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THE SOLVABILITY OF NON L^2 SOLVABLE OPERATORS

NILS DENCKER

1. INTRODUCTION

Lerner proved in [4] that there are first order pseudodifferential operators of principal type satisfying condition (Ψ) , that are not solvable in L^2 in any neighborhood of the origin. This was quite unexpected, since for first order differential operators of principal type, condition (Ψ) is equivalent to local L^2 solvability.

In this paper, we shall show that the counterexamples in [4] are locally solvable in C^∞ , and that we lose at most one derivative in the estimate for the adjoint operators. In some cases we only lose ε derivatives in the estimate, for any $\varepsilon > 0$.

By local solvability in L^2 we mean that the equation $Pu = f$ has a local solution $u \in L^2(\mathbf{R}^n)$ for any $f \in L^2(\mathbf{R}^n)$ satisfying a finite number of compatibility conditions. We say that P is locally solvable in C^∞ if the equation has a solution $u \in \mathcal{D}'$ for any $f \in C^\infty$ satisfying a finite number of compatibility conditions. Recall that an operator is of principal type if the Hamilton field H_p of the principal symbol p is independent of the Liouville vector field.

Condition (Ψ) means that the imaginary part of the principal symbol does not change sign from $-$ to $+$ along the oriented bicharacteristics of the real part, see Definition 26.4.6 in [2]. This condition is invariant under multiplication of the principal symbol by non-vanishing factors.

It was conjectured by Nirenberg and Treves [5] that condition (Ψ) was equivalent to local solvability for operators of principal type, and they proved this in several cases. The necessity of (Ψ) for local solvability in the C^∞ category was proved by Moyers in two dimensions and by Hörmander in general, see Corollary 26.4.8 in [2]. In the analytic category, the sufficiency of condition (Ψ) for solvability of microdifferential operators acting on microfunctions was proved by Trépau [6]. The sufficiency of (Ψ) for local L^2 solvability for first order pseudodifferential operators in two dimensions, was proved by Lerner [3].

For differential operators, condition (Ψ) is equivalent to condition (P) , which rules out any sign changes of the imaginary part of the principal symbol along the bicharacteristics of the real part. The sufficiency of (P) for local L^2 solvability for first order pseudodifferential operators was proved by Nirenberg and Treves [5] in the case when the principal symbol is real analytic, and by Beals and Fefferman [1] in the general case.

2. STATEMENT OF RESULTS

We shall consider the following type of operators, which includes the operators Lerner used in his counter-examples. First, let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $n \geq 2$, and

$$(2.1) \quad P = D_t + i \sum_{\nu \in \mathbf{Z}_+} Q_\nu(t, x_1, D_x) + R(t, x, D_x)$$

where $R(t, x, D_x) \in C^\infty(\mathbf{R}, \Psi_{1,0}^0(T^*\mathbf{R}^n))$ and $\sum_\nu Q_\nu(t, x_1, D_x) \in C^\infty(\mathbf{R}, \Psi_{1,0}^1)$ is on the form

$$(2.2) \quad Q_\nu(t, x_1, D_x) = \alpha_\nu(t)(D_{x_1} + H(t)\nu^k W(\nu^k x_1))\Psi_\nu(D_x), \quad \nu \in \mathbf{Z}_+.$$

Here $0 \leq \alpha_\nu(t) \in C^\infty(\mathbf{R})$ uniformly, such that $0 \notin \text{supp } \alpha_\nu$ and $\alpha_\nu(t)H(t)$ is non-decreasing with $H(t)$ the Heaviside function, $0 \leq W(x_1) \in C^\infty(\mathbf{R})$ and $k > 0$. We also have $0 \leq \Psi_\nu(\xi) \in S_{1,0}^0(T^*\mathbf{R}^n)$ uniformly, having non-overlapping interiors of the supports and $0 < c \leq |\xi|2^{-\nu} \leq C$ in $\text{supp } \Psi_\nu$. Since $0 \notin \text{supp } \alpha_\nu$ we may write $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$, where $\beta_\nu(t) \in C^\infty$ (but not uniformly) such that $0 \leq \beta_\nu(t) \leq 1$ and $0 \leq \partial_t \beta_\nu$. We find that $\sum_\nu \nu^k W(\nu^k x_1) \Psi_\nu(D_x) \in C^\infty(\mathbf{R}, \Psi_{1,0}^\varepsilon)$, for any $\varepsilon > 0$. Since $0 \leq \alpha_\nu(t)$ and $W(\nu^k x_1) \Psi_\nu(\xi) \geq 0$, it is clear that P satisfies condition (Ψ^*) , i. e., the adjoint P^* satisfies condition (Ψ) . In what follows, we shall suppress the t dependence and write S^m instead of $C^\infty(\mathbf{R}, S^m)$ for example. We shall use the classical calculus of pseudo-differential operators, but with the general metrics and weights of the Weyl calculus. For notation and calculus results, see chapter 18 in

We define the norms

$$(2.3) \quad \|u\|_{(s,k)}^2 = \int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} (\log \langle \xi \rangle + 1)^{2k} d\xi \quad s, k \in \mathbf{R},$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$. Then $\|u\|_{(s,0)} \cong \|u\|_{(s)}$, the usual Sobolev norm, and $\forall s, k \in \mathbf{R}$ we have

$$(2.4) \quad c_{k,\varepsilon} \|u\|_{(s-\varepsilon)} \leq \|u\|_{(s,k)} \leq C_{k,\varepsilon} \|u\|_{(s+\varepsilon)} \quad \forall \varepsilon > 0.$$

We find that $\|u\|_{(s,k)}$ is equivalent to $\sum_\nu \langle \xi_\nu \rangle^{2s} (\log \langle \xi_\nu \rangle + 1)^{2k} \|\psi_\nu(D_x)u\|^2$ if $\{\psi_\nu(\xi)\}_\nu$ is a partition of unity: $\sum_\nu |\psi_\nu|^2 = 1$ such that $\langle \xi \rangle \approx \langle \xi_\nu \rangle$ only varies with a fixed factor in $\text{supp } \psi_\nu$.

THEOREM 2.1. Let P be given by (2.1). Then, for any $s \in \mathbf{R}$ there exists positive T_s and C_s such that

$$(2.5) \quad \int \|u\|_{(s)}^2(t) dt \leq C_s T^2 \int \|Pu\|_{(s,2k)}^2(t) dt$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_s$.

Thus, we obtain for any $s \in \mathbf{R}$ that

$$(2.6) \quad \int \|u\|_{(s)}^2(t) dt \leq C_{s,\varepsilon} T^2 \int \|Pu\|_{(s+\varepsilon)}^2 ds \quad \forall \varepsilon > 0$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_s$. This shows that P^* is locally solvable in C^∞ , with loss of ε derivatives, $\forall \varepsilon > 0$.

We shall also consider the following operators, which includes the operators Lerner used in his counter-example with homogeneous symbols. Let

$$(2.7) \quad P = D_t + i \sum_{\nu \in J} Q_\nu(t, x, D_x) + R(t, x, D_x)$$

where J is a subset of $\mathbf{Z}_+$ and $\sum_\nu Q_\nu(t, x, D_x) \in \Psi_{1,0}^1$ is given by

$$(2.8) \quad Q_\nu(t, x, D_x) = \alpha_\nu(t) C(D_x) \chi_\nu(x_2) (D_{x_1} + H(t) \nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2}) \quad \nu \in J.$$

Here we have the same conditions on α_ν , W and R as before. Also, $0 \leq C(\xi)$ is homogeneous, supported where $|\xi_1| \leq C\xi_2$ and $0 \leq \chi_\nu(x_2) \in S(1, dx_2^2)$ uniformly with non-overlapping supports. In fact, there exists a function $\mu(\nu)$ on $\mathbf{Z}_+$ such that $\mu(\nu) \leq C_N \nu^N$, for some $N > 0$, and there exists $\tilde{\chi}_\nu \in S(1, \mu^2(\nu) dx_2^2)$ uniformly, with disjoint supports such that $0 \leq \tilde{\chi}_\nu(x_2) \leq 1$ and $\tilde{\chi}_\nu = 1$ on $\text{supp } \chi_\nu$. As before, we find that P satisfies condition (Ψ^*) .

THEOREM 2.2. Let P be given in (2.7). Then, for every $s \in \mathbf{R}$ we find $T_s > 0$ and $C_s > 0$ such that

$$(2.9) \quad \int \|u\|_{(s)}^2(t) dt \leq C_s T^2 \int \|Pu\|_{(s+1)}^2(t) dt \quad \forall s$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_s$.

Thus P^* is locally solvable in C^∞ , with loss of one derivative. The theorems are going to be proved in the next sections.

3. PROOF OF THEOREM 2.1

Clearly, by conjugating with $\langle D_x \rangle^s$ we may assume that $s = 0$, which only changes $R(t, x, D_x) \in \Psi_{1,0}^0$ (dependingly on s). Next, we shall eliminate $R(t, x, D_x)$. We choose $E_\pm(t, x, D_x) \in \Psi_{1,0}^0$ with principal symbols

$$(3.1) \quad e_\pm(t, x, \xi) = \exp(\pm \int_0^t iR(t, x, \xi) dt),$$

such that $E_- E_+ \cong E_+ E_- \cong \text{Id}$ modulo $\Psi^{-\infty}$. Then by conjugating with $E_\pm$ we obtain $R \in \Psi_{1,0}^{-1}$, but this changes Q_ν into

$$(3.2) \quad Q_\nu(t, x, D_x) = \alpha_\nu(t) \left((D_{x_1} + H(t)\nu^k W(\nu^k x_1)) \Psi_\nu(D_x) + \varrho_\nu(t, x, D_x) \right)$$

where $\{\varrho_\nu(t, x, \xi)\}_\nu \in S_{1,0}^0$. Since we may skip terms in Ψ^{-1} in P in the estimate (2.5), we may assume that $\text{supp } \varrho_\nu \subseteq \text{supp } \Psi_\nu$.

We shall localize in $S_{1/2,0}^0$ in order to separate the different Q_ν terms. Let $\{\phi_j(\xi)\}_j \in S_{1/2,0}^0$ be a partition of unity such that ϕ_j is supported where $|\xi - \xi_j| \leq c\langle \xi_j \rangle^{1/2}$, and $\text{supp } \phi_j$ is connected, $\forall j$. Let $J \subset \mathbf{Z}_+$ be the set of those j for which $\text{supp } \phi_j$ intersects $\bigcap_\nu \text{supp } \Psi_\nu$. Since the principal symbol of $\sum_\nu Q_\nu \in \Psi_{1,0}^1$ vanishes of infinite order somewhere in $\text{supp } \phi_j$ when $j \in J$, and $\phi_j(\xi) \in S_{1/2,0}^0$, we find that

$$(3.3) \quad \phi_j(D_x)Pu = \phi_j(D_x)D_t u + R_j(t, x, D_x)u$$

with $\{R_j\}_{j \in J} \in \Psi_{1,0}^0$ (with values in ℓ^2). We have

$$(3.4) \quad \begin{aligned} \int \|\phi_j(D_x)u\|^2(t) dt &\leq CT^2 \int \|D_t \phi_j(D_x)u\|^2(t) dt \\ &\leq CT^2 \int \|\phi_j(D_x)Pu\|^2(t) + \|R_j u\|^2(t) dt \end{aligned}$$

for $j \in J$. Since $\sum_{j \in J} \|R_j u\|^2 \leq C\|u\|^2$, we get the result for small enough T , providing that we also have an estimate for the other terms.

Thus we only have to consider the case when $\text{supp } \phi_j$ does not intersect $\bigcap_\nu \text{supp } \Psi_\nu$, i. e. $j \notin J$. Since $\text{supp } \phi_j$ is connected, we find that $\text{supp } \phi_j$ is contained in the interior of $\text{supp } \Psi_\nu$ for some unique $\nu = \nu_j$ when $j \notin J$. Observe that this gives $|\xi_j| \approx 2^{\nu_j}$ in $\text{supp } \phi_j$. Clearly, since $\text{supp } Q_\nu \subseteq \text{supp } \Psi_\nu$ we have $P\phi_j(D_x)u = P_{\nu_j}\phi_j(D_x)u$ where we define

$$(3.5) \quad P_\nu = D_t + iQ_\nu(t, x_1, D_x).$$

Now we use the following

Lemma 3.1. Let P_ν be given by (3.5). Then we find

$$(3.6) \quad \int \|u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \leq CT^2 \nu^{4k} \int \|P_\nu u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1} dt$$

uniformly in ν , if $u \in \mathcal{S}$ has support in $|t| \leq T$, for T small enough.

By substituting $\phi_j(D_x)u$, taking $\nu = \nu_j$ in (3.6), and replacing P_{ν_j} by P , we obtain for $j \notin J$ that

$$(3.7) \quad \int \|\phi_j(D_x)u\|^2(t) dt \leq CT^2 \nu_j^{4k} \int \|P\phi_j(D_x)u\|^2(t) dt \\ \leq CT^2 \nu_j^{4k} \int \|\phi_j(D_x)Pu\|^2(t) + \|[P, \phi_j(D_x)]u\|^2(t) dt.$$

Now $\{\nu_j^{2k}[P, \phi_j(D_x)]\}_{j \notin J} \in \Psi_{1/2,0}^{\varepsilon-1/2}$ with values in ℓ^2 , $\forall \varepsilon > 0$. In fact, we find that $\sum_\nu \nu^k W(\nu^k x_1) \Psi_\nu(D_x) \in C^\infty(\mathbf{R}, \Psi_{1,0}^\varepsilon)$ and $\{\nu_j^{2k} \phi_j(\xi)\}_{j \notin J} \in S_{1/2,0}^\varepsilon$, $\forall \varepsilon > 0$, since $\phi_j(\xi)$ is supported where $|\xi| \approx 2^{\nu_j}$ when $j \notin J$. Thus by summing up (3.4) and (3.7) we obtain (2.5) for $s = 0$ and small enough T . This completes the proof of Theorem 2.1.

Proof. [Proof of Lemma 3.1] We may assume ν is fixed in what follows. In the proof, we are going to localize in $|\xi_1| \gtrsim \nu^{2k}$. For that purpose we use the metric

$$(3.8) \quad g_\nu = \nu^{2k}|dx|^2 + |d\xi|^2/(\nu^{4k} + \xi_1^2) \quad \nu \in \mathbf{Z}_+$$

which is uniformly slowly varying, σ temperate and

$$(3.9) \quad g_\nu/g_\nu^\sigma = h_\nu^2 = \nu^{2k}/(\nu^{4k} + \xi_1^2)$$

which makes $h_\nu^{-2} = |\xi_1|^2 \nu^{-2k} + \nu^{2k} \geq 2|\xi_1|$. We find that $Q_\nu \in \text{Op } S(h_\nu^{-2}, g_\nu)$ but $\nu^k W(\nu^k x_1) \in S(h_\nu^{-1}, g_\nu)$ uniformly.

Now we localize with $\chi_0(\xi_1) = \chi(\xi_1 \nu^{-2k}) \in S(1, g_\nu)$ where $\chi \in C_0^\infty$ is equal to 1 near 0, and with $\chi_\pm(\xi_1) = H(\pm \xi_1)(1 - \chi_0(\xi_1)) \in S(1, g_\nu)$ which has support where $\pm \xi_1 > c\nu^{2k}$ so that $\chi_0 + \chi_+ + \chi_- \equiv 1$. We also choose non-negative $\tilde{\chi}_\pm(\xi_1)$ and $\tilde{\chi}_0(\xi_1) \in S(1, g_\nu)$ such $\tilde{\chi}_\pm \chi_\pm = \chi_\pm$ and $\tilde{\chi}_0 \chi_0 = \chi_0$. This can be done so that $\tilde{\chi}_\pm$ have support where $\pm \xi_1 > c_0 \nu^{2k}$, $c_0 > 0$, and $\tilde{\chi}_0$ has support where $|\xi_1| \leq C\nu^{2k}$.

First we estimate the $\chi_\pm(D_{x_1})u$ terms by Lemma 5.1 with the operator

$$(3.10) \quad P_\pm = D_t + Q_\nu \tilde{\chi}_\pm(D_{x_1}),$$

where

$$(3.11) \quad \pm \text{Re } Q_\nu \tilde{\chi}_\pm(D_{x_1}) \geq \mp C \quad \text{on } u \in \mathcal{S},$$

by the Fefferman–Phong inequality, where $\text{Re } F = (F + F^*)/2$. In fact, the symbol of

$$(3.12) \quad \pm \alpha_\nu(t) \text{Re} \left(D_{x_1} + H(t) \nu^k W(\nu^k x_1) \right) \Psi_\nu(D_x) \tilde{\chi}_\pm(D_{x_1})$$

is bounded from below, modulo terms in $S(1, g_\nu)$. Thus Lemma 5.1 gives (after changing t to $-t$ for P_-)

$$(3.13) \quad \int \|u\|^2(t) dt \leq CT^2 \int \|P_\pm u\|^2(t) dt$$

if $u \in \mathcal{S}$ is supported where $|t| \leq T$ and T is small enough. Now, by substituting $\chi_\pm(D_{x_1})u$ into (3.13) and using that $P_\pm \chi_\pm(D_{x_1}) = P_\nu \chi_\pm(D_{x_1})$ and that $[P_\nu, \chi_\pm(D_{x_1})] \in \text{Op } S(1, g_\nu)$ is uniformly L^2 bounded, we find

$$(3.14) \quad \int \|\chi_\pm(D_{x_1})u\|^2(t) dt \leq C_0 T^2 \int \|P_\nu u\|^2(t) + \|u\|^2(t) dt$$

if $u \in \mathcal{S}$ is supported where $|t| \leq T$ and T is small enough.

Next, we shall estimate $\|\chi_0(D_{x_1})u\|^2$. Let

$$(3.15) \quad B_\nu = D_{x_1} \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) + \beta_\nu(t) \left(\nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) + \varrho \right) \in \text{Op } S(h_\nu^{-1}, g_\nu),$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \leq \beta_\nu(t) \leq 1$, $0 \leq \partial_t \beta_\nu$ and $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$. Since $\nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) \in \text{Op } S(h_\nu^{-1}, g_\nu)$ has positive principal symbol, we find

$$(3.16) \quad \partial_t B_\nu = \partial_t \beta_\nu(t) \left(\nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) + \varrho \right) \geq 0$$

for large enough ϱ . We also find $B_\nu \in \text{Op } S(\nu^{2k}, g_\nu)$ uniformly, thus $\|B_\nu\| \leq C\nu^{2k}$. Applying Lemma 5.2 on $\chi_0(D_{x_1})u$, with $P_0 = D_t + \alpha_\nu(t)(B_\nu + r_\nu)$, $r_\nu = \varrho_\nu(t, x, D_x) \tilde{\chi}_0(D_{x_1}) - \beta_\nu(t)\varrho$ and $M = C\nu^{2k}$, we find

$$(3.17) \quad \int \|\chi_0(D_{x_1})u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \leq C_1 \nu^{4k} T^2 \int \|P_0 \chi_0(D_{x_1})u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1} dt$$

if $u \in \mathcal{S}$ is supported where $|t| \leq T$ and T is small enough. As before, we find $P_0 \chi_0(D_{x_1}) = P_\nu \chi_0(D_{x_1})$ and we have $[P_\nu, \chi_0(D_{x_1})] = \alpha_\nu(t)f_\nu$, where $f_\nu \in \text{Op } S(1, g_\nu)$ is uniformly L^2 bounded. Since

$$(3.18) \quad \nu^{4k} \alpha_\nu^2(t) / (\nu^{2k} \alpha_\nu(t) + 1) \leq \nu^{2k} \alpha_\nu(t) + 1,$$

we obtain

$$(3.19) \quad \int \|\chi_0(D_x)u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \\ \leq C_1 T^2 \left(\int \nu^{4k} \|P_\nu u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1} dt + \int \|u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1) dt \right)$$

if u is supported where $|t| \leq T$ and T is small enough. Combining (3.14) and (3.19), we obtain (3.6) for small enough T . ■

4. PROOF OF THEOREM 2.2

First, we conjugate with $\langle D_x \rangle^{s+1/2}$ to reduce to the case $s = -1/2$ (this only changes $R(t, x, D_x)$ dependently on s). We choose $E_\pm(t, x, D_x) \in \Psi_{1,0}^0$ with principal symbols

$$(4.1) \quad e_\pm(t, x, \xi) = \exp(\pm \int_0^t i R(t, x, \xi) dt),$$

such that $E_- E_+ \cong E_+ E_- \cong \text{Id}$ modulo $\Psi^{-\infty}$. As before, the calculus gives $R \in \Psi_{1,0}^{-1}$ for the new operator, but changes Q_ν into

$$(4.2) \quad Q_\nu(t, x, D_x) = \alpha_\nu(t) \left(C(D_x) \chi_\nu(x_2) (D_{x_1} + H(t) \nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2}) + \varrho_\nu(t, x, D_x) \right)$$

where $\varrho_\nu(t, x, \xi) \in S_{1,0}^0$ uniformly, with $\text{supp } \varrho_\nu \subseteq \text{supp } \chi_\nu$. Thus, we may assume $R \equiv 0$ since the term $CT\|Ru\|_{(1/2)}$ can be estimated by the left hand side of (2.9) for $s = -1/2$ and small enough T .

Next, we localize in x_2 to separate the different Q_ν terms. By assumption there exists $\tilde{\chi}_\nu(x_2) \in S(1, \mu^2(\nu)dx_2^2)$ uniformly when $\nu \in J$, with disjoint supports, such that $0 \leq \tilde{\chi}_\nu(x_2) \leq 1$ and $\tilde{\chi}_\nu \chi_\nu = \chi_\nu$. We also localize in ξ : let $\{\psi_j(\xi)\}_j$ and $\{\phi_j(\xi)\}_j \in S_{1,0}^0$ (with values in ℓ^2) such that $\sum_j \psi_j(\xi)^2 = 1$, $\phi_j(\xi)$ and $\psi_j(\xi)$ are non-negative, $\phi_j \psi_j = \psi_j$ and ψ_j, ϕ_j are supported where $0 < c \leq |\xi| 2^{-\nu} \leq C$. We may also assume that for some fixed $N > 0$ we have $\sum_{|j-k| \leq N} \psi_k^2(\xi) \equiv 1$ on $\text{supp } \psi_j$, $\forall j$.

Since $\tilde{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2)$ we find that $\{\psi_j(\xi)\tilde{\chi}_\nu(x_2)\}_{\nu,j}$ is not in a good symbol class. Therefore, we put

$$(4.3) \quad \tilde{\chi}_{0j}(x_2) = 1 - \sum_{\substack{0 < \nu \leq j^2 \\ \nu \in J}} \tilde{\chi}_\nu(x_2).$$

Since ψ_j is supported where $|\xi| \approx 2^j$ and $\mu(\nu) \leq C_N \nu^N$ for some $N > 0$, it is easy to see that $\{\tilde{\chi}_\nu(x_2)\psi_j(\xi)\}_{J \ni \nu \leq j^2}$ and $\{\tilde{\chi}_{0j}(x_2)\psi_j(\xi)\}_j \in \Psi_{1,\varepsilon}^0$, $\forall \varepsilon > 0$. Let

$$(4.4) \quad \alpha_{\nu j}(t) = \sqrt{\alpha_\nu(t) + 2^{-j}} \quad \forall j \in J, \quad \forall \nu,$$

in what follows. Now, we are going to use the following

Lemma 4.1. We find that

$$(4.5) \quad \int \sum_{\substack{J \ni \nu \leq j^2 \\ j}} \|\alpha_{\nu j}(t)\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t) + \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) dt \\ \leq CT \int \sum_{\substack{J \ni \nu \leq j^2 \\ j}} \|\alpha_{\nu j}^{-1}(t)\tilde{\chi}_\nu(x_2)\psi_j(D_x)Pu\|^2(t) \\ + \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)Pu\|^2(t) + \|u\|_{(-1/2)}^2(t) dt.$$

if $u \in \mathcal{S}$ has support in $|t| \leq T$ for T small enough.

Since $2^{-j/2} \leq \alpha_{\nu j}$, $|\xi| \approx 2^j$ in $\text{supp } \psi_j$, the supports of $\tilde{\chi}_\nu$ are disjoint and $\sum_{J \ni \nu \leq j^2} \tilde{\chi}_\nu + \tilde{\chi}_{0j} \equiv 1$, $\forall j$, it is easy to see that the left hand side of (4.5) is greater than $c \int \|u\|_{(-1/2)}^2(t) dt$ for some $c > 0$, and the right hand side is less than $CT \int \|Pu\|_{(1/2)}^2(t) + \|u\|_{(-1/2)}^2(t) dt$. Thus (4.5) implies (2.9) for the case $s = -1/2$ for small T , and completes the proof of Theorem 2.2.

Proof. [Proof of Lemma 4.1] Since $\psi_j(1 - \phi_j) \equiv 0 \quad \forall j$, the calculus gives that we may replace P by $P_j = D_t + i \sum_{\nu \in J} Q_\nu \phi_j(D_x)$ for the terms containing the factor $\psi_j(D_x)$ in (4.5).

For the terms $\|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2$ we use the fact that $\nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2} \phi_j(D_x) \in \Psi^{-\infty}$ uniformly when $(\log |\xi|)^2 \approx j^2 < \nu$. Thus we use Nirenberg–Treves estimate in [2, Theorem 26.8.1] with $B = D_{x_1} \phi_j(D_x)$ bounded, and $0 \leq A \in \Psi_{1,0}^0$ such that

$$(4.6) \quad A \cong \sum_{J \ni \nu > j^2} \alpha_\nu(t) C(D_x) \chi_\nu(x_2) \quad \text{mod } \Psi_{1,0}^{-1}.$$

By perturbing this estimate with L^2 bounded operators, and substituting the term $\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u$, we find for small enough T that

$$(4.7) \quad \int \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) dt \leq CT^2 \int \|\tilde{P}_j \tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) dt \quad \forall j$$

when $|t| \leq T$ in $\text{supp } u$. Here

$$(4.8) \quad \tilde{P}_j = D_t + i \sum_{J \ni \nu > j^2} \alpha_\nu(t) (C(D_x) \chi_\nu(x_2) D_{x_1} + \varrho_\nu(t, x, D_x)) \phi_j(D_x) \\ \cong D_t + i \sum_{J \ni \nu > j^2} Q_\nu \phi_j(D_x) \quad \text{modulo } \Psi^{-\infty}.$$

Thus $\tilde{P}_j$ satisfies condition (P), i. e., the imaginary part of the principal symbol has no sign changes for fixed (x, ξ) .

Since $\alpha_\nu \leq C\alpha_{\nu j}$ and $\text{supp } \varrho_\nu \subseteq \text{supp } \chi_\nu$, the calculus gives that

$$(4.9) \quad \left\{ [\tilde{P}_j, \tilde{\chi}_{0j}(x_2)\psi_j(D_x)] \right\}_j \cong \left\{ \sum_{\nu > j^2} \alpha_{\nu j}(t) f_{\nu j}(x, D_x) \right\}_j \mod \Psi_{1,\varepsilon}^{-1/2}$$

where $\{f_{\nu j}\}_{\nu j} \in \Psi_{1,0}^0$ with values in ℓ^2 , and $\text{supp } f_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$. In order to estimate these terms we need the following

Lemma 4.2. If $\{f_{\nu j}(x, D_x)\}_{\nu j} \in \Psi_{1,0}^0$ with values in ℓ^2 , and $\text{supp } f_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$, $\forall \nu j$, then

$$(4.10) \quad \sum_{\substack{\nu \in J \\ j}} \|\alpha_{\nu j}(t) f_{\nu j}(x, D_x) u\|^2 \leq C \left(\sum_{\nu \leq j^2} \|\alpha_{\nu j}(t) \tilde{\chi}_j(x_2) \psi_j(D_x) u\|^2 \right. \\ \left. + \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(D_x) u\| + \|u\|_{(-1/2)}^2 \right)$$

for $u \in \mathcal{S}$.

Since $\tilde{\chi}_{0j} \equiv 0$ on $\text{supp } \chi_\nu$ when $J \ni \nu \leq j^2$, we find that $\{\tilde{\chi}_{0j}(x_2) \psi_j(D_x) (\tilde{P}_j - P_j)\}_j \in \Psi^{-\infty}$, where as before $P_j = D_t + i \sum_{\nu \in J} Q_\nu \phi_j(D_x) \in \Psi_{1,0}^1$. Thus we find

$$(4.11) \quad \int \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(D_x) \tilde{P}_j u\|^2(t) dt \\ \leq CT \int \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(D_x) P_j u\|^2(t) + \|u\|_{(-1/2)}^2(t) dt.$$

This gives the estimate (4.5) for the terms $\|\tilde{\chi}_{0j}(x_2) \psi_j(D_x) u\|^2$ for small T , providing we can estimate the other terms.

As before, we are going to use Lemma 5.2 with $a(t) = \alpha_\nu(t)$ and

$$(4.12) \quad B_t = \text{Re } C(D_x) \chi_\nu(x_2) \left(D_{x_1} \phi_j(D_x) + \beta_\nu(t) \left(\nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2} \phi_j(D_x) + \varrho \right) \right),$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \leq \beta_\nu(t) \leq 1$, $0 \leq \partial_t \beta_\nu$ and $\alpha_\nu(t) H(t) \equiv \alpha_\nu(t) \beta_\nu(t)$. We have $\|B_t\| \leq C2^j$, $\partial_t B_t \geq 0$ for large ϱ and $R_t \in \Psi^0$. By substituting $\tilde{\chi}_\nu(x_2) \psi_j(D_x) u$ in this Lemma, we find for small T that

$$(4.13) \quad \int \|\tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) (2^j \alpha_\nu(t) + 1) dt \\ \leq CT^2 2^{2j} \int \|(D_t + iQ_\nu \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) (2^j \alpha_\nu(t) + 1)^{-1} dt$$

when $J \ni \nu \leq j^2$, providing $|t| \leq T$ in $\text{supp } u$. This is equivalent to

$$(4.14) \quad \int \|\alpha_{\nu j}(t) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) dt \\ \leq CT^2 \int \|\alpha_{\nu j}^{-1}(t) (D_t + iQ_\nu \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) dt.$$

Now, it follows from the asymptotic expansion that

$$(4.15) \quad \{[Q_\nu \phi_j(D_x), \tilde{\chi}_\nu(x_2) \psi_j(D_x)]\}_{J \in \nu \leq j^2} \cong \{\alpha_\nu(t) \tilde{f}_{\nu j}(t, x, D_x)\}_{J \in \nu \leq j^2}$$

modulo $\Psi_{1,\varepsilon}^{-1/2}$, where $\{ \tilde{f}_{\nu j}(t, x, D_x) \}_{\nu j} \in \Psi_{1,0}^0$ with values in ℓ^2 , $\text{supp } \tilde{f}_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$, $\forall t$. Thus, we may estimate the commutator terms by Lemma 4.2:

$$(4.16) \quad \sum_{\substack{J \ni \nu \leq j^2 \\ j}} \|\alpha_{\nu j}(t) \tilde{f}_{\nu j}(t, x, D_x) u\|^2 \\ \leq C \left(\sum_{\nu \leq j^2} \|\alpha_{\nu j} \tilde{\chi}_j \psi_j u\|^2 + \sum_j \|\tilde{\chi}_{0j} \psi_j u\|^2 + \|u\|_{(-1/2)}^2 \right) \quad \forall t.$$

Since the supports of $\tilde{\chi}_\nu$ are disjoint, and $\sum_{J \ni \mu \neq \nu} Q_\mu \phi_j(D_x) \in \Psi_{1,0}^1$ uniformly, we obtain that

$$(4.17) \quad \left\{ \tilde{\chi}_\nu(x_2) \psi_j(D_x) \sum_{J \ni \mu \neq \nu} Q_\mu \phi_j(D_x) \right\}_{J \ni \nu \leq j^2} \in \Psi^{-\infty}$$

with values in ℓ^2 . Thus we may replace $D_t + iQ_\nu \phi_j(D_x)$ by P_j in the estimate, which proves (4.5). ■

Proof. [Proof of Lemma 4.2] Since $\sum_{|j-k| \leq N} \psi_k^2(\xi) \equiv 1$ on $\text{supp } f_{\nu j}$ and $\{ f_{\nu j} \}_{\nu j} \in S_{1,0}^0$, we may use the calculus to write

$$(4.18) \quad \sum_{\nu, j} \|\alpha_{\nu j}(t) f_{\nu j}(x, D_x) u\|^2 \leq \sum_{\substack{\nu, j \\ |k-j| \leq N}} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u\|^2 + C \|u\|_{(-1)}^2,$$

where $\{ e_{\nu j k} \}_{\nu j k} \in \Psi_{1,0}^0$ with values in ℓ^2 , and $\text{supp } e_{\nu j k} \subseteq \text{supp } f_{\nu j} \psi_k$. Since $\tilde{\chi}_{0k} + \sum_{\mu \leq k^2} \tilde{\chi}_\mu \equiv 1$, we find

$$(4.19) \quad \sum_{\substack{\nu, j \\ |k-j| \leq N}} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u\|^2 \leq 2 \sum_{\substack{\nu, j \\ |k-j| \leq N}} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 \\ + 2 \sum_{\substack{\nu, j \\ |k-j| \leq N}} \left\| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u \right\|^2.$$

By summing up in j and ν we find

$$(4.20) \quad \sum_{\substack{\nu, j \\ |k-j| \leq N}} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 \\ \leq C_N \left(\sum_k \|\tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 + \|u\|_{(-1/2)}^2 \right),$$

since $\alpha_{\nu j} \leq c$ and $\{ e_{\nu j k} \}_{\nu j} \in \Psi_{1,0}^0$ with values in ℓ^2 , uniformly in k . Now $\alpha_{\nu j} \leq C \alpha_{\nu k}$ when $|j - k| \leq N$ which similarly gives by the calculus

$$(4.21) \quad \sum_{\substack{\nu, j \\ |k-j| \leq N}} \left\| \alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u \right\|^2 \\ \leq C \sum_{\mu \leq k^2} \|\alpha_{\mu k}(t) \tilde{\chi}_\mu(x_2) \psi_k(D_x) u\|^2 + C \|u\|_{(-1/2)}^2$$

since $\text{supp } e_{\mu j k} \subseteq \text{supp } \chi_\mu \forall j, k$. ■

5. SOME ESTIMATE LEMMAS

We assume that

$$(5.1) \quad P = D_t + iQ_t + R_t$$

where Q_t is a closed, densely defined operator on $L^2(\mathbf{R}^n)$ such that $\mathcal{S} \subset D(Q_t) \cap D(Q_t^*)$ $\forall t$, $t \mapsto \langle Q_t u, u \rangle$ is continuous for $u \in \mathcal{S}$, and

$$(5.2) \quad \operatorname{Re} Q_t \geq -C_1 \quad \text{on } \mathcal{S} \quad \forall t,$$

where $2 \operatorname{Re} Q_t = Q_t + Q_t^*$. We also assume that $\|R_t\| \leq C_0$ on $L^2(\mathbf{R}^n)$. Let $\|u\|$ be the L^2 norm of $u \in L^2(\mathbf{R}^n)$ and $\langle u, v \rangle$ the corresponding sesquilinear form.

Lemma 5.1. There exists $T_0 > 0$ and $C > 0$ such that

$$(5.3) \quad \int \|u\|^2(t) \leq CT^2 \int \|Pu\|^2(t) dt$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_0$. Here T_0 and C only depend on C_0 and C_1 .

Proof. We only need to prove the estimate (5.1) for $R_t \equiv 0$, since we may perturb it with L^2 bounded terms for small T . We find

$$(5.4) \quad \langle Q_t u, u \rangle \geq -C_1 \|u\|^2 \quad \forall t$$

when $u \in \mathcal{S}$. Since $iP = \partial_t - Q_t$, this gives

$$(5.5) \quad \begin{aligned} \|u\|^2(t) &= - \int_t^T 2 \operatorname{Re} \langle \partial_t u, u \rangle(t) dt \\ &= - \int_t^T 2 \operatorname{Re} \langle iPu, u \rangle(t) - \int_t^T 2 \operatorname{Re} \langle Q_t u, u \rangle(t) dt \\ &\leq - \int_t^T 2 \operatorname{Re} \langle iPu, u \rangle(t) dt + 2C_1 \int_t^T \|u\|^2(t) dt \end{aligned}$$

when $u \in \mathcal{S}$, and $u \equiv 0$ when $t \geq T$.

By integrating in t we find

$$(5.6) \quad \int_{-T}^T \|u\|^2(t) dt \leq 4T \int_{-T}^T \operatorname{Im} \langle Pu, u \rangle(t) dt + 4C_1 T \int_{-T}^T \|u\|^2(t) dt$$

By using the Cauchy-Schwarz inequality we obtain

$$(5.7) \quad 2 \langle Pu, u \rangle \leq \lambda \|u\|^2/T + \|Pu\|^2 T/\lambda \quad \forall \lambda > 0.$$

This gives

$$(5.8) \quad (1 - 4CT - 2\lambda) \int \|u\|^2 \leq 2T^2/\lambda \int \|Pu\|^2 dt,$$

which gives (5.3) when $T_0 \leq 1/16C$ and $\lambda \leq 1/4$. ■

The next case we shall consider is

$$(5.9) \quad P = D_t + ia(t)(B_t + R_t)$$

where $0 \leq a(t) \leq C_0$, B_t and $\partial_t B_t$ are self-adjoint and bounded, $\partial_t B_t \geq 0$ and $\|R_t\| \leq C_1$ on $L^2(\mathbf{R}^n)$. We also assume that there exists a constant $M > 0$ such that

$$(5.10) \quad \|B_t\| \leq M \quad \forall t$$

$$(5.11) \quad \|[B_s, B_t]\| \leq M \quad \forall s, t.$$

Lemma 5.2. There exists $T_0 > 0$ and $C > 0$ such that

$$(5.12) \quad \int \|u\|^2(t)(a(t) + M^{-1}) dt \leq CT^2 \int \|Pu\|^2(t)(a(t) + M^{-1})^{-1} dt$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_0$. Here C_0 and T_0 are independent of M , and only depend on C_0 and C_1 .

Proof. First we consider the case $a(t) \geq M^{-1} > 0$. Then (5.12) is equivalent to the estimate:

$$(5.13) \quad \int \|u\|^2(t)a(t) dt \leq CT^2 \int \|Pu\|^2(t) dt/a(t)$$

if $u \in \mathcal{S}$ has support where $|t| \leq T$ is small enough. Introducing $s = \int_0^t a(t) dt$ as a new time variable and $P_0 = D_s + iB_t$, we find that it suffices to prove

$$(5.14) \quad \int \|u\|^2(s) ds \leq CT^2 \int \|P_0 u\|^2(s) ds$$

if $u \in \mathcal{S}$ has support where $|t| \leq T$, which implies $|s| \leq CT$. In fact, we may then perturb the estimate with the L^2 bounded term $iR_t u$ for small T .

Now $[P_0^*, P_0] = 2\partial_s B_t \geq 0$, which implies

$$(5.15) \quad \|P_0 u\|^2 - \|P_0^* u\|^2 = \langle [P_0^*, P_0] u, u \rangle \geq 0.$$

Since $\|D_s u\|^2 \leq 2(\|P_0 u\|^2 + \|P_0^* u\|^2)$, we find

$$(5.16) \quad \int \|u\|^2(s) ds \leq C_0 T^2 \int \|D_s u\|^2(s) ds \leq 4CT^2 \int \|P_0 u\|^2(s) ds$$

if $u \in \mathcal{S}$ has support where $|s| \leq CT$. This proves (5.13) in the case $a(t) \geq M^{-1}$.

Next we consider the case $a(t) \geq 0$. In order to reduce to the case $a \geq M^{-1}$ we conjugate with E_t solving

$$(5.17) \quad \begin{cases} \partial_t E_t = -E_t B_t / M \\ E_0 = \text{Id}. \end{cases}$$

This gives bounds on $\|E_t\|$ and $\|E_t^{-1}\|$ when t is bounded (independently of M), and the conjugation transforms P into

$$(5.18) \quad \tilde{P} = D_t + i(a(t) + M^{-1})B_t + a(t)\tilde{R}_t = D_t + i(a(t) + M^{-1})(B_t + S_t)$$

where $\tilde{R}_t = iE_t^{-1}[B_t + R_t, E_t] + iR_t$ and $S_t = a(t)\tilde{R}_t/(a(t) + M^{-1})$ are uniformly bounded on $L^2(\mathbf{R}^n)$ for bounded t . In fact, if $F_r = [B_t, E_r]$, $\forall r$, then

$$(5.19) \quad \partial_r F_r = E_r[B_r, B_t]/M - F_r B_r / M$$

and $F_0 \equiv 0$, thus $F_t = [B_t, E_t]$ is bounded on $L^2(\mathbf{R}^n)$ for bounded t (independently of M). By using (5.13) with $\tilde{P}$ and $a(t) + M^{-1}$, we obtain (5.12). ■

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