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Singular Yang-Mills Connections

Lecture given at the Partial Differential Equations Meeting

in Saint Jean de Monts, May 29 – June 2, 1995

by *Johan Råde* at Lund

First I wish to thank the organizers for inviting me to speak at this conference. I will speak about an intriguing partial differential equation that arises in gauge theory. Gauge theory is mainly concerned with the Yang-Mills equation and related equations, such as the Ginzburg-Landau equation (with a magnetic field), the Yang-Mills-Higgs equation and the Seiberg-Witten equation. I will talk about solutions to the Yang-Mills equation with singularities. In a moment I will write down the Yang-Mills equation, in full detail. First I just want to mention the origin of these singular solutions.

The Yang-Mills equation has mainly been studied by topologists and geometers, in particular in connection with the topology of smooth 4-manifolds. In the early 80's Donaldson showed that Yang-Mills equation could be used as a powerful tool in smooth 4-manifold topology. In particular he defined new invariants for smooth 4-manifolds. These invariants reflect the topology of solution spaces for Yang-Mills equation on the 4-manifold. They are now known as Donaldson polynomials. These developments were a bit of a shock for the 4-manifold topologists. They were suddenly forced to learn about partial differential equations. Many of them did so very successfully. For a brief introduction to the applications of gauge theory to 4-manifold topology see [L] and for a comprehensive text see [DK]. Both books are masterpieces of mathematical exposition.

The Donaldson polynomials were at first extremely hard to calculate. However, a few years ago Kronheimer and Mrowka discovered a method for calculating them in a large number of cases. The key was to introduce a new type of Donaldson polynomials defined using spaces of singular Yang-Mills connections, [K], [KM1], [KM2], see also [R3]. The purpose of my own work has been to understand these singular Yang-Mills connections from the point of view of partial differential equations.

In October last fall a new equation and new invariants were introduced by Seiberg and Witten. Within a few weeks several famous conjectures about 4-manifolds had been settled. Priority often was a matter of days. An interesting account of these developments is given in [T]. It is not clear if 4-manifold topologists are interested in singular Yang-Mills connections any more.

§1. The Yang-Mills equation

Recall that if

$$\sigma = \sum_{i_1 < \dots < i_p} \sigma_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

is a differential p -form, and

$$\omega = \sum_{j_1 < \dots < j_q} \omega_{j_1 \dots j_q} dx_{j_1} \wedge \dots \wedge dx_{j_q}$$

then the exterior derivative of σ is defined to be the $(p+1)$ -form

$$(1.1) \quad d\sigma = \sum_j \sum_{i_1 < \dots < i_p} \frac{\partial}{\partial x_j} \sigma_{i_1 \dots i_p} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

and the wedge product of σ and ω is defined to be the $(p+q)$ -form

$$(1.2) \quad \sigma \wedge \omega = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \sigma_{i_1 \dots i_p} \omega_{j_1 \dots j_q} dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q}.$$

These operations satisfy the identities

$$\begin{aligned} \omega \wedge \sigma &= (-1)^{pq} \sigma \wedge \omega \\ d^2 \sigma &= 0 \\ d(\sigma \wedge \omega) &= d\sigma \wedge \omega + (-1)^p \sigma \wedge d\omega. \end{aligned}$$

The adjoint of the exterior derivative (with respect to the Euclidean metric $\sum dx_i^2$) is given by

$$d^* \sigma = \sum_{\nu=1}^p \sum_{i_1 < \dots < i_p} (-1)^\nu \frac{\partial}{\partial x_{i_\nu}} \sigma_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \dots \wedge dx_{i_p}.$$

Now, let G be a compact Lie group. Let $\mathfrak{g}$ be the Lie algebra of G . I usually think of G as a group of matrices; that simplifies the notation a good deal. In particular, then the Lie bracket $[X, Y]$ is simply given by $XY - YX$. In fact, I will soon restrict my attention to the case $G = \text{SU}(2)$. This will simplify the notation even further.

In gauge theory one considers differential forms σ the coefficient $\sigma_{i_1 \dots i_p}$ take values in the Lie algebra $\mathfrak{g}$. These are called $\mathfrak{g}$ -valued forms. We can still define the exterior

derivative of σ by (1.1). However, the right hand side of (1.2) is quite meaningless if $\sigma_{i_1 \dots i_p}$ and $\omega_{j_1 \dots j_q}$ are $\mathfrak{g}$ -valued forms. Instead we define the bracket of σ and ω as

$$[\sigma, \omega] = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} [\sigma_{i_1 \dots i_p}, \omega_{j_1 \dots j_q}] dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q}.$$

Then

$$\begin{aligned} [\omega, \sigma] &= (-1)^{pq+1} [\sigma, \omega] \\ d^2 \sigma &= 0 \\ d[\sigma, \omega] &= [d\sigma, \omega] + (-1)^p [\sigma, d\omega]. \end{aligned}$$

Gauge transformations. The Lie group G acts on the Lie algebra $\mathfrak{g}$ by conjugation; for $g \in G$ and $X \in \mathfrak{g}$ we can form $gXg^{-1} \in \mathfrak{g}$. If σ is a $\mathfrak{g}$ -valued p -form and g is a G -valued function, then we can form a new $\mathfrak{g}$ -valued p -form

$$g.\sigma = \sum_{i_1 < \dots < i_p} g\sigma_{i_1 \dots i_p} g^{-1} dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

We say that σ and $g.\sigma$ are gauge-equivalent. This establishes an equivalence relation on $\mathfrak{g}$ -valued p -forms.

We can now define gauge theory; it is the study of objects that are invariant under gauge transformations. One example is the commutator of $\mathfrak{g}$ -valued forms; it is clear that

$$g.[\sigma, \omega] = [g.\sigma, g.\omega].$$

Covariant derivatives. The exterior derivative is not gauge-invariant; we have

$$g.d\sigma = \sum_j \sum_{i_1 < \dots < i_p} g \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \dots i_p} \right) g^{-1} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

but

$$d(g.\sigma) = \sum_j \sum_{i_1 < \dots < i_p} \frac{\partial}{\partial x_j} (g\sigma_{i_1 \dots i_p} g^{-1}) dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

In general these differ by terms that involve the derivatives of g . A calculation shows that

$$g.d\sigma = d(g.\sigma) + [A, g.\sigma]$$

where

$$A = -(dg)g^{-1} = - \sum_i \frac{\partial g}{\partial x_i} g^{-1} dx_i.$$

This suggest that we define the covariant exterior derivative of σ as

$$d_A = d\sigma + [A, \sigma] = \sum_j \sum_{i_1 < \dots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \dots i_p} + [A_j, \sigma_{i_1 \dots i_p}] \right) dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

The covariant derivative depends on the choice of a $\mathfrak{g}$ -valued 1-form A . We call A a G -connection. A short calculation using the Jacobi identity shows that for any connection A

$$d_A[\sigma, \tau] = [d_A \sigma] + (-1)^p [\sigma, d_A \omega].$$

Another short calculation shows that exterior covariant derivative is gauge-invariant in the sense that

$$g.d_A \sigma = d_{g.A}(g.\sigma)$$

where

$$g.A = gAg^{-1} - (dg)g^{-1} = \sum_i \left(gA_i g^{-1} - \frac{\partial g}{\partial x_i} g^{-1} \right) dx_i.$$

Note that a connection transforms differently than an ordinary $\mathfrak{g}$ -valued 1-form. As before, we say that A and $g.A$ are gauge-equivalent. This establishes an equivalence relation on the set of G -connections.

$$d_A = d\sigma + [A, \sigma] = \sum_j \sum_{i_1 < \dots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \dots i_p} + [A_j, \sigma_{i_1 \dots i_p}] \right) dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

The adjoint of d_A is given by

$$d_A^* \sigma = \sum_{\nu=1}^p \sum_{i_1 < \dots < i_p} (-1)^\nu \left(\frac{\partial}{\partial x_{i_\nu}} \sigma_{i_1 \dots i_p} + [A_{i_\nu}, \sigma_{i_1 \dots i_p}] \right) dx_{i_1} \wedge \dots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \dots \wedge dx_{i_p}.$$

Curvature. We do not have $d_A^2 \sigma = 0$. Instead a short calculation shows that

$$d_A^2 \omega = [F_A, \omega]$$

where F_A is the $\mathfrak{g}$ -valued 2-form

$$F_A = dA + \frac{1}{2}[A, A] = \frac{1}{2} \sum_{ij} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] \right) dx_i \wedge dx_j.$$

The 2-form F_A is called the curvature of the connection A . Another short calculation shows that curvature is gauge-invariant, i.e.

$$g.F_A = F_{g.A}.$$

Yet another short computation shows that

$$d_A F_A = 0.$$

This is known as the Bianchi identity.

Yang-Mills equation. Let A be a connection in a domain Ω in $\mathbb{R}^n$. One defines the energy of the connection A as

$$\mathfrak{YM}(A) = \frac{1}{2} \int_{\Omega} |F_A|^2 dx = \frac{1}{2} \sum_{ij} \int_{\Omega} \left| \frac{\partial A}{\partial x_i} - \frac{\partial A}{\partial x_j} + [A_i, A_j] \right|^2 dx.$$

A short calculation shows that the Euler-Lagrange equation for this energy functional is

$$d_A^* F_A = 0.$$

This equation is known as the Yang-Mills equation. A connection A that satisfies Yang-Mills equation is called a Yang-Mills connection. If we write out the Yang-Mills equation fully we get

$$\sum_{j=1}^n \left(\frac{\partial^2 A_i}{\partial x_j^2} - \frac{\partial^2 A_j}{\partial x_i \partial x_j} + \left[\frac{\partial A_j}{\partial x_j}, A_i \right] + \left[\frac{\partial A_j}{\partial x_i}, A_j \right] - 2 \left[\frac{\partial A_i}{\partial x_j}, A_j \right] + [A_j, [A_j, A_i]] \right) = 0$$

for $i = 1, \dots, n$. The most convenient way to write the equation is

$$d^* dA + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0.$$

Here we write $\{A \otimes \nabla A\}$ for terms that are linear in A_i and $\partial A_i / \partial x_j$ et.c.

The Yang-Mills energy is gauge invariant, i.e.

$$\mathfrak{YM}(g.A) = \mathfrak{YM}(A).$$

Hence the Yang-Mills equation is gauge-invariant. In particular, if A is a Yang-Mills connection, then $g.A$ is also a Yang-Mills connection.

To define the Yang-Mills energy and the Yang-Mills equation on a manifold, we need to choose a Riemannian metric. It is easy to verify that in four dimensions the Yang-Mills energy, and hence the Yang-Mills equation, are conformally invariant.

§2. A regularity theorem for Yang-Mills connections

The principal term in Yang-Mills equation is d^*dA . The operator d^*d is not elliptic. Thus we can not expect solutions to be smooth. This is also clear from the gauge invariance. Given a smooth solution we can manufacture a non-smooth solution by applying a suitable non-smooth gauge transformation. Conversely, the best we could hope for is that any solution to Yang-Mills equation is gauge-equivalent to a smooth solution. Such a result was proven by K. Uhlenbeck.

Before discussing her theorem, I want to review a classical geometric result. A connection A is said to be trivial if it is gauge-equivalent to 0. A connection A is said to be flat if $F_A = 0$. Clearly any trivial connection is flat.

Lemma 2.1. *If A is a connection defined in a simply connected domain Ω and A is flat, then A is trivial.*

Proof. A connection A is trivial if we can solve the equation $g.A = 0$ for g . Fully written out, this equation takes the form

$$(2.1) \quad \frac{\partial g}{\partial x_i} = g A_i.$$

This implies

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial g}{\partial x_j} A_i + g \frac{\partial A_i}{\partial x_j} = g A_j A_i + g \frac{\partial A_i}{\partial x_j}.$$

The identity

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i}$$

gives rise to the integrability condition

$$A_j A_i + \frac{\partial A_i}{\partial x_j} = A_i A_j + \frac{\partial A_j}{\partial x_i},$$

which is equivalent to

$$F_A = 0.$$

This condition is clearly necessary for the existence of a solution g . By Frobenius theorem it is also sufficient, as long as Ω is simply connected. $\square$

We will not actually use this Lemma. It only serves as a motivation for Uhlenbeck's good gauge theorem. In fact, Uhlenbeck's theorem can be viewed as an analyst's version of Lemma 2.1; it says that if A is a connection, on the unit ball, with small curvature, then there exists a gauge transformation g such that $g.A$ is small.

For simplicity we now restrict our attention to 4-dimensions. Let B_1 denote the unit ball in $\mathbb{R}^4$. Let ν denote the outward unit normal of ∂B_1 . Let $L^{p,k}(B_1)$ denote the Sobolev space of functions with k derivatives in L^p . We say that a form or a connection is in $L^{p,k}(B_1)$ if all its components are in $L^{p,k}(B_1)$. It is natural to consider connection $A \in L^{2,1}(B_1)$. It follows from the Sobolev embedding $L^{2,1} \rightarrow L^4$ that if $A \in L^{2,1}$ then $F_A \in L^2$ and $\mathfrak{YM}(A) < \infty$.

Theorem 2.2. [U1] *There exists $\varepsilon > 0$ such that if A is a connection in $L^{2,1}(B_1)$ with*

$$\|F_A\|_{L^2(B_1)} \leq \varepsilon$$

then there exists a gauge transformation g in $L^{2,2}(B_1)$ such that

$$(2.6) \quad \begin{cases} \nu \lrcorner (g.A) = \sum_i x_i (g.A)_i = 0 & \text{on } \partial B_1 \\ d^*(g.A) = \sum_i \frac{\partial}{\partial x_i} (g.A)_i = 0 & \text{on } B_1 \end{cases}$$

and

$$\|g.A\|_{L^{2,1}(B_1)} \leq c \|F_A\|_{L^2(B_1)}.$$

The conditions (2.6) are called gauge conditions.

The theorem is proven as follows. Assume that A satisfies the gauge conditions. Let $A+b$ be a small perturbation of A . We want to show that $A+b$ can be transformed to a connection that satisfies the gauge conditions. This amounts to solving the non-linear boundary value problem

$$\begin{cases} d^*(g.(A+b)) = 0 & \text{on } B_1 \\ \nu \lrcorner (g.(A+b)) = 0 & \text{on } \partial B_1. \end{cases}$$

for g . If we let $g = \exp \varphi$ and linearize around $\varphi = 0$ and $b = 0$ then we get the linear boundary value problem

$$(2.4) \quad \begin{cases} \Delta \varphi + \sum_i \left[A_i, \frac{\partial \varphi}{\partial x_i} \right] = -d^*b & \text{on } B_1 \\ \nu \lrcorner d\varphi = -\nu \lrcorner b & \text{on } \partial B_1. \end{cases}$$

This system can clearly be solved if A is small enough; then it is a small perturbation of the Neumann problem for the Laplace operator. It then follows from the implicit function theorem that the non-linear boundary value problem can be solved if b is small enough. The theorem can then be proven by the continuity method. See [U1] for more details.

Theorem 2.3. [U1] *There exist constants c_k such that if A in addition to the assumptions in Theorem 2.2 satisfies Yang-Mills equations, then $g.A$ is smooth on the interior of B_1 and*

$$\|g.A\|_{C^k(B_1/2)} \leq c_k \|F_A\|_{L^2(B_1)}.$$

This is seen as follows. Assume that A is Yang-Mills and $d^*A = 0$. We now have that $\Delta A = dd^*A + d^*dA$. Hence it follows that

$$(2.5) \quad \Delta A + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0.$$

This is a semi-linear elliptic equation. If $A \in L^{2,1}$, then we can estimate higher derivatives of A by bootstrapping. In the first iteration step we have to use the usual trick of estimating the difference quotient of A .

This situation is common in gauge theory. In order to prove regularity for an equation, one has to supplement it with gauge conditions. Thus, when facing a new equation, the first question is, what is the right gauge condition.

§3. Singular connections

According to a theorem by K. Uhlenbeck, point singularities of finite energy connections are removable. The precise statement is as follows:

Theorem 3.1. [U2], [U3] *If A is a connection in $L_{\text{loc}}^{2,1}(B_1 \setminus \{0\})$ and $F_A \in L^2(B_1 \setminus \{0\}) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}(B_1)$ such that $g.A \in L^{2,1}(B_1)$.*

This theorem was originally proven under the extra assumption that A be Yang-Mills, [U2]. Later it was discovered that finite energy sufficed, [U3].

According to a theorem of mine, singularities along embedded curves are removable. It suffices to consider the connections on $B_1 \setminus L_1$ where $L_1 = \{(x_1, 0, 0, 0) \mid |x_1| \leq 1\}$.

Theorem 3.2. [R2] *If A is a connection in $L_{\text{loc}}^{2,1}(B_1 \setminus L_1)$ and $F_A \in L^2(B_1 \setminus L_1) = L^2(B_1)$, then there exists a gauge transformation $g \in L_{\text{loc}}^{2,2}(B_1 \setminus L_1)$ such that $g.A \in L^{2,1}(B_1)$.*

The next case is connections on a 4-manifold with singularities along an embedded surface. The local model are then connections on B_1 with singularities along $D_1 = \{(x_1, x_2, 0, 0) \mid x_1^2 + x_2^2 = 1\}$. It is not true that finite energy connections on $B_1 \setminus D_1$ can be extended to connections on B_1 . Unlike $B_1 \setminus \{0\}$ and $B_1 \setminus L_1$, the domain $B_1 \setminus D_1$ is not simply connected. Hence Lemma 2.1 does not apply to $B_1 \setminus D_1$. Thus, before we attempt to generalize the theorems of §2 and §3 to $B_1 \setminus D_1$ we need to generalize Lemma 2.1 to non-simply-connected domains Ω . This requires the notion of holonomy.

Holonomy and flat connections. Let A be a connection in a region Ω in $\mathbb{R}^4$. Let $x_0 \in \Omega$. Let $\gamma : [0, 1] \rightarrow \Omega$ be a closed smooth curve in Ω with $\gamma(0) = \gamma(1) = x_0$. The initial value problem

$$(2.3) \quad \begin{cases} \frac{\partial h}{\partial t} + h \sum_i A_i \frac{d\gamma_i}{dt} = 0. \\ h(0) = 1 \end{cases}$$

has a unique solution. The element $h(1) \in G$ is called the holonomy of A around γ . This initial value problem is gauge-invariant in the sense that

$$(g.h)(t) = g(x(t))h(t)g(x(t))^{-1}$$

is a solution for $g.A$. Thus the conjugacy class of the holonomy is gauge-invariant.

If the connection is trivial, then (2.1) has a solution g with $g(x_0) = 1$. Then the solution to (2.3) is given by $h(t) = g(\gamma(t))$. It follows that the holonomy is $h(1) = g(\gamma(1)) = g(x_0) = g(\gamma(0)) = h(0) = 1$. Thus we get another condition for a connection to be trivial; the holonomy around each loop has to be the identity.

One can show that if A is flat, then the holonomy of A is invariant under smooth deformations of γ . Thus the holonomy only depends the homotopy class of γ . Hence it gives a map $\pi_1(\Omega, x_0) \rightarrow G$. Here $\pi_1(\Omega, x_0)$ denotes the fundamental group of Ω with base point x_0 . It is easily seen that that this map is a homomorphism. If we apply a gauge transformation g to A or if we change the base point, then this homomorphism gets conjugated by an element of G .

Theorem 3.3. *There is a 1-1 correspondence between gauge equivalence classes of flat G -connections on Ω and conjugacy classes of homomorphisms $\pi_1(\Omega) \rightarrow G$.*

The proof is not hard; see for instance [KN] Prop. 9.3.

In our special case of $B_1 \setminus D_1$, the fundamental group is generated by any loop that goes around D_1 once. It follows that flat connections are classified by the holonomy around this loop.

Corollary 3.4. *There is a 1-1 correspondence between gauge equivalence classes of flat G -connections on $B_1 \setminus D_1$ and conjugacy classes in G .*

Limit Holonomy. As we have seen, flat connections on $B_1 \setminus D_1$ are classified by their holonomy. A non-flat connection does not have a uniquely defined holonomy. However, any connection on $B_1 \setminus D_1$ with curvature in L^2 has a well-defined limit holonomy.

We introduce cylindrical coordinates (x_1, x_2, r, θ) on B_1 , with $x_3 = r \cos \theta$ and $x_4 = r \sin \theta$. In these coordinates D_1 is given by $r = 0$.

Theorem 3.5. [SS] *If A is a G -connection in $L_{\text{loc}}^{2,1}(B_1 \setminus D_1)$ with $F_A \in L^2(B_1 \setminus D_1)$, then the holonomy of A around the loop $\gamma(t) = (x_1, x_2, r \cos(2\pi t), r \sin(2\pi t))$ exists for almost all x_1, x_2 and r . The limit of this holonomy as $r \rightarrow 0$ exists for almost all x_1 and x_2 . This limit is independent of x_1 and x_2 for almost all x_1 and x_2 .*

This unique limit is called the limit holonomy of the connection.

We can now state the correct analog of Theorem 3.1 and Theorem 3.2 for $B_1 \setminus D_1$. Note that if G is connected, then $\exp : \mathfrak{g} \rightarrow G$ is surjective. (Proof: On a complete Riemannian manifold any two points can be connected by a geodesic curve. On a Lie group with an invariant metric, in particular any compact Lie group, the geodesic curves through the identity are precisely the 1-parameter subgroups.)

Theorem 3.6. [R2] *If A is a G -connection in $L_{\text{loc}}^{2,1}(B_1 \setminus D_1)$ with limit holonomy $\exp(-2\pi X)$, then there exists a gauge transformation $g \in L_{\text{loc}}^{2,2}(B_1 \setminus D_1)$ such that*

$$g.A = X d\theta + a$$

where $a, \nabla_{X d\theta} a \in L_{X d\theta}(B_1)$.

Here

$$\nabla_A \sigma = \sum_j \sum_{i_1 < \dots < i_p} \left(\frac{\partial}{\partial x_j} \sigma_{i_1 \dots i_p} + [A_j, \sigma_{i_1 \dots i_p}] \right) dx_j \otimes dx_{i_1} \wedge \dots \wedge dx_{i_p}.$$

Note that the connection $X d\theta + a$ has curvature $d_{X d\theta} a + \frac{1}{2}[a, a]$. Hence the condition $a \in L_{X d\theta}^{2,1}$ ensures that the curvature lies in L^2 .

As a consequence of Thm. 3.6, a singularity along a surface of a finite energy connection is removable is and only if the limit holonomy is trivial.

The Yang-Mills connections used by Kronheimer and Mrowka are Yang-Mills connections on a 4-manifold with singularities along an embedded surface. Near any point of the surface they are of the form $X d\theta + a$ with $a \in L_{X d\theta}^{2,1}(B_1)$.

§4. A regularity theorem for singular Yang-Mills connections

To keep the notation simple, we will now restrict our attention to the Lie group $SU(2)$. This is the group of all unitary 2×2 matrices with determinant one. These are precisely the matrices

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

where z and w are complex numbers with $|z|^2 + |w|^2 = 1$.

The corresponding Lie algebra $\mathfrak{su}(2)$ consists of the skew-hermitian 2×2 matrices with trace zero. These are precisely the matrices

$$\begin{pmatrix} it & z \\ -\bar{z} & -it \end{pmatrix}$$

with t real and z complex.

Each conjugacy class in $SU(2)$ contains exactly one element of the form

$$\begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\alpha) \end{pmatrix}$$

with $0 \leq \alpha \leq 1/2$. It then follows from Theorem 3.6 that the natural class of connections on $B_1 \setminus D_1$ are connections of the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a$$

with $0 \leq \alpha \leq 1/2$. Here I will only discuss the case $0 < \alpha < 1/2$. In the case of $\alpha = 0$, the singularity is removable, and we are back to the case discussed in §2. In the case $\alpha = 1/2$, the singularity is removable as far as the local analysis is concerned; however there can be topological obstructions to removing the singularity globally on a 4-manifold, see [KM1].

If σ is an $\mathfrak{su}(2)$ -valued p -form, then we can decompose σ as

$$\sigma = \begin{pmatrix} i\sigma_D & \sigma_T \\ -\bar{\sigma}_T & -i\sigma_D \end{pmatrix}$$

where σ_D is a real valued p -form and σ_T is a complex valued p -form. We have

$$\nabla_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta} \sigma = \nabla \sigma + \left[\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta, \sigma \right] = \begin{pmatrix} i\nabla \sigma_D & \nabla_{2i\alpha d\theta} \sigma_T \\ -\overline{\nabla_{2i\alpha d\theta} \sigma_T} & -i\nabla \sigma_D \end{pmatrix}$$

Thus $\nabla_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}$ acts on σ_D as ∇ and on σ_T as $\nabla_{2i\alpha d\theta}$. Let $d_{2i\alpha d\theta}$ denote the covariant exterior derivative given by the connection $2i\alpha d\theta$. Let $d_{2i\alpha d\theta}^*$ denote the adjoint of $d_{2i\alpha d\theta}$.

We then have the following analog of Theorem 2.2.

Theorem 4.1. [R1] For any α with $2\alpha \notin \mathbb{Z}$ there exists $\varepsilon > 0$ such that if $A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a$ is a connection with $a \in L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ and

$$\|F_A\|_{L^2(B_1 \setminus D_1)} \leq \varepsilon,$$

then there exists a gauge transformation $g \in L^{2,2}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ such that

$$g.A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta + a'$$

where

$$\begin{cases} d^* a'_D = 0 & \text{on } B_1 \\ d_{2i\alpha d\theta}^*(r^{-2} a'_T) = 0 & \text{on } B_1 \setminus D_1 \\ \nu \lrcorner a' = 0 & \text{on } \partial B_1 \end{cases}$$

and

$$\|a'\|_{L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)} \leq c \|F_A\|_{L^2(B_1)}.$$

We also have the following analog of Theorem 2.3.

Theorem 4.2. [R1] If in addition to the assumptions of Theorem 4.1 the connection is Yang-Mills, then

$$\begin{cases} |a'_D| + |\nabla a'_D| \leq c \|F_A\|_{L^2(B_1)} & \text{on } B_{1/2} \\ |\nabla^k a'_T| \leq c r^{2 \min\{2\alpha, 1-2\alpha\}-k} \|F_A\|_{L^2(B_1)} & \text{on } B_{1/2} \end{cases}.$$

These seemingly strange theorems demand an explanation. The key is to understand the function space $L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1)$ in more detail. It follows from (1.1) that

$$a \in L^{2,1}_{\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}_{2i\alpha d\theta}(B_1) \end{cases}$$

Here $a_T \in L^{2,1}_{2i\alpha d\theta}$ means that $a_T, \nabla_{2i\alpha d\theta} a_T \in L^2(B_1)$. Fully written out

$$\|\nabla_{2i\alpha d\theta} \sigma_T\|_{L^2(B_1)}^2 = \int_{B_1} \left(\left| \frac{\partial \sigma_T}{\partial x_1} \right|^2 + \left| \frac{\partial \sigma_T}{\partial x_2} \right|^2 + \left| \frac{\partial \sigma_T}{\partial r} \right|^2 + r^{-2} \left| \frac{\partial \sigma_T}{\partial \theta} + 2i\alpha \sigma_T \right|^2 \right) r dx_1 dx_2 dr d\theta$$

Now,

$$\int_{S^1} f^2 d\theta \leq (\min\{2\alpha, 1-2\alpha\})^{-1} \int_{S^1} (df/d\theta + 2i\alpha f)^2 d\theta.$$

It follows that

$$\|r^{-1}\sigma_T\|_{L^2(B_1)} \leq c\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)}.$$

On the other hand, it is clear that

$$\|\nabla_{2i\alpha d\theta}\sigma_T\|_{L^2(B_1)} \leq c(\|\nabla\sigma_T\|_{L^2(B_1)} + \|r^{-1}\sigma_T\|_{L^2(B_1)}).$$

Hence

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i_\alpha & 0 \\ 0 & -i_\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ \nabla a_T, r^{-1}a_T \in L^2(B_1) \end{cases}$$

So now you think I'm going to talk about analysis on weighted Sobolev spaces with singular weights. I'm not.

As I mentioned before, the finite energy condition and the Yang-Mills equation are conformally invariant in 4 dimensions. Thus we can replace the standard metric

$$\sum_i dx_i^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2$$

with any conformal metric. A natural choice is the metric

$$r^{-2} \sum_i dx_i^2 = r^{-2}(dx_1^2 + dx_2^2 + dr^2) + d\theta^2.$$

With this metric D_1 is moved out to infinity. We recognize $r^{-2}(dx_1^2 + dx_2^2 + dr^2)$ as the upper half space model of hyperbolic 3-space. Thus $\mathbb{R}^4$ with the metric $r^{-2} \sum dx_i^2$ is isometric with $H^3 \times S^1$, the cartesian product of hyperbolic 3-space and the unit circle. The unit ball with this metric is isometric with $H_+^3 \times S^1$, the cartesian product of one half of hyperbolic 3-space and the unit circle. Thus we can view σ_T as a differential form on $H_+^3 \times S^1$. A short calculation shows that $r^{-1}\sigma_T, \nabla\sigma_T \in L^{2,1}$ if and only if $\sigma_T \in L^{2,1}(H_+^3 \times S^1)$. Thus

$$a \in L^{2,1}_{\left(\begin{smallmatrix} i_\alpha & 0 \\ 0 & -i_\alpha \end{smallmatrix}\right)d\theta}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}(H_+^3 \times S^1) \end{cases}$$

Thus we should view a_D as a differential form on B_1 and a_T as a differential form on $H_+^3 \times S^1$. Let $d_{h,2\alpha d\theta}^*$ denote the adjoint of $d_{2\alpha d\theta}$ with respect to the metric $r^{-1} \sum dx_i^2$. Moreover, a short calculation shows that

$$d_{h,2i\alpha d\theta}^* a_T = r^4 d_{2i\alpha d\theta}^* (r^{-2} a_T).$$

In other words, the gauge condition says that a_D is coclosed on B_1 and a_T is coclosed on $H_+^3 \times S^1$.

Theorem 4.1 is now proven along the same lines as Thm. 2.1. Instead of the equation (2.4) we get the equations

$$\left\{ \begin{array}{ll} \Delta \varphi_D + \sum_i (a_T)_i \frac{\partial \varphi_T}{\partial x_i} = -d^* b_D & \text{on } B_1 \\ \Delta_{h,2i\alpha d\theta} \varphi_T + \operatorname{Re} \sum_i ((a_T)_i (d\varphi_D)_i + (a_D)_i (d_{2i\alpha d\theta} \varphi_T)_i) \\ \quad = -d_{h,2i\alpha d\theta}^* b_T & \text{on } H_+^3 \times S^1. \\ \nu \lrcorner d\varphi_D = \nu \lrcorner b_D & \text{on } \partial B_1 \\ \nu_h \lrcorner d_{2i\alpha d\theta} \varphi_T = \nu_h \lrcorner b_T & \text{on } \partial H_+^3 \times S^1 \end{array} \right.$$

where $\Delta_{h,2i\alpha d\theta} = d_{2i\alpha d\theta} d_{h,2i\alpha d\theta}^* + d_{h,2i\alpha d\theta}^* d_{2i\alpha d\theta}$ is the covariant Hodge Laplacian for 1-forms on $H_+^3 \times S^1$ given by the connection $2i\alpha d\theta$, and $\nu_h = r\nu$ is the outward unit normal of $H_+^3 \times S^1$. Thus we get a small perturbation of the Neumann problem for Δ on B_1 and the Neumann problem for $\Delta_{h,2i\alpha d\theta}$ on $H_+^3 \times S^1$. The theory for the former is well known. The latter is analyzed in [R1] by elementary methods.

Theorem 4.2 is now proven along the same lines as Thm. 2.3. Instead of the equation (2.5) we get the system

$$\left\{ \begin{array}{ll} \Delta a_D + \{a_T \otimes \nabla_{2i\alpha d\theta} a_T\} + \{a_D \otimes a_T \otimes a_T\} = 0 & \text{on } B_1 \\ \Delta_{h,2i\alpha d\theta} a_T + \{a_D \otimes \nabla_{h,2i\alpha d\theta} a_T\} + \{a_T \otimes \nabla_h a_D\} \\ \quad + \{a_T \otimes a_T \otimes a_T\} + \{a_D \otimes a_D \otimes a_T\} = 0 & \text{on } H_+^3 \times S^1. \end{array} \right.$$

The 1-form a can now be estimated by a bootstrapping procedure. On the first equation we apply standard elliptic estimates for the usual Laplacian Δ on B_1 . On the second equation we apply decay estimates at infinity for the covariant Hodge Laplacian $\Delta_{h,2i\alpha d\theta}$ on $H_+^3 \times S^1$. These decay estimates are derived in [R1] by elementary methods.

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More references on singular Yang-Mills fields can be found in [R1].

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