JOHN SYLVESTER

# Some new results in 1D inverse scattering

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Some new results in 1D inverse scattering.

John Sylvester [1]
Mathematics Department
University of Washington

In this talk we discuss one dimensional scattering and inverse scattering on the half line from the point of view of the layer stripping. We follow the approach in [6], see [1], [2], and [3] for other approaches to layer stripping. We consider the Helmholtz equation

$$\frac{d^2v}{dy^2} + \omega^2 n^2(y)u = 0 \tag{1}$$

which we transform into travel time coordinates by introducing the new independent variable

$$x = \int_0^y n(s)ds.$$

The equation (1) becomes, with $' = \frac{d}{dx}$ and $\alpha(x) = \frac{n'(x)}{n(x)}$,

$$u'' + \alpha(x)u' + \omega^2 u = 0 \tag{2}$$

We consider the unique soltution to (2) which has the assymptotics

$$u(x,\omega) \sim e^{-i\omega x}$$

as $x \to -\infty$. We assume that $n(x)$ varies only for $x < 0$ and $n \equiv 1$ for $x > 0$, hence $\alpha \equiv 0$ for $x > 0$ and, for $x > 0$, $u$ has the representation

$$u(x,\omega) = \frac{1}{T(\omega)}\left(e^{-i\omega x} + R(\omega)e^{i\omega x}\right)$$

The coefficient $R(\omega)$ is called the reflection coefficient, and we denote the scattering map by $S$

$$\alpha \overset{S}{\mapsto} R$$

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The layer stripping approach is based on the observation that the unique solution to the ODE

$$\begin{array}{rcl} r' & = & 2i\omega r + \frac{\alpha}{2}(1-r^2) \\ & & r(-\infty,\omega) = 0 \end{array} \tag{3}$$

satisfies

$$r(0,\omega) = R(\omega).$$

To verify this, we suggest that he reader first verify that $\lambda = \frac{u'}{-i\omega u}$ satisfies a similar equation, and then use the identity $\lambda = \frac{1-r}{1+r}$ to obtain (3).

The Frechet derivative, at $\alpha \equiv 0$, of the scattering map $S$ is called the Born approximation and turns out to be the Fourier transform. To see this, let $\alpha = \varepsilon a$ and $\rho = \frac{dr}{d\varepsilon}|_{\varepsilon=0}$, then $\rho$ satisfies

$$\begin{array}{rcl} \rho' & = & 2i\omega\rho + \frac{a}{2} \\ & & \rho(-\infty,\omega) = 0 \end{array} \tag{4}$$

which has the explicit solution

$$\begin{aligned} \rho(x,\omega) &= \int_{-\infty}^{x} e^{2i\omega(x-y)}\frac{a}{2}(y)dy \\ &= \left(H_{y<0}\frac{a}{2}(y+x)\right)^{\wedge} \end{aligned}$$

where $H_{y<0}$ dentoes the indicator function of the left half line. At $x=0$, this is just the Fourier transform (we use $-2i\omega$ in the exponent for convenience) of $H_{y<0}\frac{a}{2}$.

Our first point in this lecture is that several well known theorems about the Fourier transform have analogs for the nonlinear map $S$. Specifically,

THEOREM 1 (PLANCHEREL EQUALITY)

$$\begin{aligned} \frac{4}{\pi}\int_{-\infty}^{0}\alpha^2 dx &= -\int_{-\infty}^{\infty}\log(1-|R|^2)d\omega \\ &= \sum_{k=1}^{\infty}\frac{\int_{-\infty}^{\infty}|R|^{2k}d\omega}{k} =: E(R) \end{aligned}$$

The point of the second equality is to emphasize the *definiteness* of $E(R)$, which we refer to the energy of the reflection coefficient. As $E(R)$ is a positive sum of $L^p$ norms, weakly convergent sequences whose energies converge converge strongly, etc..

There are many theorems about the Fourier transform which are attributed to Paley and Weiner. The one we refer to here is the theorem that relates the support of a function to the growth rate of its transform along the imaginary axis. If $\alpha$ has compact support, the reflection coefficient itself does not necessarily extend to be holomorphic in all of $\mathbb{C}$. however, certain rational functions of $R$ and its conjugate do have this property.

THEOREM 2 (PALEY-WEINER) *The following are equivalent:*

*(i)* $\operatorname{supp}\alpha \subset [A-W, A]$ *for some* $A \leq 0$ *and some* $W > 0$

*(ii)* $\frac{|R|^2}{1-|R|^2}$ *extends to be holomorphic in* $\mathbb{C}$ *and there is a constant* $K$ *such that*

$$\frac{|R|^2}{1-|R|^2} \leq K\left(e^{W\mathrm{Im}(\omega)} + e^{-W\mathrm{Im}(\omega)}\right)$$

*(iii)* $\frac{R}{1-|R|^2}$ *extends to be holomorphic in* $\mathbb{C}$ *and and there is a constant* $K$ *such that,*

$$\frac{|R|}{1-|R|^2} \leq K\left(e^{(A+W)\mathrm{Im}(\omega)} + e^{(A-W)\mathrm{Im}(\omega)}\right)$$

It follows from the theorem above that $\frac{1}{1-|R|^2}$ and $\frac{R}{1-|R|^2}$ have Fourier transforms supported in an interval of width $W$. Therefore, the classical sampling theorem implies that they are exactly determined by their sampled values. Since their ratio is $R$, we have

THEOREM 3 (SAMPLING) *If* $\operatorname{supp}\alpha \subset [A-W, A]$, *then* $R(\omega)$ *is exactly determined by its values at* $\omega_n = \frac{n\pi}{W}$.

The second point in this lecture is to describe an inverse scattering method, a new (the first mathematically precise) implementation of layer stripping. We begin with the definition of the Hardy space $H^E(\mathbb{C}^+)$

$$H^E(\mathbb{C}^+) = \{\rho \mid \rho \text{ holomorphic in } \mathbb{C}^+ \text{ and } \sup_{b>0} E(\rho(\cdot + ib)) < \infty\}$$

where $E$ is as defined in the Plancherel theorem above. We shall add a subscript and write $H^E_{\mathbb{R}}$ to denote those functions which satisfy the symmetry condition $\rho(\overline{-\omega}) = \overline{\rho(\omega)}$. These are all Fourier transforms of functions in $L^2_{\mathbb{R}}$, i.e. real valued $L^2$ functions.

Our description of scattering theory and inverse scattering theory will rely completely on the Ricatti equation (3). We begin by noting that there is a distinct difference between propagation in the upward and downward directions. We use the notation $C((x_0,x_1);H^E_{\mathbb{R}})$ to denote continuous maps from the interval $(x_0,x_1)$ into $H^E_{\mathbb{R}}$.

THEOREM 4 *Consider the initial value problem:*

$$
\begin{aligned}
r' &= 2i\omega r + \alpha(1-r^2) \\
r(x_0,\omega) &= r_0(\omega) \in H^E_{\mathbb{R}}
\end{aligned}
\tag{5}
$$

*If $x_1 > x_0$, then*

*for every $\alpha \in L^2_{\mathbb{R}}$, there exists a unique $r(x,\omega) \in C((x_0,x_1);H^E_{\mathbb{R}})$ which solves (5).*

*If $x_1 < x_0$, then*

*there exists a unique pair $(\alpha,r) \in L^2_{\mathbb{R}}(x_1,x_0) \oplus C((x_1,x_0);H^E_{\mathbb{R}})$ which solves (5).*

COROLLARY 1 (FORWARD SCATTERING) *If $\alpha \in L^2_{\mathbb{R}}(-\infty,0)$, then $R(\omega) \in H^E_{\mathbb{R}}$.*

Proof: Let $x_0 = -\infty$ and $r_0 = 0$ in the previous theorem, then $r(0,\omega) = R(\omega)$.

COROLLARY 2 (INVERSE SCATTERING) *If $R(\omega) \in H^E_{\mathbb{R}}$, then $R$ is the reflection coefficient for a unique $\alpha \in L^2_{\mathbb{R}}(-\infty,0)$.*

Proof: Let $x_0 = 0$ and $r_0 = R$ in the previous theorem, then there exists a unique $\alpha \in L^2(-\infty,0)$.

In the remainder of this lecture we sketch the proofs of these theorems. We start with (3), multiply by $\overline{r}$, and take real parts to obtain

$$
|r|^{2'} = \frac{\alpha}{2}(r+\overline{r})(1-|r|^2)
\tag{6}
$$

Dividing by $1-|r|^2$ gives

$$
-\log(1-|r|^2)' = \frac{\alpha}{2}(r+\overline{r}).
\tag{7}
$$

The formal expansion $r(x,\omega) = \frac{\alpha(x)}{2i\omega} + O\left(\frac{1}{\omega^2}\right)$ for large $\omega$, suggests that

$$\int_{\infty}^{\infty} (r(\omega) + \overline{r}(\omega))d\omega = \frac{\alpha(x)}{2\pi}$$

so that, integrating (7) with respect to $\omega$ gives

$$\left(-\int_{\infty}^{\infty} -\log(1 - |r(x,\omega)|^2)d\omega\right)' = \frac{\alpha^2}{4\pi}.$$

Integrating with respect to $x$ gives the Plancherel equality

$$\int_{-\infty}^{\infty} -\log(1 - |r(x,\omega)|^2)d\omega = \frac{1}{4\pi}\int_{-\infty}^{x} \alpha^2(y)dy.$$

To obtain the Paley-Weiner theorem, we divide (6) by $(1-|r|^2)^2$ to obtain

$$\left(\frac{|r|^2}{1-|r|^2}\right)' = \alpha\left(\frac{r}{1-|r|^2} + \frac{\overline{r}}{1-|r|^2}\right).$$

We may also check that

$$\left(\frac{r}{1-|r|^2}\right)' = 2i\omega\left(\frac{r}{1-|r|^2}\right) + \alpha\left(\frac{|r|^2}{1-|r|^2}\right) + \frac{\alpha}{2}$$

so that the system of ODE's for $\frac{|r|^2}{1-|r|^2}$, $\frac{r}{1-|r|^2}$, and $\frac{\overline{r}}{1-|r|^2}$ is closed and linear. If we inverse Fourier transform $(\omega \mapsto t)$, then the system obtained is hyperbolic with characteristic slopes $-1, 0, 1$. The finite speed of propagation for this system implies the easy direction of the Paley-Weiner theorem in the case $A = 0$; we don't discuss the rest.

In order to sketch the proof of theorem 4, we write out the integral equation equivalent to (3),

$$r(x,\omega) = e^{2i\omega(x-x_0)}\int_{x_0}^{x} e^{2i\omega(x-y)}\frac{\alpha(y)}{2}(1 - r^2(y,\omega))dy \tag{8}$$

When $x > x_0$, the exponential $e^{2i\omega(x-y)}$ decays in $\mathbb{C}^+$, so that the solution $r$ will naturally be in $H^E \subset H^2(\mathbb{C}^+)$ (assuming we prove the necessary estimates). However, when $x < x_0$, the exponential decays in $\mathbb{C}^-$, in particular,

$$\int_{x_0}^{x} e^{2i\omega(x-y)}\alpha(y)dy \in H^2(\mathbb{C}^-)$$

We insist that $r \in H^2(\mathbb{C}^+)$ and recall that

$$L^2(\mathbb{R}) = H^2(\mathbb{C}^+) \oplus H^2(\mathbb{C}^-)$$

with $P^+$ and $P^-$ denoting the orthogonal projections onto each factor. Applying $P^-$ to (8) gives

$$0 = P^- r(x,\omega) = \int_{x_0}^{x} e^{2i\omega(x-y)} \frac{\alpha(y)}{2} dy + P^-(e^{2i\omega(x-x_0)} r_0 - \int_{x_0}^{x} e^{2i\omega(x-y)} \alpha(y) r^2 dy)$$

Rearranging this equation and applying $P^+$ to (8) gives the system

$$\begin{aligned} \int_{x_0}^{x} e^{2i\omega(x-y)} \alpha(y) dy &= P^-(e^{2i\omega(x-y)} r_0 - \int_{x_0}^{x} e^{2i\omega(x-y)} \alpha(y) r^2 dy) \\ r(x,\omega) &= P^+(e^{2i\omega(x-y)} r_0 - \int_{x_0}^{x} e^{2i\omega(x-y)} \alpha(y) r^2 dy) \end{aligned}$$

which exhibits the downward flow as an evolution for the Fourier transform of $\alpha$ and $r$ together.

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