

JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

CHARLES L. FEFFERMAN

LUIS A. SECO

The mathematics of large atoms

Journées Équations aux dérivées partielles (1995), p. 1-12

http://www.numdam.org/item?id=JEDP_1995____A11_0

© Journées Équations aux dérivées partielles, 1995, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

The Mathematics of Large Atoms

C. L. Fefferman and L. A. Seco

After the initial success to explain the hydrogen atom, one of the early challenges of quantum mechanics was to create a simplified theory to study larger atoms. One of the successful pictures is Thomas–Fermi theory. According to it, the atomic energy $E(Z)$ and density $\rho^Z(x)$ of an atom of charge Z (to be defined a little later) behave as

$$E(Z) \approx c_{\text{TF}} Z^{7/3}, \quad \rho^Z(x) \sim \rho_{\text{TF}}^Z(x) = Z^2 \rho_{\text{TF}}^1(Z^{1/3} x), \quad (1)$$

for a suitable explicit constant c_{TF} and universal (Z -independent) function ρ_{TF}^1 . Associated with the density there is also the Thomas–Fermi potential V_{TF}^Z , satisfying also a perfect scaling condition

$$V_{\text{TF}}^Z(r) = Z^{4/3} V_{\text{TF}}^1(Z^{1/3} \cdot r).$$

Here we content ourselves knowing that Thomas–Fermi theory is simpler than the Schrödinger equation. We will postpone the derivation of the Thomas–Fermi equations until later.

The problem to understand Thomas–Fermi theory mathematically was tackled in 1973 with the work of Lieb and Simon (see [LS] and [Li]), which is now a central piece in modern mathematical physics. In their setting, large atoms were viewed as a limit $Z \rightarrow \infty$. Since then, large- Z asymptotics have become the mathematical paradigm of large atoms. In particular, the work of Lieb and Simon proves that (1) is the leading expression as $Z \rightarrow \infty$.

The purpose of this presentation is to give an updated account of some more or less recent mathematical issues related to the theory of large atoms in the context of large- Z asymptotics.

We begin with some definitions. An atom of nuclear charge Z fixed at the origin, and N quantized electrons at positions $x_i \in \mathbb{R}^3$ is described by the hamiltonian

$$\sum_{i=1}^N \left(-\Delta_{x_i} - \frac{Z}{|x_i|} \right) + \frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|},$$

acting on antisymmetric functions in $L^2(\mathbb{R}^{3N})$ (in this discussion we omit spin considerations). The atomic energies are defined as

$$E(Z, N) = \inf_{\|\psi\|=1} \langle \psi, H_{Z,N} \psi \rangle, \quad E(Z) = \inf_{N \geq 0} E(Z, N).$$

The energy $E(Z, N)$ is non-increasing as a function of N , and it attains a minimum at some critical N , which represents the largest number of electrons an atom can bind.

The ground state Ψ is the eigenfunction with eigenvalue $E(Z)$. Its density is defined as

$$\rho(x) = N \int_{\mathbb{R}^{3(N-1)}} |\psi(x, x_2, \dots, x_N)|^2 dx_2 \dots dx_N.$$

Mean-Field Theory. The first attempt at a simplification of the original hamiltonian $H_{Z,N}$ involves replacing the repulsion term above by more manageable expressions of the type

$$\frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|} \sim \sum_{i=1}^N V(x_i) - \text{constants},$$

modulo small error terms.

Physically, one would argue for such formulas as follows: assume that the electronic distribution is given by a density function ρ : one would then expect the repulsion term to be approximately

$$\sum_{i=1}^N \int_{\mathbb{R}^3} \frac{\rho(y)}{|x_i - y|} dy - \frac{1}{2} \int_{\mathbb{R}^6} \frac{\rho(x) \rho(y)}{|x - y|} dx dy.$$

the last term appears to neutralize the non-quantum effect appearing in the right hand side of electrons interacting with themselves: for this reason it is referred to as the self-energy term. There are several delicate points in this physical argument, one of them being that we need to choose the density in a clever way. We postpone the discussion of this problem until later, and content ourselves with knowing that the Thomas-Fermi density achieves this purpose very well.

The first such rigorous results go back to Lieb and his collaborators, and are by now well-known in the mathematical physics folklore as Lieb's formulas. Here is one:

Theorem (Lieb's inequality): Assume $\psi(x_1, \dots, x_N)$, ($Z \leq N \leq 2Z$) is such that

$$\|\nabla\psi\|_2^2 \leq C Z^{7/3}.$$

Then, we have that

$$\langle H_{Z,N} \psi, \psi \rangle \geq \langle H_{Z,N}^{\text{ind}} \psi, \psi \rangle - \frac{1}{2} \iint \frac{\rho_{\text{TF}}(x) \rho_{\text{TF}}(y)}{|x-y|} dx dy - C' Z^{5/3},$$

where

$$H_{Z,N}^{\text{ind}} = \sum_{i=1}^N \{-\Delta_{x_i} - V_{\text{TF}}(x_i)\}.$$

The proof of this result can be found in Lemma 2 in [SW2]. The role of this inequality is that it reduces the analysis of systems with interaction to a system without it.

The interpretation of these formulas is that one can replace the true quantum electronic interaction by the classical interaction of the electrons with a more or less accurate, but *fixed* electric field, at the expense of error terms of the order $Z^{5/3}$, which are of course related to the inter-electron correlations.

We remark that this procedure can be viewed as an operator that, given a density choice $\rho(x)$, produces a total effective potential felt by electrons, given by

$$\rho \mapsto -V_\rho^{\text{eff}}(x) = -\frac{Z}{|x|} + \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy.$$

This will be useful in determining the equations satisfied by the Thomas-Fermi density and improvements on it.

In order to obtain a more refined analysis of the interactions, one then needs a better understanding of the electron correlations: we refer the reader to [FS3], [Ba] and [GS] for improved estimates.

Semiclassical asymptotics. The most immediate consequence of mean field theory is that the original hamiltonian $H_{Z,N}$ can be replaced by the much simpler $H_{Z,N}^{\text{ind}}$. This is an immense simplification, since this hamiltonian can be studied by separation of variables, and one easily sees that

$$E(Z) \geq \text{SNEG}(-\Delta - V_{\text{TF}}^Z(x)),$$

where

$$\text{SNEG}(H) = \text{Tr}(H_-),$$

denotes the sum of the negative eigenvalues of H .

The question is thus reduced to obtaining accurate formulas for the sum of the negative eigenvalues of a fixed operator in \mathbb{R}^3 , namely $-\Delta - V_{\text{TF}}^Z$. We also need estimates for the one-electron density of that same operator. Note that this problem is doubly singular: first, there is the singularity of the Thomas–Fermi potential; second, and more important, there is the singularity in the energy, due to the non-smooth restriction of the trace to the negative spectrum.

The density of $H_{Z,N}^{\text{ind}}$ equals

$$\rho(x) = \sum_k |\psi_k(x)|^2.$$

The semiclassical approximations to these quantities are

$$\begin{aligned} \text{SNEG}(-\Delta + V) &\approx -\frac{1}{15\pi^2} \int_{V<0} |V|^{5/2} + \frac{1}{8} Z^2 + \frac{1}{48\pi^2} \int_{V<0} |V|^{1/2} \Delta V. \\ \rho(x) &\approx \frac{1}{6\pi^2} |V_{\text{TF}}(x)|^{3/2}. \end{aligned}$$

We omit a discussion of these formulas, which, at least in the case for smooth potentials, are well known to experts in semiclassical asymptotics. We point out that despite that the formula for SNEG was guessed by Schwinger and Scott by comparison with the hydrogen atom and the harmonic oscillator, it is easily seen that in fact they fail for these two potentials; thus, that it continues to be true for the Thomas–Fermi potential must be a deep fact. This is the well-known relation between semiclassics and classical periodic trajectories. Aperiodicity of zero-energy Hamiltonian paths is well-known to

play a crucial role in the study of eigenvalues and eigenfunctions. In our setting, we find trigonometric expressions of the form

$$S = \sum_{1 \leq \ell < Z^{1/3} \Omega_c} \beta \left(\frac{Z^{1/3}}{\pi} F(Z^{-1/3} \ell) \right)$$

for elementary functions such as $\beta(t) = t - [t] - \frac{1}{2}$, (Here $[t]$ is the greatest integer in t) and a certain function F which depends on the potential. Since β is bounded, we obtain trivially the estimate $S = \mathcal{O}(Z^{1/3})$. If $F(\Omega) = \pi\mu\Omega + \nu$ with μ rational, then the trivial estimate for S is easily seen to be the best possible. On the other hand, if $d^2 F/d\Omega^2 < c < 0$, then one can prove that the numbers

$$\phi_l = Z^{1/3} F(Z^{-1/3} l)$$

are equidistributed modulo π . (The argument is close to Hardy's estimates on the number of lattice points in a disc.) Since $\beta(t)$ is periodic and has average zero, it follows that $S = \mathcal{O}(Z^\gamma)$ with $\gamma < \frac{1}{3}$. In the case of the hydrogen atom and harmonic oscillator, the function F degenerates to the case $F(\Omega) = \pi\mu\Omega + \nu$ with μ rational, and the oscillating sum gives a contribution comparable with the last term, and the asymptotic expansion breaks down.

We will study a little more closely the form of this trigonometric terms in the sections below.

Self-consistent Mean Fields We now come to the basic issue: our previous discussion was based in a fundamental way on the Thomas-Fermi density and potentials, although their nature has been left intentionally in the dark.

Recall that given a charge density ρ we formed the effective potential it generates V_ρ^{eff} . Also, given a potential V we constructed the density arising from the Schrödinger operator it gives rise to, ρ_V .

A natural requirement for a "reasonable" guess for a charge distribution would be that it satisfies the equation

$$\rho_V = V_\rho^{\text{eff}}. \quad (2)$$

We call this the self-consistent mean field model, which has been proved by Solovej, that it has a number of physically interesting properties.

The Thomas–Fermi density and potentials arise as the solutions to the *semiclassical* self-consistent mean field density: we replace ρ_V in (2) by its semiclassical approximation,

$$\rho_V \approx \frac{q}{6\pi^2} V_+^{3/2}(x), \quad (3)$$

and we recall that

$$\Delta V_\rho(x) = 4\pi \rho(x),$$

to arrive at the usual equation for the Thomas–Fermi potential

$$\Delta V = \frac{3}{2\pi} V_+^{3/2}. \quad (4)$$

Energy Asymptotics The starting point in this paper lies in the asymptotic formula, as Z goes to infinity, for the atomic energy

$$E(Z) = -c_{\text{TF}} Z^{7/3} + \frac{1}{8} Z^2 - c_s Z^{5/3} + \mathcal{O}\left(Z^{\frac{5}{3}-a}\right), \quad a > 0. \quad (5)$$

The first term above was introduced by Thomas and Fermi in [Th], [Fe], and proved rigorously in [LS] (See also [Li] for a review of Thomas–Fermi theory). The Z^2 term was discovered by Scott in [Sc] and proved to be true in a series of papers by Hughes–Siedentop–Weikard, in [Hu], [SW1], [SW2] and [SW3]. Its generalization to molecules was obtained by Ivrii–Sigal ([IS]). The $Z^{5/3}$ term was obtained by Schwinger in [Sch], and proved to be correct in [FS1], [FS2], [FS3], [FS4], [FS5], [FS6], [FS7] and [FS8].

The expansion in powers of Z almost surely stop here; in fact, interesting mathematical phenomena start to take place in the error terms above. In view of (11 – 18), it is naturally conjectured (see [F1]) that the next term in the energy asymptotics for $E(Z)$ above is given by the following sum

$$\Psi_Q(Z) = \sum_{\ell=1}^{\ell_{\text{TF}}} \frac{2\ell+1}{\frac{1}{\pi} \int \left(V_{\text{TF}}^Z(r) - \frac{\ell(\ell+1)}{r^2} \right)_+^{-1/2} dr} \mu \left(\frac{1}{\pi} \int \left(V_{\text{TF}}^Z(r) - \frac{\ell(\ell+1)}{r^2} \right)_+^{1/2} dr \right)$$

where $\text{dist}(x, \mathbb{Z})^2 - \frac{1}{12}$ and ℓ_{TF} is the greatest integer such that $V_{\text{TF}}^Z(r) - \ell(\ell+1)/r^2$ is positive somewhere.

The book of Englert ([En]; see also references thereof) contains a discussion of oscillatory terms in the asymptotics of $E(Z)$.

The sum $\Psi_Q(Z)$ turns out to be an adaptation of similar expressions well-known in analytic number theory, related to the circle and the divisor problem, among others. It was proved in [CFS1] and [CFS2] that this sum Ψ_Q corresponds to a sum of classical data of a certain classical hamiltonian, which would then suggest that the expansion for $E(Z)$ is a trace formula which one would expect from a path integral picture.

Next, we include a few remarks about analytic number theory which may clarify the nature of the sum Ψ_Q above.

Number Theory. Consider sums of the form

$$S(\lambda) = \sum_{l=1}^{[\lambda]} f\left(\frac{l}{\lambda}\right) \cdot \mu\left(\lambda \cdot \phi\left(\frac{l}{\lambda}\right)\right)$$

where λ is a large number, μ is a periodic function with average 0, f is an amplitude function which can be viewed as constant and ϕ is a smooth function which satisfies the crucial non-degeneracy condition

$$|\phi''(x)| \geq c_0 > 0.$$

Particular cases of sums of this kind give rise to two well-known problems in analytic number theory, namely

1. $f \equiv 1$, $\mu(x) = e^{2\pi i x}$, $\phi(x) = x^2$. In this case, $S(\lambda)$, for λ integer, corresponds to the Gauss sums. The value of S is then known explicitly, and satisfies the estimate

$$S(\lambda) = \mathcal{O}\left(\lambda^{1/2}\right)$$

2. $f \equiv 1$, $\mu(x) = x - [x] - \frac{1}{2}$, $\phi(x) = \sqrt{1 - x^2}$. In this case, S is related to the error $E(\lambda)$ in the lattice point problem for the circle in \mathbb{R}^2 , which can be defined as follows: take a large circle on \mathbb{R}^2 of radius λ , and denote by $N(\lambda)$ the number of lattice points in \mathbb{Z}^2 which fall inside this circle. Then

$$E(\lambda) = N(\lambda) - \pi\lambda^2$$

and it is an old problem in number theory to prove that

$$E(\lambda) = \mathcal{O}(\lambda^\alpha)$$

for the best possible value of α . It was observed very early, by Gauss and Dirichlet, that one can take $\alpha = 1$ which is an obvious geometric fact, and is also obviously satisfied by $S(\lambda)$. Different probabilistic approaches (as the one by Cramer, for instance) indicate that α above will not be smaller than $\frac{1}{2}$. What follows is a *brief* historic overview of the estimates for α (see [GK] for details).

- $\alpha = 1$, Gauss–Dirichlet, 1849.
- $\cdot \frac{2}{3} = 0.666\dots$, Voronoi 1904, Hardy, 1917.
- $\cdot \frac{66}{100} = 0.6600$, Van der Corput 1922.
- $\cdot \frac{163}{247} = 0.659919\dots$, Walfisz 1927.
- $\cdot \frac{27}{41} = 0.6585\dots$, Nieland–Van der Corput 1928.
- $\cdot \frac{15}{23} = 0.6521\dots$, Tichmarsh 1935.
- $\cdot \frac{13}{20} = 0.6500$, Loo Keng Hua 1942.
- $\cdot \frac{24}{37} = 0.6486\dots$, Kolesnik–Yin Wen Lin 1962.
- $\cdot \frac{35}{54} = 0.6481\dots$, Kolesnik 1971.
- $\cdot \frac{278}{429} = 0.648018\dots$, Kolesnik 1985.
- $\cdot \frac{7}{11} = 0.636636\dots$, Iwaniec–Mozzochi 1988. (improved by Huxley 1992.)

Note now that the perfect scaling condition of the Thomas–Fermi potential shows that our sum Ψ_Q is (almost exactly) of the form $S(\lambda)$ as defined above, where $\mu(x) = \text{dist}(x, \mathbb{Z})^2 - \frac{1}{12}$, $\lambda = Z^{1/3}$, and a certain explicit ϕ . The proof of the non-degeneracy condition for ϕ was done in [FSS], and it has the peculiarity that its is a computer assisted proof.

A natural question then arises: what is the level of difficulty in analyzing the size of Ψ_Q ? Is it as simple as the analysis of the gauss sums above? Or so hard as the analysis of the lattice point problem?

A method devised by Van der Corput (or at least, a variant of it), in his attempts to understand the lattice point problem provides the answer: we compute our sum using Poisson summation, and then we expand each Fourier integral using stationary phase.

In doing this, we end up with a sum in which μ is replaced by its Fourier coefficients $\hat{\mu}(n)$. If they decrease fast enough (like $|n|^{-3/2}$, it so happens), the sum is bounded by $\lambda^{1/2}$. In our case, $\mu(n) \sim |n|^{-2}$, therefore, after realizing that the size of our amplitude function is $Z^{4/3}$, we can conclude that $\Psi_Q \sim Z^{\frac{4}{3} + \frac{1}{6}}$.

A density model. Compared to the atomic energy, the ground state density continues to be largely a mystery. In addition to the well-known classical results of M. and T. Hoffmann-Ostenhof, there is a number of recent interesting results, namely the proof of the strong Scott conjecture in [ILS] or the computation of the full hydrogen density in [HL].

In this short and entirely speculative section we present a conjectured density model which appears to have a number of physically interesting features. The main idea is to translate the previous considerations for oscillatory terms in the ground state energy asymptotics, to the density.

If one knew better semiclassical formulas for the density of one-particle hamiltonians, we would be able to improve on (3), and thus obtain a different, hopefully more accurate density functional equation in (4). It was conjectured in [FS6] that

$$\rho_V \approx \frac{1}{6\pi^2} V_+^{3/2}(x) + \rho_{NT}(x),$$

for

$$\rho_{NT}(x) = \frac{1}{4\pi|x|^2} \sum_{\ell} (V_{\ell}^Z(x))^{-1/2} \cdot \frac{(2\ell + 1) \chi(\phi_{\ell})}{\int_{\mathbb{R}} (V_{\ell}^Z(x))_+^{-1/2} dx}.$$

Notice the close similarities between this ρ_{NT} and the oscillating function in the energy asymptotics. This correction ρ_{NT} was studied in [CFS3].

The natural conjecture then for a refined Thomas-Fermi density is

$$V = V_{TF} + V_{\text{correction}},$$

where $V_{\text{correction}}$ is determined as the solution of the PDE

$$\left. \begin{aligned} \Delta V_{\text{correction}} &= \frac{1}{\pi} V_{TF}^{1/2} V_{\text{correction}} + 4\pi \rho_{NT}, \\ V_{\text{correction}} &\sim \text{const.}, \quad x \rightarrow 0. \end{aligned} \right\}$$

It is not very hard to see that the density obtained in this way has certain properties which remind us of the shell structure we expect atoms to have.

Acknowledgments. C. Fefferman is supported by an N.S.F. grant. L. Seco was supported by a N.A.T.O research grant no. CRG921184, by an NSERC grant and by a CICYT grant.

References

- [Ba] Bach, V. “*Accuracy of Mean Field Approximations for Atoms and Molecules*” *Comm. Math. Phys.* 155 no. 2, 295 – 310 (1993).
- [CFS1] Córdoba, A., Fefferman, C., Seco, L., “*A Trigonometric Sum relevant to the Non-relativistic Theory of Atoms*” *Proc. Nat. Acad. Sci. USA*, Vol. 91, 5776 – 5778, June 1994.
- [CFS2] Córdoba, A., Fefferman, C., Seco, L., “*Weyl Sums and Atomic Energy Oscillations*” *Revista Matemática Iberoamericana*, Vol 11, no. 1. (1995); pp 167 – 228.
- [CFS2] Córdoba, A., Fefferman, C., Seco, L., “*A Number-Theoretic Estimate for the Thomas-Fermi Density*” To appear.
- [En] Englert, B. G. “*Semiclassical Theory of the Atom*” Springer Verlag Lecture Notes in Physics, vol 301.
- [FS1] Fefferman, C. and Seco, L. “*The Ground-State Energy of a Large Atom*” *Bull. A.M.S.*, Vol 23 no. 2, 525—530, 1990
- [FS2] Fefferman, C. and Seco, L. “*Eigenvalues and Eigenfunctions of Ordinary Differential Operators*” *Adv. Math.* Vol 95 no. 2, 145 – 305 (1992).
- [FS3] Fefferman, C. and Seco, L. “*The Eigenvalue Sum for a One-Dimensional Potential*” *Advances in Math.*, Vol. 108 no. 2, Oct 1994; 263 – 335.
- [FS4] Fefferman, C. and Seco, L. “*The Density in a One-Dimensional Potential*” *Advances in Math.*, Vol 107 no. 2, Sep 1994, 187 – 364.
- [FS5] Fefferman, C. and Seco, L. “*The Eigenvalue Sum for a Three-Dimensional Radial Potential*” To appear in *Adv. Math.*

- [FS6] Fefferman, C. and Seco, L. “*The Density in a Three-Dimensional Radial Potential*” *Advances in Math.*, Vol 111 no. 1, March 1995, 88–161.
- [FS7] Fefferman, C. and Seco, L. “*On the Dirac and Schwinger Corrections to the Ground-State Energy of an Atom*” *Advances in Math.*, Vol 107 no. 1, Aug 1994, 1 – 185.
- [FS8] Fefferman, C. and Seco, L. “*Aperiodicity of the Hamiltonian Flow in the Thomas–Fermi Potential*” *Revista Matemática Iberoamericana*, Vol 9 no. 3, 409 – 551 (1993).
- [F1] Fefferman, C. “*Atoms and Analytic Number Theory*” *A.M.S. Centennial Publication Vol. II* (1992), 27 – 36.
- [Fe] Fermi, E. (1927) “*Un Metodo Statistico per la Determinazione di alcune Proprietà dell’Atome*” *Rend. Accad. Naz. Lincei* **6**, 602—607.
- [GS] Graf, G. M. and Solovej, J. P. “*A Correlation Estimate with Applications to Quantum Systems with Coulomb Interactions*” *Reviews in Math. Phys.* Vol 6, No 5a (1994), 977–997.
- [GK] S. W. Graham and G. Kolesnik “*Van der Corput’s Method of Exponential Sums*” Cambridge University Press. London Math. Soc. Lecture Notes Series, 126.
- [HL] Heilmann and Lieb “*Preprint.*”
- [Hu] Hughes, W. “*An Atomic Energy Lower Bound that Agrees with Scott’s Correction.*” *Advances in Mathematics*, **79**, 213–270, 1990.
- [ILS] A. Iantchenko, E. Lieb and H. Siedentop “*Proof of the Strong Scott Conjecture for Atomic and Molecular Cores*” Preprint.
- [IS] Ivrii, V. and Sigal, I. M. “*Asymptotics of the Ground State Energies of Large Coulomb Systems*” *Annals of Math.* Vol. 138 no. 2, 243 – 335 (1993)
- [LS] Lieb, E. and Simon, B. (1977) “*Thomas–Fermi Theory of Atoms, Molecules and Solids*” *Adv. Math.* **23**, pp 22—116.
- [Li] Lieb, E. (1981) “*Thomas–Fermi and Related Theories of Atoms and Molecules*” *Reviews of Modern Physics* Vol 53 no. 4, 603–641.
- [Sch] Schwinger, J. (1981) “*Thomas–Fermi Model: The Second Correction*” *Physical Review A* **24**, **5**, 2353—2361.

- [Sc] Scott, J. M. C. (1952) “*The Binding Energy of the Thomas–Fermi Atom*” *Phil. Mag.* **43** 859—867.
- [SW1] Siedentop, H., Weikard, R. “*On the Leading Energy Correction for the Statistical Model of the Atom: Interacting Case*” *Communications in Mathematical Physics* **112** 471-490 (1987).
- [SW2] Siedentop, H., Weikard, R. “*On the Leading Correction of the Thomas–Fermi Model: Lower Bound*” and an appendix by A.M.K. Müller. *Inv. Math.*, Vol., 97, pp 159—193 (1989).
- [SW3] Siedentop, H., Weikard, R. (1990) “*A New Phase Space Localization Technique with Applications to the Sum of Negative Eigenvalues of Schrödinger Operators.*” *Ann. Scient. Ecole Normale Supérieure* 24, 215 – 225, (1991).
- [Th] Thomas, L. H. (1927) “*The Calculation of Atomic Fields*” *Proc. Cambridge Philos. Soc.* **23** 542—548.