JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1995), p. 1-15
<http://www.numdam.org/item?id=JEDP_1995____A8_0>

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Lecture given at the Partial Differential Equations Meeting in Saint Jean de Monts, May 29 – June 2, 1995

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First I wish to thank the organizers for inviting me to speak at this conference. I will speak about an intriguing partial differential equation that arises in gauge theory. Gauge theory is mainly concerned with the Yang-Mills equation and related equations, such as the Ginzburg-Landau equation (with a magnetic field), the Yang-Mills-Higgs equation and the Seiberg-Witten equation. I will talk about solutions to the Yang-Mills equation with singularities. In a moment I will write down the Yang-Mills equation, in full detail. First I just want to mention the origin of these singular solutions.

The Yang-Mills equation has mainly been studied by topologists and geometers, in particular in connection with the topology of smooth 4-manifolds. In the early 80's Donaldson showed that Yang-Mills equation could be used as a powerful tool in smooth 4-manifold topology. In particular he defined new invariants for smooth 4-manifolds. These invariants reflect the topology of solution spaces for Yang-Mills equation on the 4-manifold. They are now known as Donaldson polynomials. These developments were a bit of a shock for the 4-manifold topologists. They were suddenly forced to learn about partial differential equations. Many of them did so very succesfully. For a brief introduction to the applications of gauge theory to 4-manifold topology see [L] and for a comprehensive text see [DK]. Both books are masterpieces of mathematical exposition.

The Donaldson polynomials were at first extremely hard to calculate. However, a few years ago Kronheimer and Mrowka discovered a method for calculating them in a large number of cases. The key was to introduce a new type of Donaldson polynomials defined using spaces of singular Yang-Mills connections, [K], [KM1], [KM2], see also [R3]. The purpose of my own work has been to understand these singular Yang-Mills connections from the point of view of partial differential equations.

In October last fall a new equation and new invariants were introduced by Seiberg and Witten. Within a few weeks several famous conjectures about 4-manifolds had been settled. Priority often was a matter of days. An interesting account of these developments is given in [T]. It is not clear if 4-manifold topologists are interested in singular Yang-Mills connections any more.

§1. The Yang-Mills equation

Recall that if

$$\sigma = \sum_{i_1<\cdots<i_p} \sigma_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

is a differential p-form, and

$$\omega = \sum_{j_1<\cdots<j_q} \omega_{j_1\ldots j_q} dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

then the exterior derivative of σ is defined to be the $(p+1)$-form

$$d\sigma = \sum_j \sum_{i_1<\cdots<i_p} \frac{\partial}{\partial x_j}\sigma_{i_1\ldots i_p} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \tag{1.1}$$

and the wedge product of σ and ω is defined to be the $(p+q)$-form

$$\sigma \wedge \omega = \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \sigma_{i_1\ldots i_p}\omega_{j_1\ldots j_q} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}. \tag{1.2}$$

These operations satisfy the identities

$$\begin{aligned} \omega \wedge \sigma &= (-1)^{pq}\sigma \wedge \omega \\ d^2\sigma &= 0 \\ d(\sigma \wedge \omega) &= d\sigma \wedge \omega + (-1)^p \sigma \wedge d\omega. \end{aligned}$$

The adjoint of the exterior derivative (with respect to the Euclidean metric $\sum dx_i^2$) is given by

$$d^*\sigma = \sum_{\nu=1}^{p} \sum_{i_1<\cdots<i_p} (-1)^\nu \frac{\partial}{\partial x_{i_\nu}}\sigma_{i_1\ldots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \cdots \wedge dx_{i_p}.$$

Now, let G be a compact Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. I usually think of G as a group of matrices; that simplifies the notation a good deal. In particular, then the Lie bracket $[X, Y]$ is simply given by $XY - YX$. In fact, I will soon restrict my attention to the case $G = \mathrm{SU}(2)$. This will simplify the notation even further.

In gauge theory one considers differential forms σ the coefficient $\sigma_{i_1\ldots i_p}$ take values in the Lie algebra $\mathfrak{g}$. These are called $\mathfrak{g}$-valued forms. We can still define the exterior derivative of σ by by (1.1). However, the right hand side of (1.2) is quite meaningless if $\sigma_{i_1\dots i_p}$ and $\omega_{j_1\dots j_q}$ are $\mathfrak{g}$-valued forms. Instead we define the bracket of σ and ω as

$$[\sigma,\omega] = \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} [\sigma_{i_1\dots i_p}, \omega_{j_1\dots j_q}] dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

Then

$$\begin{aligned} [\omega,\sigma] &= (-1)^{pq+1}[\sigma,\omega] \\ d^2\sigma &= 0 \\ d[\sigma,\omega] &= [d\sigma,\omega] + (-1)^p[\sigma, d\omega]. \end{aligned}$$

Gauge transformations. The Lie group G acts on the Lie algebra $\mathfrak{g}$ by conjugation; for $g \in G$ and $X \in \mathfrak{g}$ we can form $gXg^{-1} \in \mathfrak{g}$. If σ is a $\mathfrak{g}$-valued p-form and g is a G-valued function, then we can form a new $\mathfrak{g}$-valued p-form

$$g.\sigma = \sum_{i_1<\cdots<i_p} g\sigma_{i_1\dots i_p} g^{-1} dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

We say that σ and $g.\sigma$ are gauge-equivalent. This establishes an equivalence relation on $\mathfrak{g}$-valued p-forms.

We can now define gauge theory; it is the study of objects that are invariant under gauge transformations. One example is the commutator of $\mathfrak{g}$-valued forms; it is clear that

$$g.[\sigma,\omega] = [g.\sigma, g.\omega].$$

Covariant derivatives. The exterior derivative is not gauge-invariant; we have

$$g.d\sigma = \sum_j \sum_{i_1<\cdots<i_p} g(\frac{\partial}{\partial x_j}\sigma_{i_1\dots i_p})g^{-1} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

but

$$d(g.\sigma) = \sum_j \sum_{i_1<\cdots<i_p} \frac{\partial}{\partial x_j}(g\sigma_{i_1\dots i_p} g^{-1}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

In general these differ by terms that involve the derivatives of g. A calculation shows that

$$g.d\sigma = d(g.\sigma) + [A, g.\sigma]$$

where

$$A = -(dg)g^{-1} = -\sum_i \frac{\partial g}{\partial x_i} g^{-1} dx_i.$$

This suggest that we define the covariant exterior derivative of σ as

$$d_A = d\sigma + [A, \sigma] = \sum_j \sum_{i_1 < \cdots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p} + [A_j, \sigma_{i_1 \ldots i_p}] \right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The covariant derivative depends on the choice of a $\mathfrak{g}$-valued 1-form A. We call A a G-connection. A short calculation using the Jacobi identity shows that for any connection A

$$d_A[\sigma, \tau] = [d_A\sigma] + (-1)^p[\sigma, d_A\omega].$$

Another short calculation shows that exterior covariant derivative is gauge-invariant in the sense that

$$g.d_A\sigma = d_{g.A}(g.\sigma)$$

where

$$g.A = gAg^{-1} - (dg)g^{-1} = \sum_i \left(gA_i g^{-1} - \frac{\partial g}{\partial x_i} g^{-1} \right) dx_i.$$

Note that a connection transforms differently than an ordinary $\mathfrak{g}$-valued 1-form. As before, we say that A and $g.A$ are gauge-equivalent. This establishes an equivalence relation on the set of G-connections.

$$d_A = d\sigma + [A, \sigma] = \sum_j \sum_{i_1 < \cdots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p} + [A_j, \sigma_{i_1 \ldots i_p}] \right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The adjoint of d_A is given by

$$d_A^*\sigma = \sum_{\nu=1}^{p} \sum_{i_1 < \cdots < i_p} (-1)^\nu \left(\frac{\partial}{\partial x_{i_\nu}} \sigma_{i_1 \ldots i_p} + [A_{i_\nu}, \sigma_{i_1 \ldots i_p}] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \cdots \wedge dx_{i_p}.$$

Curvature. We do not have $d_A^2\sigma = 0$. Instead a short calculation shows that

$$d_A^2\omega = [F_A, \omega]$$

where F_A is the $\mathfrak{g}$-valued 2-form

$$F_A = dA + \tfrac{1}{2}[A, A] = \frac{1}{2} \sum_{ij} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] \right) dx_i \wedge dx_j.$$

The 2-form F_A is called the curvature of the connection A. Another short calculation shows that curvature is gauge-invariant, i.e.

$$g.F_A = F_{g.A}.$$

Yet another short computation shows that

$$d_A F_A = 0.$$

This is known as the Bianchi identity.

Yang-Mills equation. Let A be a connection in a domain Ω in $\mathbb{R}^n$. One defines the energy of the connection A as

$$\mathfrak{YM}(A) = \frac{1}{2}\int_\Omega |F_A|^2 dx = \frac{1}{2}\sum_{ij}\int_\Omega \left|\frac{\partial A}{\partial x_i} - \frac{\partial A}{\partial x_j} + [A_i, A_j]\right|^2 dx.$$

A short calculation shows that the Euler-Lagrange equation for this energy functional is

$$d_A^* F_A = 0.$$

This equation is known as the Yang-Mills equation. A connection A that satisfies Yang-Mills equation is called a Yang-Mills connection. If we write out the Yang-Mills equation fully we get

$$\sum_{j=1}^n \left(\frac{\partial^2 A_i}{\partial x_j^2} - \frac{\partial^2 A_j}{\partial x_i \partial x_j} + \left[\frac{\partial A_j}{\partial x_j}, A_i\right] + \left[\frac{\partial A_j}{\partial x_i}, A_j\right] - 2\left[\frac{\partial A_i}{\partial x_j}, A_j\right] + [A_j, [A_j, A_i]]\right) = 0$$

for $i = 1, \ldots, n$. The most convenient way to write the equation is

$$d^* dA + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0.$$

Here we write $\{A \otimes \nabla A\}$ for terms that are linear in A_i and $\partial A_i/\partial x_j$ et.c.

The Yang-Mills energy is gauge invariant, i.e.

$$\mathfrak{YM}(g.A) = \mathfrak{YM}(A).$$

Hence the Yang-Mills equation is gauge-invariant. In particular, if A is a Yang-Mills connection, then $g.A$ is also a Yang-Mills connection.

To define the Yang-Mills energy and the Yang-Mills equation on a manifold, we need to choose a Riemannian metric. It is easy to verify that in four dimensions the Yang-Mills energy, and hence the Yang-Mills equation, are conformally invariant.

§2. A regularity theorem for Yang-Mills connections

The pricipal term in Yang-Mills eqaution is d^*dA. The operator d^*d is not elliptic. Thus we can not expect solutions to be smooth. This is also clear from the gauge invariance. Given a smooth solution we can manufacture a non-smooth solution by applying a suitable non-smooth gauge transformation. Conversely, the best wecould hope for is that any solution to Yang-Mills equation is gauge-equivalent to a smooth solution. Such a result was proven by K. Uhlenbeck.

Before discussing her theorem, I want to review a classical geometric result. A connection A is said be trivial if it gauge-equivalent to 0. A connection A is said to be flat if $F_A = 0$. Clearly any trivial connection is flat.

Lemma 2.1. *If A is a connection defined in a simply connected domain Ω and A is flat, then A is trivial.*

Proof. A connection A is trivial if we can solve the equation $g.A = 0$ for g. Fully written out, this equation takes the form

$$\frac{\partial g}{\partial x_i} = gA_i. \tag{2.1}$$

This implies

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial g}{\partial x_j} A_i + g\frac{\partial A}{\partial x_j} = gA_jA_i + g\frac{\partial A}{\partial x_j}.$$

The identity

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i}$$

gives rise to the integrability condition

$$A_jA_i + \frac{\partial A_i}{\partial x_j} = A_iA_j + \frac{\partial A_j}{\partial x_i},$$

which is equivalent to

$$F_A = 0.$$

This condition is clearly necessary for the existence of a solution g. By Frobenius theorem it is also sufficient, as long as Ω is simply connected. □

We will not actually use this Lemma. It only serves as a motivation for Uhlenbeck's good gauge theorem. In fact, Uhlenbeck's theorem can be viewed as an analyst's version of Lemma 2.1; it says that if A is a connection, on the unit ball, with small curvature, then there exists a gauge transformation g such that $g.A$ is small.

For simplicity we now restrict out attention to 4-dimensions. Let B_1 denote the unit ball in $\mathbb{R}^4$. Let ν denote the outward unit normal of ∂B_1. Let $L^{p,k}(B_1)$ denote the Sobolev space of functions with k derivatives in L^p. We say that a form or a connection is in $L^{p,k}(B_1)$ if all its components are in $L^{p,k}(B_1)$. It is natural to consider connection $A \in L^{2,1}(B_1)$. It follows from the Sobolev embedding $L^{2,1} \to L^4$ that if $A \in L^{2,1}$ then $F_A \in L^2$ and $\mathfrak{YM}(A) < \infty$.

Theorem 2.2. [U1] *There exists $\varepsilon > 0$ such that if A is a connection in $L^{2,1}(B_1)$ with*

$$\|F_A\|_{L^2(B_1)} \leq \varepsilon$$

then there exists a gauge transformation g in $L^{2,2}(B_1)$ such that

$$(2.6) \qquad \begin{cases} \nu \lrcorner (g.A) = \sum_i x_i (g.A)_i = 0 & \text{on } \partial B_1 \\ d^*(g.A) = \sum_i \frac{\partial}{\partial x_i}(g.A)_i = 0 & \text{on } B_1 \end{cases}$$

and

$$\|g.A\|_{L^{2,1}(B_1)} \leq c\|F_A\|_{L^2(B_1)}.$$

The conditions (2.6) are called gauge conditions.

The theorem is proven as follows. Assume that A satsifies the gauge conditions. Let $A+b$ be a small perturbation of A. We want to show that $A+b$ can be transformed to a connection that satisfies the gauge conditions. This amounts to solving the non-linear boundary value problem

$$\begin{cases} d^*(g.(A+b)) = 0 & \text{on } B_1 \\ \nu \lrcorner (g.(A+b)) = 0 & \text{on } \partial B_1. \end{cases}$$

for g. If we let $g = \exp \varphi$ and linearize around $\varphi = 0$ and $b = 0$ then we get the linear boundary value problem

$$(2.4) \qquad \begin{cases} \Delta\varphi + \sum_i \left[A_i, \frac{\partial \varphi}{\partial x_i}\right] = -d^*b & \text{on } B_1 \\ \nu \lrcorner d\varphi = -\nu \lrcorner b & \text{on } \partial B_1. \end{cases}$$

This system can clearly be solved if A is small enough; then it is a small perturbation of the Neumann problem for the Laplace operator. It then follows from the implicit function theorem that the non-linear boundary value problem can be solved if b is small enough. The theorem can then be proven by the continuity method. See [U1] for more details.

Theorem 2.3. [U1] *There exist constants c_k such that if A in addition to the assumptions in Theorem 2.2 satisfies Yang-Mills equations, then $g.A$ is smooth on the interior of B_1 and*

$$\|g.A\|_{C^k(B_1/2)} \leq c_k \|F_A\|_{L^2(B_1)}.$$

This is seen as follows. Assume that A is Yang-Mills and $d^*A = 0$. We now have that $\Delta A = dd^*A + d^*dA$. Hence it follows that

$$\Delta A + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0. \tag{2.5}$$

This is a semi-linear elliptic equation. If $A \in L^{2,1}$, then we can estimate higher derivatives of A by bootstrapping. In the first iteration step we have to use to usual trick of estimating the difference quotient of A.

This siuation is common in gauge theory. In order to prove regularity for an equation, one has to supplement it with gauge conditions. Thus, when facing a new equation, the first question is, what is the right gauge condition.

§3. Singular connections

According to a theorem by K. Uhlenbeck, point singularities of finite energy connections are removable. The precise statement is as follows:

Theorem 3.1. [U2], [U3] *If A is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus \{0\})$ and $F_A \in L^2(B_1 \setminus \{0\}) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}(B_1)$ such that $g.A \in L^{2,1}(B_1)$.*

This theorem was originally proven under the extra assumption that A be Yang-Mills, [U2]. Later it was discovered that finite energy sufficed, [U3].

According to a theorem of mine, singularities along embedded curves are removable. It suffices to consider the connections on $B_1 \setminus L_1$ where $L_1 = \{(x_1, 0, 0, 0) \mid |x_1| \leq 1\}$.

Theorem 3.2. [R2] *If A is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus L_1)$ and $F_A \in L^2(B_1 \setminus L_1) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus L_1)$ such that $g.A \in L^{2,1}(B_1)$.*

The next case is connections on a 4-manifold with singularities along an embedded surface. The local model are then connections on B_1 with singularities along $D_1 = \{(x_1, x_2, 0, 0) \mid x_1^2 + x_2^2 = 1\}$. It is not true that finite energy connections on $B_1 \setminus D_1$ can be extended to connections on B_1. Unlike $B_1 \setminus \{0\}$ and $B_1 \setminus L_1$, the domain $B_1 \setminus D_1$ is not simply connected. Hence Lemma 2.1 does not apply to $B_1 \setminus D_1$. Thus, before we attempt to generalize the theorems of §2 and §3 to $B_1 \setminus D_1$ we need to generalize Lemma 2.1 to non-simply-connected domains Ω. This requires the notion of holonomy.

Holonomy and flat connections. Let A be a connection in a region Ω in $\mathbb{R}^4$. Let $x_0 \in \Omega$. Let $\gamma : [0,1] \to \Omega$ be a closed smooth curve in Ω with $\gamma(0) = \gamma(1) = x_0$. The initial value problem

$$(2.3) \qquad \begin{cases} \dfrac{\partial h}{\partial t} + h \sum_i A_i \dfrac{d\gamma_i}{dt} = 0. \\ h(0) = 1 \end{cases}$$

has a unique solution. The element $h(1) \in G$ is called the holonomy of A around γ. This initial value problem is gauge-invariant in the sense that

$$(g.h)(t) = g(x(t))h(t)g(x(t))^{-1}$$

is a solution for $g.A$. Thus the conjugacy class of the holonomy is gauge-invariant.

If the connection is trivial, then (2.1) has a solution g with $g(x_0) = 1$. Then the solution to (2.3) is given by $h(t) = g(\gamma(t))$. It follows that that the holonomy is $h(1) = g(\gamma(1)) = g(x_0) = g(\gamma(0)) = h(0) = 1$. Thus we get another condition for a connection to be trivial; the holonomy around each loop has to be the identity.

One can show that if A is flat, then the holonomy of A is invariant under smooth deformations of γ. Thus the holonomy only depends the homotopy class of γ. Hence it gives a map $\pi_1(\Omega, x_0) \to G$. Here $\pi_1(\Omega, x_0)$ denotes the fundamental group of Ω with base point x_0. It is easily seen that that this map is a homomorphism. If we apply a gauge transformation g to A or if we change the base point, then this homomorphism gets conjugated by an element of G.

Theorem 3.3. *There is a 1–1 correspondence between gauge equivalence classes of flat G-connections on Ω and conjugacy classes of homomorphisms $\pi_1(\Omega) \to G$.*

The proof is not hard; see for instance [KN] Prop. 9.3.

In our special case of $B_1 \setminus D_1$, the fundamental group is generated by any loop that goes around D_1 once. It follows that flat connections are classified by the holonomy around this loop.

Corollary 3.4. *There is a 1-1 correspondence between gauge equivalence classes of flat G-connections on $B_1 \setminus D_1$ and conjugacy classes in G.*

Limit Holonomy. As we have seen, flat connections on $B_1 \setminus D_1$ are classified by their holonomy. A non-flat connection does not a uniquely defined holonomy. However, any connection on $B_1 \setminus D_1$ with curvature in L^2 has a well-defined limit holonomy.

We introduce cylindrical coordinates (x_1, x_2, r, θ) on B_1, with $x_3 = r\cos\theta$ and $x_4 = r\sin\theta$. In these coordinates D_1 is given by $r = 0$.

Theorem 3.5. [SS] *If A is a G-connection in $L^{2,1}_{\text{loc}}(B_1 \backslash D_1)$ with $F_A \in L^2(B_1 \backslash D_1)$, then the holonomy of A around the loop $\gamma(t) = (x_1, x_2, r\cos(2\pi t), r\sin(2\pi t))$ exists for almost all x_1, x_2 and r. The limit of this holonomy as $r \to 0$ exists for almost all x_1 and x_2. This limit is independent of x_1 and x_2 for almost all x_1 and x_2.*

This unique limit is called the limit holonomy of the connection.

We can now state the correct analog of Theorem 3.1 and Theorem 3.2 for $B_1 \setminus D_1$. Note that if G is connected, then $\exp : \mathfrak{g} \to G$ is surjective. (Proof: On a complete Riemannian manifold any two points can be connected by a geodesic curve. On a Lie group with an invariant metric, in particular any compact Lie group, the geodesic curves through the identity are precisely the 1-parameter subgroups.)

Theorem 3.6. [R2] *If A is a G-connection in $L^{2,1}_{\text{loc}}(B_1 \setminus D_1)$ with limit holonomy $\exp(-2\pi X)$, then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus D_1)$ such that*

$$g.A = X\, d\theta + a$$

where $a, \nabla_{X\,d\theta} a \in L_{X\,d\theta}(B_1)$.

Here

$$\nabla_A \sigma = \sum_j \sum_{i_1 < \cdots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p} + [A_j, \sigma_{i_1 \ldots i_p}] \right) dx_j \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Note that the connection $X\, d\theta + a$ has curvature $d_{X\,d\theta} a + \frac{1}{2}[a, a]$. Hence the condition $a \in L^{2,1}_{X\,d\theta}$ ensures that the curvature lies in L^2.

As a consequence of Thm. 3.6, a singularity along a surface of a finite energy connection is removable is and only if the limit holonomy is trivial.

The Yang-Mills connections used by Kronheimer and Mrowka are Yang-Mills connections on a 4-manifold with singularities along an embedded surface. Near any point of the surface they are of the form $X\, d\theta + a$ with $a \in L^{2,1}_{X\,d\theta}(B_1)$.

§4. A regularity theorem for singular Yang-Mills connections

To keep the notation simple, we will now restrict our attention to the Lie group SU(2). This is the group of all unitary 2×2 matrices with determinant one. These are precisely the matrices

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

where z and w are complex numbers with $|z|^2 + |w|^2 = 1$.

The corresponding Lie algebra $\mathfrak{su}(2)$ consists of the skew-hermitian 2×2 matrices with trace zero. These are precisely the matrices

$$\begin{pmatrix} it & z \\ -\overline{z} & -it \end{pmatrix}$$

with t real and z complex.

Each conjugacy class in SU(2) contains exactly one element of the form

$$\begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\alpha) \end{pmatrix}$$

with $0 \le \alpha \le 1/2$. It then follows from Theorem 3.6 that the natural class of connections on $B_1 \setminus D_1$ are connections of the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a$$

with $0 \le \alpha \le 1/2$. Here I will only discuss the case $0 < \alpha < 1/2$. In the case of $\alpha = 0$, the singularity is removable, and we are back to the case discussed in §2. In the case $\alpha = 1/2$, the singularity is removable as far as the local analysis is concerned; however there can be topological obstructions to removing the singularity globally on a 4-manifold, see [KM1].

If σ is an $\mathfrak{su}(2)$-valued p-form, then we can decompose σ as

$$\sigma = \begin{pmatrix} i\sigma_D & \sigma_T \\ -\overline{\sigma}_T & -i\sigma_D \end{pmatrix}$$

where σ_D is a real valued p-form and σ_T is a complex valued p-form. We have

$$\nabla_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}\sigma = \nabla\sigma + [\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta, \sigma] = \begin{pmatrix} i\nabla\sigma_D & \nabla_{2i\alpha d\theta}\sigma_T \\ -\overline{\nabla_{2i\alpha d\theta}\sigma_T} & -i\nabla\sigma_D \end{pmatrix}$$

Thus $\nabla_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right) d\theta}$ acts on σ_D as ∇ and on σ_T as $\nabla_{2i\alpha d\theta}$. Let $d_{2i\alpha d\theta}$ denote the covariant exterior derivative given by the connection $2\alpha\, d\theta$. Let $d^*_{2i\alpha d\theta}$ denote the adjoint of $d_{2i\alpha d\theta}$.

We then have the following analog of Theorem 2.2.

Theorem 4.1. [R1] *For any α with $2\alpha \notin \mathbb{Z}$ there exists $\varepsilon > 0$ such that if $A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a$ is a connection with $a \in L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ and*

$$\|F_A\|_{L^2(B_1 \setminus D_1)} \le \varepsilon,$$

then there exists a gauge transformation $g \in L^{2,2}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ such that

$$g.A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a'$$

where

$$\begin{cases} d^* a'_D = 0 & \text{on } B_1 \\ d^*_{2i\alpha d\theta}(r^{-2} a'_T) = 0 & \text{on } B_1 \setminus D_1 \\ \nu \lrcorner a' = 0 & \text{on } \partial B_1 \end{cases}$$

and

$$\|a'\|_{L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)} \le c\|F_A\|_{L^2(B_1)}.$$

We also have the following analog of Theorem 2.3.

Theorem 4.2. [R1] *If in additional to the assumptions of Theorem 4.1 the connection is Yang-Mills, then*

$$\begin{cases} |a'_D| + |\nabla a'_D| \le c\|F_A\|_{L^2(B_1)} & \text{on } B_{1/2} \\ |\nabla^k a'_T| \le c r^{2\min\{2\alpha, 1-2\alpha\} - k} \|F_A\|_{L^2(B_1)} & \text{on } B_{1/2}\ . \end{cases}$$

These seemingly strange theorems demand an explanation. The key is to understand the function space $L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ in more detail. It follows from (1.1) that

$$a \in L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}_{2i\alpha d\theta}(B_1) \end{cases}$$

Here $a_T \in L^{2,1}_{2i\alpha d\theta}$ means that $a_T, \nabla_{2i\alpha d\theta} a_T \in L^2(B_1)$. Fully written out

$$\|\nabla_{2i\alpha d\theta}\sigma_T\|^2_{L^2(B_1)} = \int_{B_1} \left(\left|\frac{\partial \sigma_T}{\partial x_1}\right|^2 + \left|\frac{\partial \sigma_T}{\partial x_2}\right|^2 + \left|\frac{\partial \sigma_T}{\partial r}\right|^2 + r^{-2}\left|\frac{\partial \sigma_T}{\partial \theta} + 2i\alpha\sigma_T\right|^2 \right) r\, dx_1\, dx_2\, dr\, d\theta$$

Now,

$$\int_{S^1} f^2\, d\theta \le \left(\min\{2\alpha, 1-2\alpha\}\right)^{-1} \int_{S_1} (df/d\theta + 2i\alpha f)^2 d\theta.$$

It follows that

$$\|r^{-1}\sigma_T\|_{L^2(B_1)} \le c\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)}.$$

On the other hand, it is clear that

$$\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)} \le c(\|\nabla\sigma_T\|_{L^2(B_1)} + \|r^{-1}\sigma_T\|_{L^2(B_1)}).$$

Hence

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ \nabla a_T, r^{-1}a_T \in L^2(B_1) \end{cases}$$

So now you think I'm going to talk about analysis on weighted Sobolev spaces with singular weights. I'm not.

As I mentioned before, the finite energy condition and the Yang-Mills equation are conformally invariant in 4 dimensions. Thus we can replace the standard metric

$$\sum_i dx_i^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2$$

with any conformal metric. A natural choice is the metric

$$r^{-2}\sum_i dx_i^2 = r^{-2}(dx_1^2 + dx_2^2 + dr^2) + d\theta^2.$$

With this metric D_1 is moved out to infinity. We recognize $r^{-2}(dx_1^2 + dx_2^2 + dr^2)$ as the upper half space model of hyperbolic 3-space. Thus $\mathbb{R}^4$ with the metric $r^{-2}\sum dx_i^2$ is isometric with $H^3 \times S^1$, the cartesian product of hyperbolic 3-space and the unit circle. The unit ball with this metric is isometric with $H^3_+ \times S^1$, the cartesian product of one half of hyperbolic 3-space and the unit circle. Thus we can view σ_T as a differntial form on $H^3_+ \times S^1$. A short calculation shows that $r^{-1}\sigma_T, \nabla\sigma_T \in L^{2,1}$ if and only if $\sigma_T \in L^{2,1}(H^3_+ \times S^1)$. Thus

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}(H^3_+ \times S^1) \end{cases}$$

Thus we should view a_D as a differential form on B_1 and a_T as a differential form on $H^3_+ \times S^1$. Let $d^*_{\mathrm{h},2\alpha d\theta}$ denote the adjoint of $d_{2\alpha d\theta}$ with respect to the metric $r^{-1}\sum dx_i^2$. Moreover, a short calculation shows that

$$d^*_{\mathrm{h},2i\alpha d\theta}a_T = r^4 d^*_{2i\alpha d\theta}(r^{-2}a_T).$$

In other words, *the gauge condition says that a_D is coclosed on B_1 and a_T is coclosed on $H^3_+ \times S^1$.*

Theorem 4.1 is now proven along the same lines as Thm. 2.1. Instead of the equation (2.4) we get the equations

$$\begin{cases} \Delta\varphi_D + \sum_i (a_T)_i \dfrac{\partial \varphi_T}{\partial x_i} = -d^* b_D & \text{on } B_1 \\ \Delta_{\mathrm{h},2i\alpha d\theta}\varphi_T + \operatorname{Re}\sum_i \big((a_T)_i (d\varphi_D)_i + (a_D)_i (d_{2i\alpha d\theta}\varphi_T)_i\big) & \\ \quad = -d^*_{\mathrm{h},2i\alpha d\theta} b_T & \text{on } H^3_+ \times S^1. \\ \nu \lrcorner d\varphi_D = \nu \lrcorner b_D & \text{on } \partial B_1 \\ \nu_\mathrm{h} \lrcorner d_{2i\alpha d\theta}\varphi_T = \nu_\mathrm{h} \lrcorner b_T & \text{on } \partial H^3_+ \times S^1 \end{cases}$$

where $\Delta_{\mathrm{h},2i\alpha d\theta} = d_{2i\alpha d\theta} d^*_{\mathrm{h},2i\alpha d\theta} + d^*_{\mathrm{h},2i\alpha d\theta} d_{2i\alpha d\theta}$ is the covariant Hodge Laplacian for 1-forms on $H^3_+ \times S^1$ given by the connection $2i\alpha\, d\theta$, and $\nu_\mathrm{h} = r\nu$ is the outward unit normal of $H^3_+ \times S^1$. Thus we get a small perturbation of the Neumann problem for Δ on B_1 and the Neumann problem for $\Delta_{\mathrm{h},2i\alpha d\theta}$ on $H^3_+ \times S^1$. The theory for the former is well known. The letter is analyzed in [R1] by elementary methods.

Theorem 4.2 is now proven along the same lines as Thm. 2.3. Instead of the equation (2.5) we get the system

$$\begin{cases} \Delta a_D + \{a_T \otimes \nabla_{2i\alpha d\theta} a_T\} + \{a_D \otimes a_T \otimes a_T\} = 0 & \text{on } B_1 \\ \Delta_{\mathrm{h},2i\alpha d\theta} a_T + \{a_D \otimes \nabla_{\mathrm{h},2i\alpha d\theta} a_T\} + \{a_T \otimes \nabla_\mathrm{h} a_D\} & \\ \quad + \{a_T \otimes a_T \otimes a_T\} + \{a_D \otimes a_D \otimes a_T\} = 0 & \text{on } H^3_+ \times S^1 \ . \end{cases}$$

The 1-form a can now be estimated by a bootstrapping procedure. On the first equation we apply standard elliptic estimates for the usual Laplacian Δ on B_1. On the second equation we apply decay estimates at infinity for the covariant Hodge Laplacian $\Delta_{\mathrm{h},2i\alpha d\theta}$ on $H^3_+ \times S^1$. These decay estimates are derived in [R1] by elementary methods.