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1. INTRODUCTION

Lerner proved in [4] that there are first order pseudodifferential operators of principal type satisfying condition ($\Psi$), that are not solvable in $L^2$ in any neighborhood of the origin. This was quite unexpected, since for first order differential operators of principal type, condition ($\Psi$) is equivalent to local $L^2$ solvability.

In this paper, we shall show that the counterexamples in [4] are locally solvable in $C^\infty$, and that we lose at most one derivative in the estimate for the adjoint operators. In some cases we only lose $\varepsilon$ derivatives in the estimate, for any $\varepsilon > 0$.

By local solvability in $L^2$ we mean that the equation $Pu = f$ has a local solution $u \in L^2(\mathbb{R}^n)$ for any $f \in L^2(\mathbb{R}^n)$ satisfying a finite number of compatibility conditions. We say that $P$ is locally solvable in $C^\infty$ if the equation has a solution $u \in \mathcal{D}'$ for any $f \in C^\infty$ satisfying a finite number of compatibility conditions. Recall that an operator is of principal type if the Hamilton field $H_p$ of the principal symbol $p$ is independent of the Liouville vector field.

Condition ($\Psi$) means that the imaginary part of the principal symbol does not change sign from $-$ to $+$ along the oriented bicharacteristics of the real part, see Definition 26.4.6 in [2]. This condition is invariant under multiplication of the principal symbol by non-vanishing factors.

It was conjectured by Nirenberg and Treves [5] that condition ($\Psi$) was equivalent to local solvability for operators of principal type, and they proved this in several cases. The necessity of ($\Psi$) for local solvability in the $C^\infty$ category was proved by Moyers in two dimensions and by Hörmander in general, see Corollary 26.4.8 in [2]. In the analytic category, the sufficiency of condition ($\Psi$) for solvability of microdifferential operators acting on microfunctions was proved by Trépreau [6]. The sufficiency of ($\Psi$) for local $L^2$ solvability for first order pseudodifferential operators in two dimensions, was proved by Lerner [3].

For differential operators, condition ($\Psi$) is equivalent to condition ($P$), which rules out any sign changes of the imaginary part of the principal symbol along the bicharacteristics of the real part. The sufficiency of ($P$) for local $L^2$ solvability for first order pseudodifferential operators was proved by Nirenberg and Treves [5] in the case when the principal symbol is real analytic, and by Beals and Fefferman [1] in the general case.

2. STATEMENT OF RESULTS

We shall consider the following type of operators, which includes the operators Lerner used in his counter-examples. First, let $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, and

$$P = D_t + i \sum_{\nu \in \mathbb{Z}_+} Q_\nu(t, x_1, D_x) + R(t, x, D_x)$$

where $R(t, x, D_x) \in C^\infty(\mathbb{R}, \Psi^0_1(T^*\mathbb{R}^n))$ and $\sum_\nu Q_\nu(t, x_1, D_x) \in C^\infty(\mathbb{R}, \Psi^1_1)$ is on the form

$$Q_\nu(t, x_1, D_x) = \alpha_\nu(t)(D_{x_1} + H(t)\nu^kW(\nu^kx_1))\Psi_\nu(D_x), \quad \nu \in \mathbb{Z}_+.$$
Here \( 0 \leq \alpha_\nu(t) \in C^\infty(\mathbb{R}) \) uniformly, such that \( 0 \notin \text{supp} \alpha_\nu \) and \( \alpha_\nu(t)H(t) \) is non-decreasing with \( H(t) \) the Heaviside function, \( 0 \leq W(x_1) \in C^\infty(\mathbb{R}) \) and \( k > 0 \). We also have \( 0 \leq \Psi_\nu(\xi) \in S^0_{1,0}(\mathbb{R}^n) \) uniformly, having non-overlapping interiors of the supports and \( 0 < c \leq |\xi|2^{-\nu} \leq C \) in \( \text{supp} \Psi_\nu \). Since \( 0 \notin \text{supp} \alpha_\nu \) we may write \( \alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t) \), where \( \beta_\nu(t) \in C^\infty \) (but not uniformly) such that \( 0 \leq \beta_\nu(t) \leq 1 \) and \( 0 \leq \partial_t \beta_\nu \). We find that \( \sum_\nu \nu^kW(\nu^kx_1)\Psi_\nu(D_x) \in C^\infty(\mathbb{R}, \Psi^r_{1,0}) \), for any \( \varepsilon > 0 \). Since \( 0 \leq \alpha_\nu(t) \) and \( W(\nu^kx_1)\Psi_\nu(\xi) \geq 0 \), it is clear that \( P \) satisfies condition \((\Psi^*)\), i.e., the adjoint \( P^* \) satisfies condition \((\Psi)\). In what follows, we shall suppress the \( t \) dependence and write \( S^m \) instead of \( C^\infty(\mathbb{R}, S^m) \) for example. We shall use the classical calculus of pseudo-differential operators, but with the general metrics and weights of the Weyl calculus. For notation and calculus results, see chapter 18 in

We define the norms

\[
(2.3) \quad \|u\|_{(s,k)}^2 = \int |\hat{u}(\xi)|^2(\xi^2)2^s(\log(\xi) + 1)^{2k}d\xi \quad s, k \in \mathbb{R},
\]

where \( \langle \xi \rangle^2 = 1 + |\xi|^2 \). Then \( \|u\|_{(s,0)} \simeq \|u\|_{(s)} \), the usual Sobolev norm, and \( \forall s, k \in \mathbb{R} \) we have

\[
(2.4) \quad c_{k,\varepsilon}\|u\|_{(s-k)} \leq \|u\|_{(s,k)} \leq C_{k,\varepsilon}\|u\|_{(s+k)} \quad \forall \varepsilon > 0.
\]

We find that \( \|u\|_{(s,k)} \) is equivalent to \( \sum_\nu \langle \xi_\nu \rangle^{2s}(\log(\xi_\nu) + 1)^{2k}||\psi_\nu(\xi_\nu)||^2 \) if \( \{ \psi_\nu(\xi) \} \nu \) is a partition of unity: \( \sum_\nu |\psi_\nu|^2 = 1 \) such that \( \langle \xi \rangle \approx \langle \xi_\nu \rangle \) only varies with a fixed factor in \( \text{supp} \psi_\nu \).

**Theorem 2.1.** Let \( P \) be given by \((2.1)\). Then, for any \( s \in \mathbb{R} \) there exists positive \( T_s \) and \( C_s \) such that

\[
(2.5) \quad \int \|u\|_{(s)}^2(t)dt \leq C_sT^2 \int \|Pu\|_{(s,2k)}^2(t)dt \quad \forall \varepsilon > 0.
\]

if \( u \in \mathcal{S} \) has support where \( |t| \leq T \leq T_s \).

Thus, we obtain for any \( s \in \mathbb{R} \) that

\[
(2.6) \quad \int \|u\|_{(s)}^2(t)dt \leq C_{s,\varepsilon}T^2 \int \|Pu\|_{(s,\varepsilon)}^2d\varepsilon \quad \forall \varepsilon > 0
\]

if \( u \in \mathcal{S} \) has support where \( |t| \leq T \leq T_s \). This shows that \( P^* \) is locally solvable in \( C^\infty \), with loss of \( \varepsilon \) derivatives, \( \forall \varepsilon > 0 \).

We shall also consider the following operators, which includes the operators Lerner used in his counter-example with homogeneous symbols. Let

\[
(2.7) \quad P = D_t + i \sum_{\nu \in J} Q_\nu(t, x, D_x) + R(t, x, D_x)
\]

where \( J \) is a subset of \( \mathbb{Z}_+ \) and \( \sum_\nu Q_\nu(t, x, D_x) \in \Psi^1_{1,0} \) is given by

\[
(2.8) \quad Q_\nu(t, x, D_x) = \alpha_\nu(t)C(D_x)\chi_\nu(x_2)(D_{x_1} + H(t)\nu^kW(\nu^kx_1)2^{-\nu}D_{x_2}) \quad \nu \in J.
\]

Here we have the same conditions on \( \alpha_\nu \), \( W \) and \( R \) as before. Also, \( 0 \leq C(\xi) \) homogeneous, supported where \( |\xi_1| \leq C\xi_2 \) and \( 0 \leq \chi_\nu(x_2) \in S(1, dx^2_2) \) uniformly with non-overlapping supports. In fact, there exists a function \( \mu(\nu) \) on \( \mathbb{Z}_+ \) such that \( \mu(\nu) \leq C_N\nu^N \), for some \( N > 0 \), and there exists \( \tilde{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2) \) uniformly, with disjoint supports such that \( 0 \leq \tilde{\chi}_\nu(x_2) \leq 1 \) and \( \tilde{\chi}_\nu = 1 \) on \( \text{supp} \chi_\nu \). As before, we find that \( P \) satisfies condition \((\Psi^*)\).
THEOREM 2.2. Let $P$ be given in (2.7). Then, for every $s \in \mathbb{R}$ we find $T_s > 0$ and $C_s > 0$ such that

$$\int \|u\|^2_{(s)}(t) dt \leq C_s T^2 \int \|Pu\|^2_{(s+1)}(t) dt \quad \forall s$$

if $u \in S$ has support where $|t| \leq T \leq T_s$.

Thus $P^*$ is locally solvable in $C^\infty$, with loss of one derivative. The theorems are going to be proved in the next sections.

3. PROOF OF THEOREM 2.1

Clearly, by conjugating with $(D_x)^s$ we may assume that $s = 0$, which only changes $R(t, x, D_x) \in \Psi_{1,0}^0$ (dependingly on $s$). Next, we shall eliminate $R(t, x, D_x)$. We choose $E_{\pm}(t, x, D_x) \in \Psi_{1,0}^0$ with principal symbols

$$e_{\pm}(t, x, \xi) = \exp(\pm \int_0^t iR(t, x, \xi) dt),$$

such that $E_- E_+ \cong E_+ E_- \cong \text{Id}$ modulo $\Psi^{-\infty}$. Then by conjugating with $E_{\pm}$ we obtain $R \in \Psi_{1,0}^{-1}$, but this changes $Q_{\nu}$ into

$$Q_{\nu}(t, x, D_x) = \alpha_{\nu}(t) \left((D_x x_1 + H(t)\nu^k W(\nu^k x_1))\Psi_{\nu}(D_x) + g_{\nu}(t, x, D_x)\right)$$

where $\{ g_{\nu}(t, x, \xi) \} \in S_{1,0}^0$. Since we may skip terms in $\Psi^{-1}$ in $P$ in the estimate (2.5), we may assume that $\sup\alpha_{\nu} \subseteq \sup\Psi_{\nu}$.

We shall localize in $S_{1/2,0}^0$ in order to separate the different $Q_{\nu}$ terms. Let $\{ \phi_j(\xi) \}_{j \in J} \in S_{1/2,0}^0$ be a partition of unity such that $\phi_j$ is supported where $|\xi - \xi_j| \leq c(\xi_j)^{1/2}$, and $\sup\phi_j$ is connected, $\forall j$. Let $J \subseteq \mathbb{Z}_+$ be the set of those $j$ for which $\sup\phi_j$ intersects $\bigcap_{\nu} \sup\Psi_{\nu}$. Since the principal symbol of $\sum_{\nu} Q_{\nu} \in \Psi_{1,0}^1$ vanishes of infinite order somewhere in $\sup\phi_j$ when $j \in J$, and $\phi_j(\xi) \in S_{1/2,0}^0$, we find that

$$\phi_j(D_x)Pu = \phi_j(D_x)Dt u + R_j(t, x, D_x)u$$

with $\{ R_j \}_{j \in J} \in \Psi_{1,0}^0$ (with values in $\ell^2$). We have

$$\int \|\phi_j(D_x)u\|^2(t) dt \leq CT^2 \int \|D_t\phi_j(D_x)u\|^2(t) dt$$

$$\leq CT^2 \int \|\phi_j(D_x)Pu\|^2(t) + \|R_ju\|^2(t) dt$$

for $j \in J$. Since $\sum_{j \in J} \|R_ju\|^2 \leq C\|u\|^2$, we get the result for small enough $T$, providing that we also have an estimate for the other terms.

Thus we only have to consider the case when $\sup\phi_j$ does not intersect $\bigcap_{\nu} \sup\Psi_{\nu}$, i.e. $j \notin J$. Since $\sup\phi_j$ is connected, we find that $\sup\phi_j$ is contained in the interior of $\sup\Psi_{\nu}$ for some unique $\nu = \nu_j$ when $j \notin J$. Observe that this gives $|\xi_j| \approx 2^{\nu_j}$ in $\sup\phi_j$.

Clearly, $\sup\bigcap_{\nu} \sup\Psi_{\nu} \subseteq \sup\Psi_{\nu}$ we have $P\phi_j(D_x)u = P_{\nu} \phi_j(D_x)u$ where we define

$$P_{\nu} = D_t + iQ_{\nu}(t, x_1, D_x).$$

Now we use the following

**Lemma 3.1.** Let $P_{\nu}$ be given by (3.5). Then we find

$$\int \|u\|^2(t)(\nu^{2k} \alpha_{\nu}(t) + 1) dt \leq CT^2 \nu^{4k} \int \|P_{\nu}u\|^2(t)(\nu^{2k} \alpha_{\nu}(t) + 1)^{-1} dt$$

uniformly in $\nu$, if $u \in S$ has support in $|t| \leq T$, for $T$ small enough.
By substituting \( \phi_j(D_x)u \), taking \( \nu = \nu_j \) in (3.6), and replacing \( P_\nu \) by \( P \), we obtain for \( j \notin J \) that
\[
(3.7) \quad \int \| \phi_j(D_x)u \|^2(t) \, dt \leq C T^2 \nu_j^{4k} \int \| P \phi_j(D_x)u \|^2(t) \, dt \\
\leq C T^2 \nu_j^{4k} \int \| \phi_j(D_x)Pu \|^2(t) + \| [P, \phi_j(D_x)]u \|^2(t) \, dt.
\]

Now \( \{ \nu_j^2([P, \phi_j(D_x)]) \}_{j \notin J} \in \Psi_{1/2,0}^{r-1/2} \) with values in \( \ell^2 \), \( \forall \varepsilon > 0 \). In fact, we find that \( \sum_\nu \nu^4 W(\nu^k x_1) \Psi_\nu(D_x) \in C^\infty(\mathbb{R}, \Psi_{1,0}^r) \) and \( \{ \nu_j^{2k} \phi_j(\xi) \}_{j \notin J} \in S_{1/2,0}^{r-1/2} \), \( \forall \varepsilon > 0 \), since \( \phi_j(\xi) \) is supported where \( |\xi| \approx 2^j \) when \( j \notin J \). Thus by summing up (3.4) and (3.7) we obtain (2.5) for \( s = 0 \) and small enough \( T \). This completes the proof of Theorem 2.1.

Proof. [Proof of Lemma 3.1] We may assume \( \nu \) is fixed in what follows. In the proof, we are going to localize in \( |\xi| \geq \nu^{2k} \). For that purpose we use the metric
\[
(3.8) \quad g_\nu = \nu^{2k}|dx|^2 + |d\xi|^2/(\nu^{4k} + \xi^2) \quad \nu \in \mathbb{Z}_+
\]
which is uniformly slowly varying, \( \sigma \) temperate and
\[
(3.9) \quad g_\nu^\sigma /g_\nu^c = h_\nu^c = \nu^{2k}/(\nu^{4k} + \xi^2)
\]
which makes \( h_\nu^{-2} = |\xi|^2 \nu^{-2k} + \nu^{2k} \geq 2|\xi| \). We find that \( Q_\nu \in \text{Op}(h_\nu^{-2}, g_\nu) \) but \( \nu^k W(\nu^k x_1) \in \text{S}(h_\nu^{-1}, g_\nu) \) uniformly.

Now we localize with \( \chi_0(\xi) = \chi(\xi^{1/2}\nu^{-2k}) \in S(1, g_\nu) \) where \( \chi \in C^\infty \) is equal to 1 near 0, and with \( \chi_\pm(\xi) = H(\pm \xi)(1 - \chi_0(\xi)) \in S(1, g_\nu) \) which has support where \( \pm \xi_\nu > c_0 \nu^{2k} \) so that \( \chi_0 + \chi_+ + \chi_- = 1 \). We also choose non-negative \( \chi_\pm(\xi) \) and \( \chi_0(\xi) \in S(1, g_\nu) \) such \( \chi_\pm \chi_\pm = \chi_\pm \) and \( \chi_0 \chi_0 = \chi_0 \). This can be done so that \( \chi_\pm \) have support where \( \pm \xi_\nu > c_0 \nu^{2k} \), \( c_0 > 0 \), and \( \chi_0 \) has support where \( |\xi| \leq C \nu^{2k} \).

First we estimate the \( \chi_\pm(D_x)u \) terms by Lemma 5.1 with the operator
\[
(3.10) \quad P_\pm = D_t + Q_\nu \chi_\pm(D_x),
\]
where
\[
(3.11) \quad \pm \Re Q_\nu \chi_\pm(D_x) \geq \mp C \quad \text{on } u \in \mathcal{S},
\]
by the Fefferman–Phong inequality, where \( \Re F = (F + F*)/2 \). In fact, the symbol of
\[
(3.12) \quad \pm \alpha_\nu(t) \Re \left(D_x + H(t) \nu^k W(\nu^k x_1)\right) \Psi_\nu(D_x) \chi_\pm(D_x)
\]
is bounded from below, modulo terms in \( S(1, g_\nu) \). Thus Lemma 5.1 gives (after changing \( t \) to \( -t \) for \( P_- \))
\[
(3.13) \quad \int \| u \|^2(t) \, dt \leq C T^2 \int \| P_\pm u \|^2(t) \, dt
\]
if \( u \in \mathcal{S} \) is supported where \( |t| \leq T \) and \( T \) is small enough. Now, by substituting \( \chi_\pm(D_x)u \) into (3.13) and using that \( P_\pm \chi_\pm(D_x) = P_\nu \chi_\pm(D_x) \) and that \( [P_\nu, \chi_\pm(D_x)] \in \text{Op}(S(1, g_\nu)) \) is uniformly \( L^2 \) bounded, we find
\[
(3.14) \quad \int \| \chi_\pm(D_x)u \|^2(t) \, dt \leq C_0 T^2 \int \| P_\nu u \|^2(t) + \| u \|^2(t) \, dt
\]
if \( u \in \mathcal{S} \) is supported where \( |t| \leq T \) and \( T \) is small enough.

Next, we shall estimate \( \| \chi_0(D_x)u \|^2 \). Let
\[
(3.15) \quad B_\nu = D_x \Psi_\nu(D_x) \tilde{\chi}_0(D_x) + \beta_\nu(t) \left( \nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_x) + \rho \right) \in \text{Op}(h_\nu^{-1}, g_\nu),
\]
where \( q > 0 \). Here \( \beta_\nu \in C^\infty \) such that \( 0 \leq \beta_\nu(t) \leq 1, 0 \leq \partial_t \beta_\nu \) and \( \alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t) \). Since \( \nu^kW(\nu^kx_1)\Psi_\nu(D_\nu)\tilde{\chi}_0(D_{\nu}) \in \text{Op } S(h_{\nu}^{-1}, g_{\nu}) \) has positive principal symbol, we find

\[
(3.16) \quad \partial_t B_\nu = \partial_t \beta_\nu(t) \left( \nu^kW(\nu^kx_1)\Psi_\nu(D_\nu)\tilde{\chi}_0(D_{\nu}) + \varrho \right) \geq 0
\]

for large enough \( \varrho \). We also find \( B_\nu \in \text{Op } S(\nu^{2k}, g_{\nu}) \) uniformly, thus \( \|B_\nu\| \leq C\nu^{2k} \). Applying Lemma 5.2 on \( \chi_0(D_{\nu})u \), with \( P_0 = D_t + \alpha_\nu(t)(B_\nu + r_\nu), r_\nu = g_{\nu}(t, x, D_\nu)\tilde{\chi}_0(D_{\nu}) - \beta_\nu(t)\varrho \) and \( M = C\nu^{2k} \), we find

\[
(3.17) \quad \int \|\chi_0(D_{\nu})u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \leq C_1\nu^{4k}T^2 \int \|P_0\chi_0(D_{\nu})u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1} dt
\]

if \( u \in \mathcal{S} \) is supported where \(|t| \leq T \) and \( T \) is small enough. As before, we find \( P_0\chi_0(D_{\nu}) = P_\nu\chi_0(D_{\nu}) \) and we have \( [P_\nu, \chi_0(D_{\nu})] = \alpha_\nu(t)f_\nu \), where \( f_\nu \in \text{Op } S(1, g_{\nu}) \) is uniformly \( L^2 \) bounded. Since

\[
(3.18) \quad \nu^{4k}\alpha_\nu^2(t)/(\nu^{2k}\alpha_\nu(t) + 1) \leq \nu^{2k}\alpha_\nu(t) + 1,
\]

we obtain

\[
(3.19) \quad \int \|\chi_0(D_{\nu})u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \\
\leq C_1T^2 \left( \int \nu^{4k}\|P_\nu u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1} dt + \int \|u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \right)
\]

if \( u \) is supported where \(|t| \leq T \) and \( T \) is small enough. Combining (3.14) and (3.19), we obtain (3.6) for small enough \( T \). ~

4. PROOF OF THEOREM 2.2

First, we conjugate with \( (D_{\nu})^{s+1/2} \) to reduce to the case \( s = -1/2 \) (this only changes \( R(t, x, D_\nu) \) dependingly on \( s \)). We choose \( E_\pm(t, x, D_\nu) \in \Psi_{1,0}^0 \) with principal symbols

\[
(4.1) \quad e_\pm(t, x, \xi) = \exp(\pm \int_0^t iR(t, x, \xi) dt),
\]

such that \( E_-E_+ \cong E_+E_- \cong \text{Id} \) modulo \( \Psi^{-\infty} \). As before, the calculus gives \( R \in \Psi_{1,0}^{-1} \) for the new operator, but changes \( Q_\nu \) into

\[
(4.2) \quad Q_\nu(t, x, D_\nu) = \alpha_\nu(t) \left( C(D_\nu)\chi_\nu(x_2)(D_{\nu}) + H(t)\nu^kW(\nu^kx_1)2^{-s}D_{x_2} + g_{\nu}(t, x, D_\nu) \right)
\]

where \( g_{\nu}(t, x, \xi) \in \mathcal{S}_{1,0}^0 \) uniformly, with \( \text{supp } g_{\nu} \subseteq \text{supp } \chi_\nu \). Thus, we may assume \( R \equiv 0 \) since the term \( CT\|Ru\|_{(1/2)} \) can be estimated by the left hand side of (2.9) for \( s = -1/2 \) and small enough \( T \).

Next, we localize in \( x_2 \) to separate the different \( Q_\nu \) terms. By assumption there exists \( \tilde{\chi}_\nu(x_2) \in \mathcal{S}(1, \mu^2(\nu)dx_2) \) uniformly when \( \nu \in J_\nu \), with disjoint supports, such that \( 0 \leq \tilde{\chi}_\nu(x_2) \leq 1 \) and \( \tilde{\chi}_\nu\chi_\nu = \chi_\nu \). We also localize in \( \xi \); let \( \{ \psi_j(\xi) \} \) and \( \{ \phi_j(\xi) \} \) \( \in \mathcal{S}_{1,0}^0 \) with values in \( \ell^2 \) such that \( \sum_j \psi_j(\xi)^2 = 1, \phi_j(\xi) \) and \( \psi_j(\xi) \) are non-negative, \( \phi_j\psi_j = \psi_j \) and \( \psi_j, \phi_j \) are supported where \( 0 < c \leq |\xi|^{2-s} \leq C \). We may also assume that for some fixed \( N > 0 \) we have \( \sum_{|j-k| \leq N} \psi_j^2(\xi) \equiv 1 \) on \( \text{supp } \psi_j, \forall j \).
Since $\tilde{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2)$ we find that $\{ \psi_j(\xi)\tilde{\chi}_\nu(x_2) \}_{\nu,j}$ is not in a good symbol class. Therefore, we put
\begin{equation}
\tilde{\chi}_{0j}(x_2) = 1 - \sum_{0 < \nu < \jmath^2} \tilde{\chi}_\nu(x_2).
\end{equation}

Since $\psi_j$ is supported where $|\xi| \approx 2^j$ and $\mu(\nu) \le C_N \nu^N$ for some $N > 0$, it is easy to see that $\{ \tilde{\chi}_\nu(x_2)\psi_j(\xi) \}_{j \mathbb{\nu} \le j^2}$ and $\{ \tilde{\chi}_{0j}(x_2)\psi_j(\xi) \}_{j} \in \Psi^0_1, \forall \varepsilon > 0$. Let
\begin{equation}
\alpha_{\nu j}(t) = \sqrt{\alpha_\nu(t) + 2^{-j}} \quad \forall j \in J, \quad \forall \nu,
\end{equation}
in what follows. Now, we are going to use the following

\textbf{Lemma 4.1.} We find that
\begin{equation}
\int \sum_{J \mathbb{\nu} \le j^2} \| \alpha_{\nu j}(t)\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t) + \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) \ dt
\le CT \int \sum_{J \mathbb{\nu} \le j^2} \|\alpha_{\nu j}^{-1}(t)\tilde{\chi}_\nu(x_2)\psi_j(D_x)Pu\|^2(t)
+ \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)Pu\|^2(t) + \|u\|^{2}_{-1/2}(t) \ dt.
\end{equation}

if $u \in \mathcal{S}$ has support in $|t| \le T$ for $T$ small enough.

Since $2^{-j/2} \le \alpha_{\nu j}, |\xi| \approx 2^j$ in supp $\psi_j$, the supports of $\tilde{\chi}_\nu$ are disjoint and $\sum_{J \mathbb{\nu} \le j^2} \tilde{\chi}_\nu + \tilde{\chi}_{0j} \equiv 1, \forall j$, it is easy to see that the left hand side of (4.5) is greater than $c \int \|u\|^{2}_{-1/2}(t) \ dt$ for some $c > 0$, and the right hand side is less than $CT \int \|Pu\|^{2}_{-1/2}(t) + \|u\|^{2}_{-1/2}(t) \ dt$. Thus (4.5) implies (2.9) for the case $s = -1/2$ for small $T$, and completes the proof of Theorem 2.2.

\textbf{Proof.} [Proof of Lemma 4.1] Since $\psi_j(1 - \phi_j) \equiv 0 \ \forall j$, the calculus gives that we may replace $P$ by $P_j = D_t + i \sum_{\nu \in J} Q_\nu \phi_j(D_x)$ for the terms containing the factor $\psi_j(D_x)$ in (4.5).

For the terms $\|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2$ we use the fact that $\nu^k W(\nu^k x_1)2^{-\nu} D_{x_2} \phi_j(D_x) \in \Psi^{-\infty}$ uniformly when $(\log |\xi|)^2 \approx j^2 < \nu$. Thus we use Nirenberg-Treves estimate in [2, Theorem 26.8.1] with $B = D_{x_1} \phi_j(D_x)$ bounded, and $0 \le A \in \Psi_{1,0}^0$ such that
\begin{equation}
A \simeq \sum_{J \mathbb{\nu} \ge j^2} \alpha_\nu(t)C(D_x)\chi_\nu(x_2) \quad \text{mod} \quad \Psi_{1,0}^{-1}.
\end{equation}

By perturbing this estimate with $L^2$ bounded operators, and substituting the term $\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u$, we find for small enough $T$ that
\begin{equation}
\int \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) \ dt \le CT^2 \int \|\tilde{P}_j\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) \ dt \quad \forall j
\end{equation}
when $|t| \le T$ in supp $u$. Here
\begin{equation}
\tilde{P}_j = D_t + i \sum_{J \mathbb{\nu} \ge j^2} \alpha_\nu(t)C(D_x)\chi_\nu(x_2)D_{x_1} + q_\nu(t, x, D_x) \phi_j(D_x)
\geq D_t + i \sum_{J \mathbb{\nu} \ge j^2} Q_\nu \phi_j(D_x) \quad \text{modulo} \quad \Psi^{-\infty}.
\end{equation}

Thus $\tilde{P}_j$ satisfies condition (P), i. e., the imaginary part of the principal symbol has no sign changes for fixed $(x, \xi)$. 

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Since $\alpha_\nu \leq C \alpha_\nu$ and $\text{supp } \varrho_\nu \subseteq \text{supp } \chi_\nu$, the calculus gives that

$$\{(\tilde{P}_j, \tilde{\chi}_{0j}(x_2)\psi_j(D_x))\} \approx \left\{ \sum_{\nu \leq j^2} \alpha_\nu(t) f_{\nu j}(x, D_x) \right\} \mod \Psi_{1, \epsilon}^{-1/2}$$

where $\{ f_{\nu j}\}_{\nu j} \in \Psi_0^0$ with values in $\ell^2$, and $\text{supp } f_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$. In order to estimate these terms we need the following

**Lemma 4.2.** If $\{ f_{\nu j}(x, D_x)\}_{\nu j} \in \Psi_0^0$ with values in $\ell^2$, and $\text{supp } f_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$, $\forall \nu j$, then

$$\sum_{\nu \leq j^2} \| \alpha_{\nu j}(t) f_{\nu j}(x, D_x) u \|^2 \leq C \left( \sum_{\nu \leq j^2} \| \alpha_{\nu j}(t) \tilde{\chi}_{0j}(x_2) \psi_j(D_x) u \|^2 \right. \right.$$

$$\left. \left. + \sum_{\nu \leq j^2} \| \tilde{\chi}_{0j}(x_2) \psi_j(D_x) u \| + \| u \|_{(-1/2)}^2 \right)$$

for $u \in S$.

Since $\tilde{\chi}_{0j} \equiv 0$ on $\text{supp } \chi_\nu$ when $J \geq \nu \leq j^2$, we find that $\{ \tilde{\chi}_{0j}(x_2) \psi_j(D_x)(\tilde{P}_j - P_j) \}_{\nu j} \in \Psi^{-\infty}$, where as before $P_j = D_t + i \sum_{\nu \in \mathbb{J}} Q_{\nu} \phi_j(D_x) \in \Psi_{1, \epsilon}$. Thus we find

$$\int \sum_{\nu \leq j^2} \| \tilde{\chi}_{0j}(x_2) \psi_j(D_x) \tilde{P}_j u \|^2(t) \, dt \leq CT \int \sum_{\nu \leq j^2} \| \tilde{\chi}_{0j}(x_2) \psi_j(D_x) P_j u \|^2(t) + \| u \|_{(-1/2)}^2(t) \, dt.$$

This gives the estimate (4.5) for the terms $\| \tilde{\chi}_{0j}(x_2) \psi_j(D_x) u \|^2$ for small $T$, providing we can estimate the other terms.

As before, we are going to use Lemma 5.2 with $\alpha(t) = \alpha_\nu(t)$ and

$$B_t = \text{Re } C(D_x) \chi_\nu(x_2) \left( D_x, \phi_j(D_x) + \beta_\nu(t) \left( \nu^k W(\nu^k x_1) 2^{-\nu} D_x \phi_j(D_x) + \varrho \right) \right),$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \leq \beta_\nu(t) \leq 1$, $0 \leq \partial_t \beta_\nu$ and $\alpha_\nu(t) H(t) \equiv \alpha_\nu(t) \beta_\nu(t)$. We have $\| B_t \| \leq C 2^j$, $\partial_t B_t \geq 0$ for large $\varrho$ and $R_t \in \Psi^0$. By substituting $\tilde{\chi}_\nu(x_2) \psi_j(D_x) u$ in this Lemma, we find for small $T$ that

$$\int \| \tilde{\chi}_\nu(x_2) \psi_j(D_x) u \|^2(t) (2^i \alpha_\nu(t) + 1) \, dt \leq CT 2^{2j} \int \| (D_t + i Q_{\nu} \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u \|^2(t) (2^i \alpha_\nu(t) + 1)^{-1} \, dt$$

when $J \geq \nu \leq j^2$, providing $|t| \leq T$ in $\text{supp } u$. This is equivalent to

$$\int \| \alpha_{\nu j}(t) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u \|^2(t) \, dt \leq CT \int \| \alpha_{\nu j}^{-1}(t) (D_t + i Q_{\nu} \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u \|^2(t) \, dt.$$

Now, it follows from the asymptotic expansion that

$$\{ [Q_{\nu} \phi_j(D_x), \tilde{\chi}_\nu(x_2) \psi_j(D_x)] \}_{j \leq \nu \leq j^2} \cong \left\{ \alpha_{\nu j}(t) \tilde{f}_{\nu j}(t, x, D_x) \right\}_{j \leq \nu \leq j^2}$$
modulo $\Psi^{-1/2}_{t_x}$, where \( \{ \tilde{f}_{\nu j}(t, x, D_x) \}_{\nu j} \in \Psi^0_{1,0} \) with values in $\ell^2$, supp $f_{\nu j} \subseteq \text{supp} \chi_\nu \psi_j$, $\forall t$. Thus, we may estimate the commutator terms by Lemma 4.2:

\[
(4.16) \quad \sum_{\nu j} \| \alpha_{\nu j}(t) \tilde{f}_{\nu j}(t, x, D_x) u \|^2 \leq C \left( \sum_{\nu j} \| \alpha_{\nu j} \chi_{\nu j} \psi_j u \|^2 + \sum_j \| \tilde{\chi}_0 \psi_j u \|^2 + \| u \|_{-1/2}^2 \right) \quad \forall t.
\]

Since the supports of $\tilde{\chi}_\nu$ are disjoint, and $\sum J_{\nu \mu} Q_{\mu} \phi_j(D_x) \in \Psi^1_{1,0}$ uniformly, we obtain that

\[
(4.17) \quad \left\{ \tilde{\chi}_\nu(x_2) \phi_j(D_x) \sum_{J_{\nu \mu} \neq \nu} Q_{\mu} \phi_j(D_x) \right\}_{J_{\nu \mu} \leq j^2} \in \Psi^{-\infty}
\]

with values in $\ell^2$. Thus we may replace $D_t + iQ_{\nu} \phi_j(D_x)$ by $P_j$ in the estimate, which proves (4.5). ■

Proof. [Proof of Lemma 4.2] Since $\sum_{\nu - k \leq N} \psi^2(x) \equiv 1$ on supp $f_{\nu j}$ and $\{ f_{\nu j} \}_{\nu j} \in S^0_{1,0}$, we may use the calculus to write

\[
(4.18) \quad \sum_{\nu j} \| \alpha_{\nu j}(t) f_{\nu j}(x, D_x) u \|^2 \leq \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u \|^2 + C \| u \|_{-1/2}^2,
\]

where $\{ e_{\nu j k} \}_{\nu j k} \in \Psi^0_{1,0}$ with values in $\ell^2$, and supp $e_{\nu j k} \subseteq \text{supp} f_{\nu j} \psi_k$. Since $\tilde{\chi}_{0k} + \sum_{\nu \leq k} \tilde{\chi}_{\nu} \equiv 1$, we find

\[
(4.19) \quad \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u \|^2 \leq 2 \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u \|^2
\]

\[
+ 2 \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k} \tilde{\chi}_{\mu}(x_2) \psi_k(D_x) u \|^2.
\]

By summing up in $j$ and $\nu$ we find

\[
(4.20) \quad \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u \|^2
\]

\[
\leq C N(\sum_k \| \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u \|^2 + \| u \|_{-1/2}^2),
\]

since $\alpha_{\nu j} \leq c$ and $\{ e_{\nu j k} \}_{\nu j} \in \Psi^0_{1,0}$ with values in $\ell^2$, uniformly in $k$. Now $\alpha_{\nu j} \leq C \alpha_{\nu k}$ when $|j - k| \leq N$ which similarly gives by the calculus

\[
(4.21) \quad \sum_{\nu j} \| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k} \tilde{\chi}_{\mu}(x_2) \psi_k(D_x) u \|^2
\]

\[
\leq C \sum_{\mu \leq k} \| \alpha_{\mu k}(t) \tilde{\chi}_{\mu}(x_2) \psi_k(D_x) u \|^2 + C \| u \|_{-1/2}^2
\]

since supp $e_{\nu j k} \subseteq \text{supp} \chi_\mu \forall j, k$. ■
5. SOME ESTIMATE LEMMAS

We assume that

\[ P = D_t + iQ_t + R_t \]

where \( Q_t \) is a closed, densely defined operator on \( L^2(\mathbb{R}^n) \) such that \( S \subset D(Q_t) \cap D(Q_t^*) \) \( \forall t \), that \( t \mapsto \langle Q_t u, u \rangle \) is continuous for \( u \in S \), and

\[ \text{Re} Q_t \geq -C_1 \quad \text{on } S \quad \forall t, \]

where \( 2 \text{Re} Q_t = Q_t + Q_t^* \). We also assume that \( \|R_t\| \leq C_0 \) on \( L^2(\mathbb{R}^n) \). Let \( \|u\| \) be the \( L^2 \) norm of \( u \in L^2(\mathbb{R}^n) \) and \( \langle u, v \rangle \) the corresponding sesquilinear form.

**Lemma 5.1.** There exists \( T_0 > 0 \) and \( C > 0 \) such that

\[ \int \|u\|^2(t) dt \leq 2CT^2 \int \|Pu\|^2(t) dt \]

if \( u \in S \) has support where \( |t| \leq T \leq T_0 \). Here \( T_0 \) and \( C \) only depend on \( C_0 \) and \( C_1 \).

**Proof.** We only need to prove the estimate (5.1) for \( R_t = 0 \), since we may perturb it with \( L^2 \) bounded terms for small \( T \). We find

\[ \langle Q_t u, u \rangle \geq -C_1\|u\|^2 \quad \forall t \]

when \( u \in S \). Since \( iP = \partial_t - Q_t \), this gives

\[ \|u\|^2(t) = -\int_t^T 2 \text{Re}(\partial_t u, u)(t) dt \]

when \( u \in S \), and \( u \equiv 0 \) when \( t \geq T \).

By integrating in \( t \) we find

\[ \int_{-T}^T \|u\|^2(t) dt \leq 4T \int_{-T}^T \text{Im}(Pu, u)(t) dt + 4C_1T \int_{-T}^T \|u\|^2(t) dt \]

By using the Cauchy-Schwarz inequality we obtain

\[ 2\langle Pu, u \rangle \leq \lambda\|u\|^2/T + \|Pu\|^2T/\lambda \quad \forall \lambda > 0. \]

This gives

\[ (1 - 4CT - 2\lambda) \int \|u\|^2 \leq 2T^2/\lambda \int \|Pu\|^2 dt, \]

which gives (5.3) when \( T_0 \leq 1/16C \) and \( \lambda \leq 1/4 \). \( \blacksquare \)

The next case we shall consider is

\[ P = D_t + i a(t)(B_t + R_t) \]

where \( 0 \leq a(t) \leq C_0 \), \( B_t \) and \( \partial_t B_t \) are self-adjoint and bounded, \( \partial_t B_t \geq 0 \) and \( \|R_t\| \leq C_1 \) on \( L^2(\mathbb{R}^n) \). We also assume that there exists a constant \( M > 0 \) such that

\[ \|B_t\| \leq M \quad \forall t \]

\[ \|[B_s, B_t]\| \leq M \quad \forall s, t. \]
**Lemma 5.2.** There exists $T_0 > 0$ and $C > 0$ such that

\[(5.12) \quad \int \|u\|^2(t)(a(t) + M^{-1}) \, dt \leq CT^2 \int \|Pu\|^2(t)(a(t) + M^{-1})^{-1} \, dt\]

if $u \in S$ has support where $|t| \leq T \leq T_0$. Here $C_0$ and $T_0$ are independent of $M$, and only depend on $C_0$ and $C_1$.

**Proof.** First we consider the case $a(t) \geq M^{-1} > 0$. Then (5.12) is equivalent to the estimate:

\[(5.13) \quad \int \|u\|^2(t)a(t) \, dt \leq CT^2 \int \|Pu\|^2(t) \, dt/a(t)\]

if $u \in S$ has support where $|t| \leq T$ is small enough. Introducing $s = \int_0^t a(t) \, dt$ as a new time variable and $P_0 = D_s + iB_t$, we find that it suffices to prove

\[(5.14) \quad \int \|u\|^2(s) \, ds \leq CT^2 \int \|P_0u\|^2(s) \, ds\]

if $u \in S$ has support where $|s| \leq CT$. This proves (5.13) in the case $a(t) \geq M^{-1}$.

Next we consider the case $a(t) \leq 0$. In order to reduce to the case $a \geq M^{-1}$ we conjugate with $E_t$ solving

\[(5.17) \quad \begin{cases} \partial_t E_t = -E_t B_t/M \\ E_0 = Id. \end{cases}\]

This gives bounds on $\|E_t\|$ and $\|E_t^{-1}\|$ when $t$ is bounded (independently of $M$), and the conjugation transforms $P$ into

\[(5.18) \quad \tilde{P} = D_t + i(a(t) + M^{-1})B_t + a(t)\tilde{R}_t = D_t + i(a(t) + M^{-1})(B_t + S_t)\]

where $\tilde{R}_t = iE_t^{-1}[B_t + R_t, E_t] + iR_t$ and $S_t = a(t)\tilde{R}_t/(a(t) + M^{-1})$ are uniformly bounded on $L^2(\mathbb{R}^n)$ for bounded $t$. In fact, if $F_r = [B_t, E_r], \forall r$, then

\[(5.19) \quad \partial_r F_r = E_r[B_r, B_t]/M - F_r B_t/M\]

and $F_0 \equiv 0$, thus $F_t = [B_t, E_t]$ is bounded on $L^2(\mathbb{R}^n)$ for bounded $t$ (independently of $M$). By using (5.13) with $\tilde{P}$ and $a(t) + M^{-1}$, we obtain (5.12).
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