave equations. We give counterexamples to local existence for the typical model equations. The counter examples we construct for semilinear wave equations are known to be sharp, i.e. one can prove that one has local existence if data has slightly more regularity. The existence part has been shown in recent papers by Klainerman-Machedon[3], Ponce-Sideris[8], Beals-Bezard[1], Lindblad[4], Lindblad-Sogge[7], using space time estimates know as Strichartz' estimates or refinement of these.

We are considering the Cauchy problem for a quasilinear wave equations :

$$\begin{aligned} &\Box u = G(u,u',u''), \quad (t,x) \in \mathbb{R}^{1+n}, \quad 0 \le t < T, \\ &u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{aligned} \tag{0.1}$$

where G is a smooth function of u and its derivatives up to second order which is assumed to be linear in the second order derivatives and vanishing to second order at the origin. (Here $\Box = \partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2$.) Let $\dot{H}^\gamma$ denote the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^\gamma} = \| \, |D_x|^\gamma f\|_{L^2}$ where $|D_x| = \sqrt{-\Delta_x}$ and let

$$||u(t,\cdot)||_\gamma^2 = \int \left(\big| \, |D_x|^{\gamma-1} u_t(t,x) \big|^2 + \big| \, |D_x|^\gamma u(t,x) \big|^2 \right) dx. \tag{0.2}$$

We are studying which is the smallest possible γ such that

$$(f,g) \in \dot{H}^\gamma(\mathbb{R}^n \times) \dot{H}^{\gamma-1}(\mathbb{R}^n), \tag{0.3}$$

and

$$\operatorname{supp} f \cup \operatorname{supp} g \in \{x; |x| \le 2\}, \quad \operatorname{sing\,supp} f \cup \operatorname{sing\,supp} g \in \{0\} \tag{0.4}$$

implies that we have a local distributional solution of (0.1) for some $T > 0$, which satisfies

$$(u, \partial_t u) \in C_b\big([0,T]; \dot{H}^\gamma(\mathbb{R}^3) \times \dot{H}^{\gamma-1}(\mathbb{R}^3) \big) \tag{0.5}$$

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To avoid certain peculiarities when it comes to uniqueness we make the following definition:

Definition 0.1. We say that u is a proper solution of (0.1) if it is a distributional solution which satisfies (0.5) and if in addition u is the weak limit of a sequence of smooth solutions u_ε to (0.1) with data $(\phi_\varepsilon * f, \phi_\varepsilon * g)$, where $\phi_\varepsilon \in C_0^\infty$, $\phi_\varepsilon \to \delta$ and $\operatorname{supp} \phi_\varepsilon \to \{0\}$. □

The problem is that even if one has smooth data and hence a smooth solution there might still be another distributional solution which satisfies initial data in the space given by the norm (0.2). This phenomena was first observed by Shatah–Shadi Tahvildar-Zadeh[10] for wave maps. An easier example is given by $u(t,x) = 2H(t-|x|)/t$ which satisfies $\Box u = u^3$ in the sense of distribution theory, (see Lemma 1.2 in Lindblad[6]). Furthermore $||u(t,\cdot)||_\gamma \to 0$, when $t \to 0$ if $\gamma < 1/2$, by homogeneity. Since $u(t,x) = 0$ is another solution with the same data it follows that we have nonuniqueness in the class (0.5) if $\gamma < 1/2$.

Our main theorem is the following:

Theorem 0.2. *Consider the problem in 3 space dimensions, $n = 3$, with*

$$\begin{aligned} \Box u = G_{kl} &= \big((\partial_t - \partial_{x_1})^l u\big)\big((\partial_t - \partial_{x_1})^{k-l} u\big) \\ u(0,x) &= f(x), \quad u_t(0,x) = g(x) \end{aligned} \tag{0.6}$$

where $0 \le l \le k \le 2$, $l = 0, 1$. *Let $\gamma = k$. Then there are data (f,g) satisfying (0.3)-(0.4), with $\|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}$ arbitrarily small, such that the following holds:*

i) (0.6) doesn't have any proper solution satisfying (0.5) for any $T > 0$.

ii) If $k - l \le 1$ then in addition there is no proper solution such that the right hand side of (0.6) makes sense as a distribution.

iii) If $k = l = 0$ then in addition there is no distributional such that the right hand side of (0.6) makes sense as a distribution.

Remark 0.3. It follows from the proof of the theorem above that the problem is illposed if γ is as above. In fact there exist a sequence of data $f_\varepsilon, g_\varepsilon \in C_0^\infty(\{x; |x| \le 1\})$ with $\|f_\varepsilon\|_{\dot H^\gamma} + \|g_\varepsilon\|_{\dot H^{\gamma-1}} \to 0$ such that if T_ε is the largest number such that (0.1) has a solution $u_\varepsilon \in C^\infty([0,T_\varepsilon) \times \mathbb{R}^3)$ we have that either $T_\varepsilon \to 0$ or else there are numbers $t_\varepsilon \to 0$, such that $0 < t_\varepsilon < T_\varepsilon$ such that $\|u_\varepsilon(t_\varepsilon,\cdot)\|_\gamma \to \infty$. □

Remark 0.4. Just by scaling, one gets a simple, less good, counter example to well-posedness with

$$\gamma < k + \frac{n-4}{2}. \tag{0.7}$$

In fact if we have u is a solution of (0.1) which blows up when $t = T$ then $u_\varepsilon(t,x) = \varepsilon^{k-2} u(t/\varepsilon, x/\varepsilon)$ is a solution of the same equation with lifespan $T_\varepsilon = \varepsilon T$ and $\|u_\varepsilon(0,\cdot)\|_\gamma = \varepsilon^{k+(n-3)/2-\gamma}\|u(0,\cdot)\|_\gamma \to 0$ if γ satisfies (0.7). The natural generalization of the counterexample to general number of space dimensions n is

$$\gamma < k + \frac{n-3}{4} \tag{0.8}$$

Presumably the argument here together with a cutoff argument as in Lindblad[4] or Lindblad-Sogge[7] would give a proof of this. $\square$

The proof of Theorem 0.2 will appear in Lindblad[5]. The special case when $k = l = 0$ was proven in Lindblad[4]. We will however give a short and explicit proof here. At least in the semilinear case, $k - l \leq 1$, Theorem 0.2 is sharp, i.e. we have local existence in H^s, for $s > k$. A related theorem was proven in Lindblad-Sogge[7]:

Theoremn 0.5. *The problem*

$$\begin{aligned} \square u = |u|^\kappa, \quad (t,x) \in \mathbb{R}^{1+n}, \quad 0 \leq t < T, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{aligned} \tag{0.1}$$

is illposed in H^γ for general data (f,g) satisfying (0.3), if $n \geq 2$ and

$$\gamma < \gamma(\kappa) = \begin{cases} \frac{n+1}{4} - \frac{1}{\kappa-1}, & \kappa \leq \frac{n+3}{n-1}, \\ \frac{n}{2} - \frac{2}{\kappa-1}, & \text{if } \kappa \geq \frac{n+3}{n-1}. \end{cases}$$

Here n is the number of space dimensions. Moreover, if $\gamma \geq \gamma(\kappa)$ and $\kappa \geq \kappa_0$, where $\kappa_0 = \frac{(n+1)^2}{(n-1)^2+4}$, if $n \geq 3$, and $\kappa_0 = 3$ for $n = 2$, then we have a local solution in H^γ.

1. Proof in the semilinear case without derivatives: $\square u = u^2$. The result of Theorem 0.2 in case $k = l = 0$ was already proven in Lindblad[4] using the asymptotic behavior of a linear solution to show that the problem was ill-posed and from that constructing data for which we had no local existence by adding up a sequence of data. Using the formulas from there together with our new way of constructing counterexamples we can now give explicit data for which there is no local solution. We will show that there are data $f \in L^2(\mathbb{R}^3)$ and $g = 0$ such that

$$\square u = u^2, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x),$$

does not have any solution $u \in L^2([0,T] \times \mathbb{R}^3)$ for any $T > 0$. A distributional solution of the above equatipon with a right hand side in $L^1([0,T] \times \mathbb{R}^3)$ is also a solution of the convolution equation (see Hörmander[2])

$$u = E * (Hu^2) + u_L, \quad \text{where } \square u_L = 0, \quad u_L(0,x) = f(x), \ \partial_t u_L(0,x) = g(x)$$

Here E is the forward fundamental solution of $\square$ and $H(t) = 1$ when $t > 0$ and 0 otherwise. First we will make the simple observation that it follows from the positivity of the fundamental solution that $u(t,x) \geq u_L(t,x)$ and if $u_L(t,x) \geq 0$ then $u(t,x)^2 \geq u_L(t,x)^2$. Hence if

$$u_L(t,x) \geq 0, \quad \text{when} \quad (t,x) \in K = \{(t,x); t + |x - (1,0,0)| < 1, \ t > 0\} \tag{1.2}$$

then

$$u(t,x) - u_L(t,x) \geq E * (\chi_K u_L^2)(t,x), \tag{1.3}$$

where χ_K is the characteristic function of K. We will show that one can find data $f \in L^2$ and $g = 0$ such that (1.2) holds and

$$\int_0^T \int \left| E * (\chi_K u_L^2)(t,x) \right|^2 dx\, dt = \infty, \quad \text{for any } T > 0. \tag{1.4}$$

By (1.3) this shows that

$$\int_0^T \int u(t,x)^2\, dx\, dt = \infty, \quad \text{for any } T > 0. \tag{1.5}$$

We claim data the following data will do:

Lemma 1.1. *Let* $3/4 < \alpha < 1$ *and set*

$$f(x) = \begin{cases} h(x_1) = \frac{\varepsilon}{x_1 |\log|x_1/4||^\alpha}, & \textit{when} \quad |x - (1,0,0)| < 1, \\ 0, & \textit{otherwise} . \end{cases}$$

Then

$$\int f(x)\, dx = C_\alpha < \infty.$$

The proof is an easy calculation.
With this data we have inside K

$$u_L(t,x) = \frac{h(x_1 - t) + h(x_1 + t)}{2}$$

and hence $u_L(t,x)^2 \geq h(x_1 - t)^2/4$ in K.

Lemma 1.2. *Let* $\mu(t,x) = \delta(t - x_1)\delta(x_2)\delta(x_3)H(t)$. *Then*

$$E * \mu(t,x) = \frac{1}{\pi} \frac{H(t - |x|)}{t - x_1}$$

and

$$\int |E * \mu(t,x)|^2\, dx = \infty, \quad \textit{if } t > 0.$$

For a proof see Lindblad[6]. We need the following consequence of the lemma:

Lemma 1.3. *Assume that* $F(t,x) = k(x_1 - t, x_2, x_3)$ *and let* $\mu_y(t,x) = \mu(t, x - y)$. *Then*

$$E * (HF)(t,x) = \int E * \mu_y(t,x) k(y)\, dy = \int \frac{H\big(t^2 - |x - y|^2\big)}{t - (x_1 - y_1)} k(y)\, dy.$$

Since the set $\{(t,x); |x - (1/2 + t, 0, 0)| \leq 1/2, 0 < t < 1/2\}$ is contained in K we conclude using the formula and writing $x = (x_1, x')$, $y = (y_1, y')$ that

$$E * (\chi_K u_L^2)(t,x) \geq \frac{1}{4} \int_0^1 \int_{|y'|^2 \leq y_1 - y_1^2} \frac{H\big(t^2 - (x_1 - y_1)^2 - |x' - y'|^2\big)}{t - (x_1 - y_1)} h(y_1)^2\, dy'\, dy_1$$

Introducing polar coordinates $y' = (r\cos\theta, r\sin\theta)$, $dy' = rdrd\theta$, and using the fact that over half the region in θ, $x'\cdot y' \geq 0$ we conclude that

$$E*(\chi_K u_L^2)(t,x) \geq \frac{\pi}{4}\int_0^1 \frac{\int_0^{\sqrt{y_1-y_1^2}} H\big(t^2-(x_1-y_1)^2-|x'|^2-r^2\big)\,rdr}{t-(x_1-y_1)} h(y_1)^2\,dy_1$$

Assuming that $0 < x_1 < t < 1/2$ and integrating only over the region $0 < y_1 < (t-x_1)/4$, $0 < r^2 < ty_1$ we obtain that $t^2-(x_1-y_1)^2-|x'|^2-r^2 \geq (t^2-x_1^2)/2-|x'|^2$. Therefore assuming that

$$(t^2-x_1^2)/2-|x'|^2 \geq 0$$

we obtain

$$E*(\chi_K u_L^2)(t,x) \geq \frac{\pi}{5}\frac{t}{t-x_1}\int_0^{(t-x_1)/4} y_1 h(y_1)^2\,dy_1.$$

Hence

$$E*(\chi_K u_L^2)(t,x) \geq \frac{Ct\varepsilon H\big(t^2-x_1^2-2|x'|^2\big)}{(t-x_1)|\log((t-x_1)/16)|^{2\alpha-1}}.$$

It follows that

$$\int |E*(\chi_K u_L^2)(t,x)|^2\,dx \geq C(t\varepsilon)^2\int_0^t \frac{\int_{|x'|^2\leq(t^2-x_1^2)/2} dx'dx_1}{(t-x_1)^2|\log((t-x_1)/16)|^{4\alpha-2}} =$$

$$C(t\varepsilon)^2\int_0^t \frac{\pi(t^2-x_1^2)/2dx_1}{(t-x_1)^2|\log((t-x_1)/16)|^{4\alpha-2}} = \infty.$$

Since by assumption $0 < 4\alpha-2 < 1$. This proves (1.4) and hence (1.5).

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