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EIGENVALUE ESTIMATES FOR A CLASS OF OPERATORS RELATED TO SELF-SIMILAR MEASURES

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Abstract. We obtain the sharp order of growth of the eigenvalue distribution function for the operator in the Sobolev space $H_0^1(X)$, generated by the quadratic form $\int_X |u|^2 d\mu$, where $X \subset \mathbb{R}^d$ is a domain and μ is a probability self-similar fractal measure on X.

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1. The notions of self-similar set and self-similar measure were introduced by Hutchinson [H]. Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a set of contractive similitudes on $\mathbb{R}^d$, $h_1, ..., h_m$ – their coefficients of contraction. Also, let a system of positive numbers ("weights") $\mathbf{p} = \{p_1, ..., p_m\}$ be given, such that $p_1 + \cdots + p_m = 1$.

Then there exists a unique non-empty compact set $\mathcal{C} = \mathcal{C}(\mathcal{S}) \subset \mathbb{R}^d$ such that $\mathcal{C} = \bigcup_{k=1}^m S_k\mathcal{C}$, and a unique boundedly supported probability Borelian measure $\mu = \mu(\mathcal{S}, \mathbf{p})$ which satisfies the self-similarity property

$$\mu = \sum_{k=1}^{m} p_k \mu \circ S_k^{-1} . \tag{1}$$

Furthermore, $\mathcal{C}(\mathcal{S}) = \operatorname{supp} \mu(\mathcal{S}, \mathbf{p})$. Measures $\mu(\mathcal{S}, \mathbf{p})$, corresponding to the same set $\mathcal{S}$ but to different families $\mathbf{p}$, are mutually singular.

It is $\mu(\mathcal{S},\mathbf{p})$ that we are interested in. For the rest of the paper $\mu = \mu(\mathcal{S},\mathbf{p})$.

We always suppose that $\mathcal{S}$ meets the "open set condition" (see [H]). This means that there exists a bounded open set $\Omega \subset \mathbb{R}^d$, such that

$$1.\ \bigcup_{k=1}^{m} S_k(\Omega) \subset \Omega\ , \qquad 2.\ S_k(\Omega) \bigcap S_\ell(\Omega) = \emptyset \ \text{ for } \ k \neq \ell\ . \tag{2}$$

Evidently, (2) implies that $C(\mathcal{S}) \subset \overline{\Omega}$. It follows also that $\sum_{k=1}^{m} h_k^d \leq 1$. In addition, we suppose that

$$\mu(\partial\Omega) = 0\ . \tag{3}$$

It is known that otherwise $\mu(\Omega) = 0$ (see [Lau W]). Assumption (3) is of a rather technical character and can often be withdrawn.

If, for example, $d = 1$, $m = 2$, $S_1(x) = \frac{x}{3}$, $S_2(x) = 1 - \frac{x}{3}$ and $p_1 = p_2 = \frac{1}{2}$, then $C(\mathcal{S})$ is the classical Cantor set, and $\mu(\mathcal{S},\mathbf{p})$ is the Cantor measure. Here (2) is satisfied, with $\Omega = (0,1)$. Evidently, (3) is satisfied too.

2. Let $X \subset \mathbb{R}^d$ be a bounded domain, such that $\Omega \subseteq X$. In the Sobolev space $H_0^1(X)$, with the Dirichlet integral $\int_X |\nabla u|^2 dx$ as the metric form, we consider the quadratic functional

$$Q_\mu[u] = \int_X |u|^2 d\mu\ , \quad \text{where} \ \ \mu = \mu(\mathcal{S},\mathbf{p})\ .$$

If $d = 1$, then Q_μ is always bounded and the corresponding embedding $H_0^1(X) \subset L_2(X,\mu)$ is compact. To ensure the same for $d \geq 2$, additional requirements on $\mathcal{S}$ and $\mathbf{p}$ are necessary. Denote

$$A_k = h_k^{2-d} p_k\ , \quad \overline{A}(\mathcal{S},\mathbf{p}) = \max_k A_k\ .$$

Lemma. *Let (2) be satisfied. Then the imbedding $H^1(\Omega) \subset L_2(\Omega, d\mu)$ is compact if and only if*

$$\overline{A}(\mathcal{S},\mathbf{p}) < 1\ . \tag{4}$$

Necessity is quite straightforward and does not need (2). Sufficiency can be easily derived from [M], §8.8.

Note that (2) and (4) imply that the Hausdorff dimension of μ satisfies
$\beta := \dim\, \mu(\mathcal{S},\mathbf{p}) > d - 2$.

If (4) is satisfied, then $Q_\mu[u]$ generates in $H_0^1(X)$ a compact, self-adjoint and positive operator, say T_μ or, in a more detailed notation, $T_{\mu,X}$. We are interested in the behaviour of its eigenvalues $\lambda_k(T_\mu)$. As usual, we express this behaviour in terms of the corresponding distribution function

$$n(t, T_\mu) = \#\{k : \lambda_k(T_\mu) > t\} .$$

If $u \in H_0^1(X)$ is an eigenfunction of T_μ and λ is the corresponding eigenvalue, then for any $v \in H_0^1(X)$,

$$\lambda \int_X \nabla u \cdot \nabla \bar{v} dx = \int_X u\bar{v} d\mu . \tag{5}$$

For general self-similar μ, it is impossible to rewrite (5) in terms of any classical boundary value problem. Notice only that always $u(x) = 0$ outside $\mathcal{C}(\mathcal{S})$.

3. We write $\Omega \in \mathcal{P}$ if the Poincaré inequality

$$\int_\Omega |u|^2 dx \leq K \left(\int_\Omega |\nabla u|^2 dx + \left| \int_\Omega u dx \right|^2 \right) \tag{6}$$

is satisfied for any $u \in H^1(\Omega)$. Any bounded Ω whose boundary is "not too bad" belongs to $\mathcal{P}$; see [M] for the general description of domains $\Omega \in \mathcal{P}$.

Theorem 1. *Suppose that $\Omega \in \mathcal{P}$ and $\mu = \mu(\mathcal{S}, \mathbf{p})$. Let (2) and (4) be satisfied. Define $\delta > 0$ as the unique solution of*

$$\sum_{k=1}^m A_k^\delta = \sum_{k=1}^m \left(h_k^{2-d} p_k\right)^\delta = 1 . \tag{7}$$

Let X be an arbitrary open set in $\mathbb{R}^d$ such that $X \supset \Omega$. Then there exists a number $C = C(S, \mathbf{p}, X)$ such that

$$n(t, T_{\mu,X}) \leq Ct^{-\delta} , \quad \text{any} \ \ t > 0 . \tag{8}$$

If, besides, (3) is satisfied, then there exist also $c > 0$ and $t_0 > 0$, such that

$$n(t, T_{\mu,X}) \geq ct^{-\delta} , \quad \text{any} \ \ t \in (0, t_0) . \tag{9}$$

Remark. If (3) is not satisfied and $X = \Omega$, then $T_\mu = 0$ and (9) fails. For open sets $X \supset \overline{\Omega}$, (9) probably remains valid without assumption (3).

4. A more detailed analysis is possible for $d = 1$. Suppose for simplicity that $X = \Omega = (0,1)$. Denote

$$c_k = \log A_k = \log(h_k p_k) , \quad k = 1, \ldots, m .$$

We suppose $m \geq 2$, because otherwise $\mu(\mathcal{S}, \mathbf{p})$ is the Dirac measure supported at a point $x_0 \in [0,1]$, and the problem considered is meaningless.

Theorem 2. *Let $d = 1$, $X = \Omega = (0,1)$, $m \geq 2$, and (2) be satisfied. If, besides, at least one of the ratios $\frac{c_k}{c_\ell}$ is irrational, then there exists a positive constant B, such that*

$$n(t; T_\mu) \sim Bt^{-\delta} \quad \text{as } \ t \to 0 . \tag{10}$$

If all the ratios $\frac{c_k}{c_\ell}$ are rational, then there exists a uniformly positive, bounded periodic function $\psi(s)$ on $\mathbb{R}$, such that

$$n(t; T_\mu) \sim \psi(\log \frac{1}{t}) t^{-\delta} \quad \text{as } \ t \to 0 . \tag{11}$$

5. Discussion

(1) The proofs of both Theorems 1, 2 use the variational approach. We reduce Theorem 1 to the analysis of a certain combinatorial problem. This one is investigated with the help of the "renewal equation", see e.g. [F]. If $d = 1$, then a direct reduction of the initial problem to renewal equation turns out to be possible. This allows to prove Theorem 2.

Renewal equation was applied to spectral problems (of another nature) involving fractals in [K Lap], [Lap] and [Lev V], see also references there. For the usage of the renewal theory in other problems related to fractals, see [Lau W] and [St 1,2].

(2) The Hausdorff dimension of μ equals (see [St1])

$$\beta = \left(\sum_{k=1}^{m} p_k \ln \, p_k \right) / \left(\sum_{k=1}^{m} p_k \ln \, h_k \right) .$$

The Hausdorff dimension α of $\mathcal{C}(\mathcal{S})$ is determined by the equation $\sum_{k=1}^{m} h_k^\alpha = 1$ (see [H]). Always $\beta \le \alpha$, and the equality takes place if and only if $p_k = h_k^\alpha$, $k = 1, ..., m$. These p_k's are called "natural weights".

The exponent δ meets the following correlations:

$$\frac{\beta}{\beta+1} \le \delta \le \frac{\alpha}{\alpha+1} \quad \text{for } d = 1 \,, \tag{12}$$

$$\delta = 1 \quad \text{for } d = 2 \,,$$

$$\frac{\alpha}{\alpha-(d-2)} \le \frac{\beta}{\beta-(d-2)} \le \delta \quad \text{for } d \ge 3 \,. \tag{13}$$

If $d \ge 3$, the value of δ can be arbitrarily large. The equalities in (12), (13) take place only in the case of natural weights.

(3) The sign "=" in the inequality $\sum_{k=1}^{m} h_k^d \le 1$ is rather rare. It corresponds to the case when $S_k(\Omega)$, $k = 1, ..., m$, constitute a tiling of Ω by a family of subdomains similar to Ω. If, in addition, p_k are the natural weights, $\mu(\mathcal{S}, \mathbf{p})$ is nothing but the d-dimensional Lebesgue measure on Ω. Here, clearly, $\delta = \frac{d}{2}$.

(4) Similar results are true for the operators on $H^1(\Omega)$, i.e. for "natural boundary conditions".

(5) Consider the case $X = \mathbb{R}^d$, which is possible for $d \ge 3$. Here we deal with the "homogeneous Sobolev space" $\mathcal{H}^1(\mathbb{R}^d)$, instead of $H_0^1(X)$. The space $\mathcal{H}^1(\mathbb{R}^d)$ is defined as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the metric given by the Dirichlet integral. Equation (5) (with u and v from $\mathcal{H}^1(\mathbb{R}^d)$) is still satisfied for any eigenpair (u, λ).

Theorem 3. *The result of Theorem 1 remains valid for $X = \mathbb{R}^d$, provided $d \ge 3$.*

(6) Under the assumptions of Theorem 3, the quadratic form

$$a_{\mu,\alpha}[u] = \int_{\mathbb{R}^d} |\nabla u|^2 dx - \alpha \int_{\mathbb{R}^d} |u|^2 d\mu \,, \quad u \in H^1(\mathbb{R}^d) \,, \tag{14}$$

is bounded from below and closed in $L_2(\mathbb{R}^d)$. In (14) $\alpha > 0$ is the "coupling parameter". The self-adjoint operator $A_{\mu,\alpha}$ in $L_2(\mathbb{R}^d)$, generated by the quadratic form (14), can be treated as Schrödinger operator with self-similar measure μ as a potential. Due to the "Birman-Schwinger principle" (see e.g. [R Sim]), Theorem 3 implies the following result.

Theorem 4. *Let the assumptions of Theorem 3 be satisfied. Then, for any* $\alpha > 0$, *the number* $N_-(\mu;\alpha)$ *of bound states of* $A_{\mu,\alpha}$ *satisfies*

$$N_-(\mu,\alpha) \leq C_{(\mu)}\alpha^{\delta}\ , \quad \text{any} \quad \alpha > 0\ ,$$

where δ *is given by (7).*

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