NILS DENCKER

The solvability of non $L^2$ solvable operators


<http://www.numdam.org/item?id=JEDP_1996____A10_0>
THE SOLVABILITY OF NON $L^2$ SOLVABLE OPERATORS
NILS DENCKER

1. INTRODUCTION

Lerner proved in [4] that there are first order pseudodifferential operators of principal type satisfying condition $(\Psi)$, that are not solvable in $L^2$ in any neighborhood of the origin. This was quite unexpected, since for first order differential operators of principal type, condition $(\Psi)$ is equivalent to local $L^2$ solvability.

In this paper, we shall show that the counterexamples in [4] are locally solvable in $C^\infty$, and that we lose at most one derivative in the estimate for the adjoint operators. In some cases we only lose $\varepsilon$ derivatives in the estimate, for any $\varepsilon > 0$.

By local solvability in $L^2$ we mean that the equation $Pu = f$ has a local solution $u \in L^2(\mathbb{R}^n)$ for any $f \in L^2(\mathbb{R}^n)$ satisfying a finite number of compatibility conditions. We say that $P$ is locally solvable in $C^\infty$ if the equation has a solution $u \in \mathcal{D}'$ for any $f \in C^\infty$ satisfying a finite number of compatibility conditions. Recall that an operator is of principal type if the Hamilton field $H_p$ of the principal symbol $p$ is independent of the Liouville vector field.

Condition $(\Psi)$ means that the imaginary part of the principal symbol does not change sign from $-\to +$ along the oriented bicharacteristics of the real part, see Definition 26.4.6 in [2]. This condition is invariant under multiplication of the principal symbol by non-vanishing factors.

It was conjectured by Nirenberg and Treves [5] that condition $(\Psi)$ was equivalent to local solvability for operators of principal type, and they proved this in several cases. The necessity of $(\Psi)$ for local solvability in the $C^\infty$ category was proved by Moyers in two dimensions and by Hörmander in general, see Corollary 26.4.8 in [2]. In the analytic category, the sufficiency of condition $(\Psi)$ for solvability of microdifferential operators acting on microfunctions was proved by Trépreau [6]. The sufficiency of $(\Psi)$ for local $L^2$ solvability for first order pseudodifferential operators in two dimensions, was proved by Lerner [3].

For differential operators, condition $(\Psi)$ is equivalent to condition $(P)$, which rules out any sign changes of the imaginary part of the principal symbol along the bicharacteristics of the real part. The sufficiency of $(P)$ for local $L^2$ solvability for first order pseudodifferential operators was proved by Nirenberg and Treves [5] in the case when the principal symbol is real analytic, and by Beals and Fefferman [1] in the general case.

2. STATEMENT OF RESULTS

We shall consider the following type of operators, which includes the operators Lerner used in his counter-examples. First, let $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, and

\begin{equation}
P = D_t + i \sum_{\nu \in \mathbb{Z}^n} Q_\nu(t, x_1, D_x) + R(t, x, D_x)
\end{equation}

where $R(t, x, D_x) \in C^\infty(\mathbb{R}, \Psi^0_{1,0}(T^*\mathbb{R}^n))$ and $\sum_\nu Q_\nu(t, x_1, D_x) \in C^\infty(\mathbb{R}, (\Psi^1_{1,0}))$ is on the form

\begin{equation}
Q_\nu(t, x_1, D_x) = \alpha_\nu(t)(D_{x_1} + H(t)\nu^k W(\nu^k x_1))\Psi_\nu(D_x), \quad \nu \in \mathbb{Z}^+.
\end{equation}
Here $0 \leq \alpha_\nu(t) \in C^\infty(\mathbf{R})$ uniformly, such that $0 \notin \text{supp } \alpha_\nu$ and $\alpha_\nu(t)H(t)$ is non-decreasing with $H(t)$ the Heaviside function, $0 \leq W(x_1) \in C^\infty(\mathbf{R})$ and $k > 0$. We also have $0 \leq \Psi_\nu(\xi) \in S^0_{1,0}(T^*\mathbf{R}^n)$ uniformly, having non-overlapping interiors of the supports and $0 < c \leq |\xi|^{2\nu} \leq C \text{ in supp } \Psi_\nu$. Since $0 \notin \text{supp } \alpha_\nu$ we may write $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$, where $\beta_\nu(t) \in C^\infty$ (but not uniformly) such that $0 \leq \beta_\nu(t) \leq 1$ and $0 \leq \partial_t \beta_\nu$. We find that $\sum_\nu \nu^k W(\nu^k x_1)\Psi_\nu(D_x) \in C^\infty(\mathbf{R}, \Psi_{1,0}^k)$, for any $\varepsilon > 0$. Since $0 \leq \alpha_\nu(t)$ and $W(\nu^k x_1)\Psi_\nu(\xi) \geq 0$, it is clear that $P$ satisfies condition $(\Psi^*)$, i.e., the adjoint $P^*$ satisfies condition $(\Psi)$. In what follows, we shall suppress the $t$ dependence and write $S^m$ instead of $C^\infty(\mathbf{R}, S^m)$ for example. We shall use the classical calculus of pseudo-differential operators, but with the general metrics and weights of the Weyl calculus. For notation and calculus results, see chapter 18 in

We define the norms

\begin{equation}
\|u\|_{(s,k)} = \int |\hat{u}(\xi)|^2 (\xi \cdot \overline{\xi})^{2s}(\log(\xi) + 1)^{2k} \, d\xi \quad s, \, k \in \mathbf{R},
\end{equation}

where $\langle \xi \rangle^2 = 1 + |\xi|^2$. Then $\|u\|_{(s,0)} \approx \|u\|_{(s)}$, the usual Sobolev norm, and $\forall s, \, k \in \mathbf{R}$ we have

\begin{equation}
c_{k,\varepsilon} \|u\|_{(s-k)} \leq \|u\|_{(s,k)} \leq C_{k,\varepsilon} \|u\|_{(s+\varepsilon)} \quad \forall \varepsilon > 0.
\end{equation}

We find that $\|u\|_{(s,k)}$ is equivalent to $\sum_\nu |\Psi_\nu(\xi)|^2(\log(\xi) + 1)^{2k}\|\psi_\nu(D_x)u\|^2$ if $\{ \psi_\nu(\xi) \}$ is a partition of unity: $\sum_\nu |\psi_\nu|^2 = 1$ such that $\langle \xi \rangle \approx \langle \xi \rangle$ only varies with a fixed factor in $\text{supp } \Psi_\nu$.

**Theorem 2.1.** Let $P$ be given by (2.1). Then, for any $s \in \mathbf{R}$ there exists positive $T_s$ and $C_s$ such that

\begin{equation}
\int \|u\|^2_{(s)}(t) \, dt \leq C_s T^2 \int \|Pu\|^2_{(s,2k)}(t) \, dt
\end{equation}

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_s$.

Thus, we obtain for any $s \in \mathbf{R}$ that

\begin{equation}
\int \|u\|^2_{(s)}(t) \, dt \leq C_{s,\varepsilon} T^2 \int \|Pu\|^2_{(s,2k)} \, ds \quad \forall \varepsilon > 0
\end{equation}

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_s$. This shows that $P^*$ is locally solvable in $C^\infty$, with loss of $\varepsilon$ derivatives, $\forall \varepsilon > 0$.

We shall also consider the following operators, which includes the operators Lerner used in his counter-example with homogeneous symbols. Let

\begin{equation}
P = D_t + i \sum_{\nu \in J} Q_\nu(t, x, D_x) + R(t, x, D_x)
\end{equation}

where $J$ is a subset of $\mathbf{Z}_+$ and $\sum_\nu Q_\nu(t, x, D_x) \in \Psi^1_{1,0}$ is given by

\begin{equation}
Q_\nu(t, x, D_x) = \alpha_\nu(t)C(D_x)\chi_\nu(x_2)(D_{x_1} + H(t)\nu^k W(\nu^k x_1)2^{-\nu}D_{x_2}) \quad \nu \in J.
\end{equation}

Here we have the same conditions on $\alpha_\nu$, $W$ and $R$ as before. Also, $0 \leq C(\xi)$ is homogeneous, supported where $|\xi| \leq C \xi_2$ and $0 \leq \chi_\nu(x_2) \in S(1,dx_2^2)$ uniformly with non-overlapping supports. In fact, there exists a function $\mu(\nu)$ on $\mathbf{Z}_+$ such that $\mu(\nu) \leq C_{N\nu}^N$, for some $N > 0$, and there exists $\chi_\nu \in S(1, \mu^2(\nu)dx_2^2)$ uniformly, with disjoint supports such that $0 \leq \chi_\nu(x_2) \leq 1$ and $\chi_\nu = 1$ on supp $\chi$. As before, we find that $P$ satisfies condition $(\Psi^*)$. 

\[X-2\]
Theorem 2.2. Let $P$ be given in (2.7). Then, for every $s \in \mathbb{R}$ we find $T_s > 0$ and $C_s > 0$ such that

$$
\int \|u\|_2^2(t) dt \leq C_s T^2 \int \|Pu\|_{(s+1)}^2(t) dt \quad \forall s
$$

if $u \in S$ has support where $|t| \leq T \leq T_s$.

Thus $P^*$ is locally solvable in $C^\infty$, with loss of one derivative. The theorems are going to be proved in the next sections.

3. Proof of Theorem 2.1

Clearly, by conjugating with $(D_x)^s$ we may assume that $s = 0$, which only changes $R(t, x, D_x) \in \Psi_{1,0}^0$ (dependingly on $s$). Next, we shall eliminate $R(t, x, D_x)$. We choose $E_\pm(t, x, D_x) \in \Psi_{1,0}^0$ with principal symbols

$$
eq_\pm(t, x, \xi) = \exp(\pm \int_0^t i R(t, x, x) dt),$$

such that $E_- E_+ \cong E_+ E_- \cong \text{Id}$ modulo $\Psi^{-\infty}$. Then by conjugating with $E_\pm$ we obtain $R \in \Psi_{1,0}^{-1}$, but this changes $Q_\nu$ into

$$Q_\nu(t, x, D_x) = \alpha_\nu(t) \left((D_{x_1} + R(t, x, x))\Psi_\nu(D_x) + g_\nu(t, x, D_x)\right)$$

where $\{ g_\nu(t, x, \xi) \}_{\nu} \in S_{1,0}^0$. Since we may skip terms in $\Psi^{-1}$ in $P$ in the estimate (2.5), we may assume that $\text{supp} \ g_\nu \subseteq \text{supp} \Psi_\nu$.

We shall localize in $S_{1/2,0}^1$ in order to separate the different $Q_\nu$ terms. Let $\{ \phi_j(\xi) \}_{j} \in S_{1/2,0}^1$ be a partition of unity such that $\phi_j$ is supported where $|\xi - \xi_j| \leq c(\xi_j)^{1/2}$, and $\text{supp} \phi_j$ is connected, $\forall j$. Let $J \subset \mathbb{Z}_+$ be the set of those $j$ for which $\text{supp} \phi_j$ intersects $\bigcap \text{supp} \Psi_\nu$. Since the principal symbol of $\sum_\nu Q_\nu \in \Psi_{1,0}^1$ vanishes of infinite order somewhere in $\text{supp} \phi_j$ when $j \in J$, and $\phi_j(\xi) \in S_{1/2,0}^0$, we find that

$$\phi_j(D_x)P u = \phi_j(D_x)D_t u + R_j(t, x, D_x) u$$

with $\{ R_j \}_{j \in J} \in \Psi_{1,0}^0$ (with values in $\ell^2$). We have

$$\int \|\phi_j(D_x)u\|^2(t) dt \leq C k^2 \int \|D_t \phi_j(D_x)u\|^2(t) dt$$

$$\leq C k^2 \int \|\phi_j(D_x)Pu\|^2(t) + \|R_j u\|^2(t) dt$$

for $j \in J$. Since $\sum_{j \in J} \|R_j u\|^2 \leq C \|u\|^2$, we get the result for small enough $T$, providing that we also have an estimate for the other terms.

Thus we only have to consider the case when $\text{supp} \phi_j$ does not intersect $\bigcap \text{supp} \Psi_\nu$, i.e. $j \notin J$. Since $\text{supp} \phi_j$ is connected, we find that $\text{supp} \phi_j$ is contained in the interior of $\text{supp} \Psi_\nu$ for some unique $\nu = \nu_j$ when $j \notin J$. Observe that this gives $|\xi_j| \approx 2^\nu$ in $\text{supp} \phi_j$.

Clearly, since $\text{supp} Q_\nu \subseteq \text{supp} \Psi_\nu$ we have $P \phi_j(D_x) u = P_\nu \phi_j(D_x) u$ where we define

$$P_\nu = D_t + iQ_\nu(t, x_1, D_x).$$

Now we use the following

Lemma 3.1. Let $P_\nu$ be given by (3.5). Then we find

$$\int \|u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1) dt \leq C T^2 k^4 \int \|P_\nu u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1)^{-1} dt$$

uniformly in $\nu$, if $u \in S$ has support in $|t| \leq T$, for $T$ small enough.
By substituting $\phi_j(D_x)u$, taking $\nu = \nu_j$ in (3.6), and replacing $P_{\nu_j}$ by $P$, we obtain for $j \notin J$ that

\[ (3.7) \quad \int \|\phi_j(D_x)u\|^2(t) \, dt \leq C T^2 \nu_j^{4k} \int \|P\phi_j(D_x)u\|^2(t) \, dt \]

\[ \leq C T^2 \nu_j^{4k} \int \|\phi_j(D_x)Pu\|^2(t) + \|P, \phi_j(D_x)u\|^2(t) \, dt. \]

Now $\{ \nu_j^2(P, \phi_j(D_x)) \}_{j \notin J} \in \Psi^{r-1/2}_{1/2,0}$ with values in $\ell^2$, $\forall \epsilon > 0$. In fact, we find that $\sum \nu_j^2 W(\nu_j x_1) \Psi_j(D_x) \in C^\infty(\mathbb{R}, \Psi^{s}_{\epsilon,0})$ and $\{ \nu_j^2 \phi_j(\xi) \}_{j \notin J} \in S^{1/2,0}_{1/2,0}$, $\forall \epsilon > 0$, since $\phi_j(\xi)$ is supported where $|\xi| \approx 2^j$ when $j \notin J$. Thus by summing up (3.4) and (3.7) we obtain (2.5) for $s = 0$ and small enough $T$. This completes the proof of Theorem 2.1.

Proof. [Proof of Lemma 3.1] We may assume $\nu$ is fixed in what follows. In the proof, we are going to localize in $|\xi| \geq \nu^{2k}$. For that purpose we use the metric

\[ (3.8) \quad g_{\nu} = \nu^{2k} |dx|^2 + |d\xi|^2 / (\nu^{4k} + \xi_i^2) \quad \nu \in \mathbb{Z}_+ \]

which is uniformly slowly varying, $\sigma$ temperate and

\[ (3.9) \quad g_{\nu}/g_{\nu}^\sigma = h_{\nu}^2 = \nu^{2k} / (\nu^{4k} + \xi_i^2) \]

which makes $h_{\nu}^{-2} = |\xi_i|^2 \nu^{-2k} + \nu^{2k} \geq 2|\xi_i|$. We find that $Q_{\nu} \in \text{Op}(h_{\nu}^{-2}, g_{\nu})$ but $\nu^k W(\nu^k x_1) \in \text{Op}(h_{\nu}^{-1}, g_{\nu})$ uniformly.

Now we localize with $\chi_0(\xi) = \chi(\xi_0 \nu^{-2k}) \in S(1, g_{\nu})$ where $\chi \in C^\infty_0$ is equal to 1 near 0, and with $\chi_{\pm}(\xi) = H(\xi_0(\xi_1 - \nu^{-2k})) \in S(1, g_{\nu})$ which has support where $\pm \xi_1 > c_0 \nu^{2k}$ so that $\chi_0 + \chi_+ + \chi_- \equiv 1$. We also choose non-negative $\tilde{\chi}_\pm(\xi)$ and $\tilde{\chi}_0(\xi) \in S(1, g_{\nu})$ such $\chi_\pm \chi_\pm = \chi_\pm$ and $\tilde{\chi}_0 \chi_0 = \chi_0$. This can be done so that $\tilde{\chi}_\pm$ has support where $\pm \xi_1 > c_0 \nu^{2k}$, $c_0 > 0$, and $\tilde{\chi}_0$ has support where $|\xi_1| \leq C \nu^{2k}$.

First we estimate the $\pm(\chi_i(D_x)u)$ terms by Lemma 5.1 with the operator

\[ (3.10) \quad P_{\pm} = D_t + Q_{\nu} \tilde{\chi}_\pm(D_x), \]

where

\[ (3.11) \quad \pm \text{Re} Q_{\nu} \tilde{\chi}_\pm(D_x) \geq \mp C \quad \text{on } u \in S, \]

by the Fefferman–Phong inequality, where $\text{Re} F = (F + F^*)/2$. In fact, the symbol of

\[ (3.12) \quad \pm \alpha_\nu(t) \text{Re} \left( D_{x_1} + H(t) \nu^k W(\nu^k x_1) \right) \Psi_{\nu}(D_x) \tilde{\chi}_\pm(D_x) \]

is bounded from below, modulo terms in $S(1, g_{\nu})$. Thus Lemma 5.1 gives (after changing $t$ to $-t$ for $P_-$)

\[ (3.13) \quad \int ||u||^2(t) \, dt \leq C T^2 \int ||P_+ u||^2(t) \, dt \]

if $u \in S$ is supported where $|t| \leq T$ and $T$ is small enough. Now, by substituting $\chi_\pm(D_x)u$ into (3.13) and using that $P_\pm \chi_\pm(D_x) = P_\nu \chi_\pm(D_x)$ and that $[P_\nu, \chi_\pm(D_x)] \in \text{Op} S(1, g_{\nu})$ is uniformly $L^2$ bounded, we find

\[ (3.14) \quad \int \|\chi_\pm(D_x)u\|^2(t) \, dt \leq C T^2 \int \|P_\nu u\|^2(t) + \|u\|^2(t) \, dt \]

if $u \in S$ is supported where $|t| \leq T$ and $T$ is small enough.

Next, we shall estimate $||\chi_0(D_x)u||^2$. Let

\[ (3.15) \quad B_\nu = D_{x_1} \Psi_{\nu}(D_x)\tilde{\chi}_0(D_x) + \beta_\nu(t) \left( \nu^k W(\nu^k x_1) \Psi_{\nu}(D_x)\tilde{\chi}_0(D_x) + \rho \right) \in \text{Op}(h_{\nu}^{-1}, g_{\nu}), \]
where \( q > 0 \). Here \( \beta_\nu \in C^\infty \) such that \( 0 \leq \beta_\nu(t) \leq 1 \), \( 0 \leq \partial_t \beta_\nu \) and \( \alpha_\nu(t) H(t) \equiv \alpha_\nu(t) \beta_\nu(t) \). Since \( \nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) \in \text{Op} \ S(h^{-1}_\nu, g_\nu) \) has positive principal symbol, we find

\[
\partial_t B_\nu = \partial_t \beta_\nu(t) \left( \nu^k W(\nu^k x_1) \Psi_\nu(D_x) \tilde{\chi}_0(D_{x_1}) + q \right) \geq 0
\]

for large enough \( q \). We also find \( B_\nu \in \text{Op} \ S(\nu^{2k}, g_\nu) \) uniformly, thus \( \|B_\nu\| \leq C \nu^{2k} \). Applying Lemma 5.2 on \( \chi_0(D_{x_1}) u \), with \( P_0 = D_t + \alpha_\nu(t)(B_\nu + r_\nu) \), \( r_\nu = g_\nu(t, x, D_x) \tilde{\chi}_0(D_{x_1}) - \beta_\nu(t) q \) and \( M = C \nu^{2k} \), we find

\[
\int \|\chi_0(D_{x_1}) u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1) dt \leq C_1 \nu^{4k} T^2 \int \|P_0 \chi_0(D_{x_1}) u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1)^{-1} dt
\]

if \( u \in \mathcal{S} \) is supported where \(|t| \leq T \) and \( T \) is small enough. As before, we find \( P_0 \chi_0(D_{x_1}) = P_\nu \chi_0(D_{x_1}) \) and we have \( [P_\nu, \chi_0(D_{x_1})] = \alpha_\nu(t) f_\nu \), where \( f_\nu \in \text{Op} \ S(1, g_\nu) \) is uniformly \( L^2 \) bounded. Since

\[
\nu^{4k} \alpha_\nu^2(t)/(\nu^{2k} \alpha_\nu(t) + 1) \leq \nu^{2k} \alpha_\nu(t) + 1,
\]

we obtain

\[
\int \|\chi_0(D_{x_1}) u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1) dt \leq C_t T^2 \left( \int \nu^{4k} \|P_\nu u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1)^{-1} dt + \int \|u\|^2(t)(\nu^{2k} \alpha_\nu(t) + 1) dt \right)
\]

if \( u \) is supported where \(|t| \leq T \) and \( T \) is small enough. Combining (3.14) and (3.19), we obtain (3.6) for small enough \( T \). \( \square \)

4. PROOF OF THEOREM 2.2

First, we conjugate with \((D_x)^{s+1/2}\) to reduce to the case \( s = -1/2 \) (this only changes \( R(t, x, D_x) \) dependingly on \( s \)). We choose \( E_\pm(t, x, D_x) \in \Psi^0_{1,0} \) with principal symbols

\[
e_\pm(t, x, \xi) = \exp(\pm \int_0^t iR(t, x, \xi) dt),
\]

such that \( E_- E_+ \cong E_+ E_- \cong \text{Id} \) modulo \( \Psi^{-\infty} \). As before, the calculus gives \( R \in \Psi^{-\frac{1}{2}}_{1,0} \) for the new operator, but changes \( Q_\nu \) into

\[
Q_\nu(t, x, D_x) = \alpha_\nu(t) \left( C(D_x) \chi_\nu(x_2)(D_{x_1} + H(t) \nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2}) + g_\nu(t, x, D_x) \right)
\]

where \( g_\nu(t, x, \xi) \in S^0_{1,0} \) uniformly, with \( \text{supp} g_\nu \subseteq \text{supp} \chi_\nu \). Thus, we may assume \( R \equiv 0 \) since the term \( CT \|Ru\|_{1/2} \) can be estimated by the left hand side of (2.9) for \( s = -1/2 \) and small enough \( T \).

Next, we localize in \( x_2 \) to separate the different \( Q_\nu \) terms. By assumption there exists \( \chi_\nu(x_2) \in S(1, \mu^2(\nu) dx^2_2) \) uniformly when \( \nu \in J \), with disjoint supports, such that \( 0 \leq \chi_\nu(x_2) \leq 1 \) and \( \chi_\nu \chi_\nu = \chi_\nu \). We also localize in \( \xi \); let \( \{ \psi_j(\xi) \}_{j} \) and \( \{ \phi_j(\xi) \}_{j} \in S^0_{1,0} \) (with values in \( L^2 \)) such that \( \sum_j \psi_j(\xi)^2 = 1 \), \( \phi_j(\xi) \) and \( \psi_j(\xi) \) are non-negative, \( \phi_j \psi_j = \psi_j \), \( \phi_j \), and \( \psi_j \) are supported where \( 0 < c \leq |\xi|^{2^{-\nu}} \leq C \). We may also assume that for some fixed \( N > 0 \) we have \( \sum_{|j-k| \leq N} \psi_k^2(\xi) \equiv 1 \) on \( \text{supp} \psi_j, \forall j \).
Since $\bar{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2)$ we find that \{ $\psi_j(\xi)\bar{\chi}_\nu(x_2)$ \}_{\nu,j}$ is not in a good symbol class. Therefore, we put
\begin{equation}
\bar{\chi}_{0j}(x_2) = 1 - \sum_{0 < \nu \leq j^2} \bar{\chi}_\nu(x_2).
\end{equation}
Since $\psi_j$ is supported where $|\xi| \approx 2^j$ and $\mu(\nu) \leq C_N \nu^N$ for some $N > 0$, it is easy to see that \{ $\bar{\chi}_\nu(x_2)\psi_j(\xi)$ \}_{j \nu \leq j^2}$ and \{ $\bar{\chi}_{0j}(x_2)\psi_j(\xi)$ \}_{j \psi_0 \leq j^2}$ are in $\Psi^0_{0,\varepsilon}$, $\forall \varepsilon > 0$. Let
\begin{equation}
\alpha_{\nu j}(t) = \sqrt{\alpha_\nu(t) + 2^{-j}} \quad \forall j \in J, \ \forall \nu,
\end{equation}
in what follows. Now, we are going to use the following
\begin{lem}
We find that
\begin{equation}
\int \sum_{J \nu \leq j^2} \| \alpha_{\nu j}(t)\bar{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t) + \sum_j \| \bar{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) dt
\leq CT \int \sum_{J \nu \leq j^2} \| \alpha_{\nu j}^{-1}(t)\bar{\chi}_\nu(x_2)\psi_j(D_x)Pu\|^2(t)
+ \sum_j \| \bar{\chi}_{0j}(x_2)\psi_j(D_x)Pu\|^2(t) + \|u\|_{-1/2}(t) dt.
\end{equation}
\end{lem}
\begin{proof}
[Proof of Lemma 4.1] Since $\psi_j(1 - \phi_j) \equiv 0 \ \forall j$, the calculus gives that we may replace $P$ by $P_j = D_t + i \sum_{\nu \in J} Q_\nu \phi_j(D_x)$ for the terms containing the factor $\psi_j(D_x)$ in (4.5).

For the terms $\| \bar{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2$ we use the fact that $\nu^j W(\nu^k x_1)2^{-j} B_{x_1} \phi_j(D_x) \in \Psi^{-\infty}$ uniformly when $(\log |\xi|)^2 \approx j^2 < \nu$. Thus we use Nirenberg-Treves estimate in [2, Theorem 26.8.1] with $B = D_{x_1} \phi_j(D_x)$ bounded, and $0 \leq A \in \Psi^0_{1,0}$ such that
\begin{equation}
A \simeq \sum_{J \nu \geq j^2} \alpha_\nu(t) C(D_x)\chi_\nu(x_2) \mod \Psi^{-1}_{1,0}.
\end{equation}
By perturbing this estimate with $L^2$ bounded operators, and substituting the term $\bar{\chi}_{0j}(x_2)\psi_j(D_x)u$, we find for small enough $T$ that
\begin{equation}
\int \| \bar{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t) dt \leq CT \int \| \bar{\chi}_{0j}(x_2)\psi_j(D_x)D_{x_1}^\nu(t, x, D_x) \phi_j(D_x) \|^2(t) dt
\end{equation}
when $|t| \leq T$ in supp $u$. Here
\begin{equation}
\bar{P}_{\nu j} = D_t + i \sum_{J \nu \geq j^2} \alpha_\nu(t)(C(D_x)\chi_\nu(x_2)D_{x_1}^\nu(t, x, D_x)) \phi_j(D_x)
\end{equation}
modulo $\Psi^{-\infty}$.

Thus $\bar{P}_{\nu j}$ satisfies condition $(P)$, i.e., the imaginary part of the principal symbol has no sign changes for fixed $(x, \xi)$. 

X-6
Since $\alpha_\nu \leq C\alpha_\nu$ and $\text{supp} \varrho_\nu \subseteq \text{supp} \chi_\nu$, the calculus gives that

$$\{[\tilde{P}_j, \tilde{\chi}_{0j}(x_2)\psi_j(Dx)]\}_j \approx \left\{ \sum_{\nu \leq j^2} \alpha_\nu(t) \tilde{f}_{\nu j}(t, Dx) \right\}_j \mod \Psi^{-1/2}$$

where $\{f_{\nu j}\}_{\nu j} \in \Psi^0_{1,0}$ with values in $\ell^2$, and $\text{supp} f_{\nu j} \subseteq \text{supp} \chi_\nu \psi_j$. In order to estimate these terms we need the following

**Lemma 4.2.** If $\{f_{\nu j}(x, D_x)\}_{\nu j} \in \Psi^0_{1,0}$ with values in $\ell^2$, and $\text{supp} f_{\nu j} \subseteq \text{supp} \chi_\nu \psi_j, \forall \nu j$, then

$$\sum_{\nu \leq j^2} \|\alpha_\nu(t) f_{\nu j}(x, D_x) u\|^2 \leq C \left( \sum_{\nu \leq j^2} \|\alpha_\nu(t) \tilde{\chi}_{0j}(x_2) \psi_j(Dx) u\|^2 + \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(Dx) u\| + \|u\|_{-1/2}^2 \right)$$

for $u \in \mathcal{S}$.

Since $\tilde{\chi}_{0j} \equiv 0$ on $\text{supp} \chi_\nu$ when $J \geq \nu \leq j^2$, we find that $\{\tilde{\chi}_{0j}(x_2) \psi_j(Dx) (\tilde{P}_j - P_j)\}_j \in \Psi^{-\infty}$, where as before $P_j = D_t + i \sum_{\nu \in J} Q_\nu \phi_j(D_x) \in \Psi^1_{1,0}$. Thus we find

$$\int \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(Dx) \tilde{P}_j u\|^2(t) \, dt \leq CT \int \sum_j \|\tilde{\chi}_{0j}(x_2) \psi_j(Dx) P_j u\|^2(t) + \|u\|_{-1/2}^2(t) \, dt.$$}

This gives the estimate (4.5) for the terms $\|\tilde{\chi}_{0j}(x_2) \psi_j(Dx) u\|^2$ for small $T$, providing we can estimate the other terms.

As before, we are going to use Lemma 5.2 with $a(t) = \alpha_\nu(t)$ and

$$B_t = \text{Re} C(D_x) \chi_\nu(x_2) \left( D_x, \phi_j(D_x) + \beta_\nu(t) \left( \nu^k W(\nu^k x_1) 2^{-\nu} D_{x_2} \phi_j(D_x) + \varrho \right) \right),$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \leq \beta_\nu(t) \leq 1$, $0 \leq \partial_t \beta_\nu$ and $\alpha_\nu(t) H(t) \equiv \alpha_\nu(t) \beta_\nu(t)$. We have $\|B_t\| \leq C2^j$, $\partial_t B_t \geq 0$ for large $\varrho$ and $R_t \in \Psi^0$. By substituting $\tilde{\chi}_\nu(x_2) \psi_j(D_x) u$ in this Lemma, we find for small $T$ that

$$\int \|\tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) (2^j \alpha_\nu(t) + 1) \, dt \leq CT^2 2^j \int \|(D_t + iQ_\nu \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) (2^j \alpha_\nu(t) + 1)^{-1} \, dt$$

when $J \geq \nu \leq j^2$, providing $|t| \leq T$ in $\text{supp} u$. This is equivalent to

$$\int \|\alpha_\nu(t) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) \, dt \leq CT^2 \int \|\alpha_\nu^{-1}(t) (D_t + iQ_\nu \phi_j(D_x)) \tilde{\chi}_\nu(x_2) \psi_j(D_x) u\|^2(t) \, dt.$$}

Now, it follows from the asymptotic expansion that

$$\{[Q_\nu \phi_j(D_x), \tilde{\chi}_\nu(x_2) \psi_j(D_x)]\}_{j \in \mathbb{Z}^2} \cong \left\{ \alpha_\nu(t) \tilde{f}_{\nu j}(t, x, D_x) \right\}_{j \in \mathbb{Z}^2}$$

X-7
modulo $\Psi^{-1/2}$, where $\{f_{\nu j}(t, x, D_x)\}_{\nu j} \in \Psi^0_{1,0}$ with values in $\ell^2$, supp $f_{\nu j} \subseteq \text{supp } \chi_\nu \psi_j$, $\forall t$. Thus, we may estimate the commutator terms by Lemma 4.2:

$$\sum_{J \ni \nu \leq j^2} \|\alpha_{\nu j}(t) f_{\nu j}(t, x, D_x) u\|^2 \leq C \left( \sum_{\nu \leq j^2} \|\alpha_{\nu j} \tilde{\chi}_j \psi_j u\|^2 + \sum_j \|\tilde{\chi}_0 \psi_j u\|^2 + \|u\|_{-1/2}^2 \right) \forall t.$$  

Since the supports of $\tilde{\chi}_\nu$ are disjoint, and $\sum_{J \ni \nu \neq \nu} Q_\mu \phi_j(D_x) \in \Psi^1_{1,0}$ uniformly, we obtain that

$$\left\{ \tilde{\chi}_\nu(x_2)\psi_j(D_x) \sum_{J \ni \nu \neq \nu} Q_\mu \phi_j(D_x) \right\}_{J \ni \nu \leq j^2} \in \Psi^{-\infty}$$

with values in $\ell^2$. Thus we may replace $D_t + iQ_\nu \phi_j(D_x)$ by $P_j$ in the estimate, which proves (4.5). $lacksquare$

Proof. [Proof of Lemma 4.2] Since $\sum_{|j-k| \leq N} \psi_k^2(\xi) \equiv 1$ on supp $f_{\nu j}$ and $\{f_{\nu j}\}_{\nu j} \in S^0_{1,0}$, we may use the calculus to write

$$\sum_{\nu j} \|\alpha_{\nu j}(t) f_{\nu j}(x, D_x) u\|^2 \leq \sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u\|^2 + C \|u\|_{-1/2}^2,$$

where $\{e_{\nu j k}\}_{\nu j k} \in \Psi^0_{1,0}$ with values in $\ell^2$, and supp $e_{\nu j k} \subseteq \text{supp } f_{\nu j} \psi_k$. Since $\tilde{\chi}_{0k} + \sum_{\mu \leq k} \tilde{\chi}_\mu \equiv 1$, we find

$$\sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \psi_k(D_x) u\|^2 \leq 2 \sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2$$

$$+ 2 \sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u\|^2.$$

By summing up in $j$ and $\nu$ we find

$$\sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2$$

$$\leq C_N (\sum_k \|\tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 + \|u\|_{-1/2}^2),$$

since $\alpha_{\nu j} \leq c$ and $\{e_{\nu j k}\}_{\nu j k} \in \Psi^0_{1,0}$ with values in $\ell^2$, uniformly in $k$. Now $\alpha_{\nu j} \leq C \alpha_{\nu k}$ when $|j - k| \leq N$ which similarly gives by the calculus

$$\sum_{\nu j} \|\alpha_{\nu j}(t) e_{\nu j k}(x, D_x) \sum_{\mu \leq k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u\|^2$$

$$\leq C \sum_{\mu \leq k^2} \|\alpha_{\mu k}(t) \tilde{\chi}_\mu(x_2) \psi_k(D_x) u\|^2 + C \|u\|_{-1/2}^2$$

since supp $e_{\nu j k} \subseteq \text{supp } \chi_\mu \forall j, k$. $lacksquare$
5. SOME ESTIMATE LEMMAS

We assume that
\[ P = D_t + iQ_t + R_t \]
where \( Q_t \) is a closed, densely defined operator on \( L^2(\mathbb{R}^n) \) such that \( \mathcal{S} \subset D(Q_t) \cap D(Q_t^*) \) \( \forall t \), \( t \mapsto (Q_t u, u) \) is continuous for \( u \in \mathcal{S} \), and
\[ \text{Re} Q_t \geq -C_1 \quad \text{on } \mathcal{S} \quad \forall t, \]
where \( 2 \text{Re} Q_t = Q_t + Q_t^* \). We also assume that \( \|R_t\| \leq C_0 \) on \( L^2(\mathbb{R}^n) \). Let \( \|u\| \) be the \( L^2 \) norm of \( u \in L^2(\mathbb{R}^n) \) and \( \langle u, v \rangle \) the corresponding sesquilinear form.

**Lemma 5.1.** There exists \( T_0 > 0 \) and \( C > 0 \) such that
\[ \int \|u(t)\|^2 dt \leq C T^2 \int \|P u(t)\|^2 dt \]
if \( u \in \mathcal{S} \) has support where \( |t| \leq T \leq T_0 \). Here \( T_0 \) and \( C \) only depend on \( C_0 \) and \( C_1 \).

**Proof.** We only need to prove the estimate (5.1) for \( R_t \equiv 0 \), since we may perturb it with \( L^2 \) bounded terms for small \( T \). We find
\[ \langle Q_t u, u \rangle \geq -C_1 \|u\|^2 \quad \forall t \]
when \( u \in \mathcal{S} \). Since \( iP = \partial_t - Q_t \), this gives
\[ \|u(t)\|^2 = -\int_t^T 2 \text{Re} \langle \partial_t u, u \rangle(t) dt \]
when \( u \in \mathcal{S} \), and \( u \equiv 0 \) when \( t \geq T \).

By integrating in \( t \) we find
\[ \int_T^\infty \|u(t)\|^2 dt \leq 4 T \int_T^\infty \text{Re} \langle Pu, u \rangle dt + 4 C_1 T \int_T^\infty \|u(t)\|^2 dt \]
By using the Cauchy–Schwarz inequality we obtain
\[ 2 \langle Pu, u \rangle \leq \lambda \|u\|^2 / T + \|Pu\|^2 T / \lambda \quad \forall \lambda > 0. \]
This gives
\[ (1 - 4 C T - 2 \lambda) \int \|u\|^2 \leq 2 T^2 / \lambda \int \|Pu\|^2 dt, \]
which gives (5.3) when \( T_0 \leq 1 / 16 C \) and \( \lambda \leq 1 / 4 \). ■

The next case we shall consider is
\[ P = D_t + ia(t)(B_t + R_t) \]
where \( 0 \leq a(t) \leq C_0 \), \( B_t \) and \( \partial_t B_t \) are self-adjoint and bounded, \( \partial_t B_t \geq 0 \) and \( \|R_t\| \leq C_1 \) on \( L^2(\mathbb{R}^n) \). We also assume that there exists a constant \( M > 0 \) such that
\[ \|B_t\| \leq M \quad \forall t \]
\[ |||B_s, B_t||| \leq M \quad \forall s, t. \]
Lemma 5.2. There exists $T_0 > 0$ and $C > 0$ such that
\begin{equation}
(5.12) \quad \int \|u\|^2(t)(a(t) + M^{-1}) \, dt \leq C T^2 \int \|P_0u\|^2(t)(a(t) + M^{-1})^{-1} \, dt
\end{equation}
if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_0$. Here $C_0$ and $T_0$ are independent of $M$, and only depend on $C_0$ and $C_1$.

Proof. First we consider the case $a(t) \geq M^{-1} > 0$. Then (5.12) is equivalent to the estimate:
\begin{equation}
(5.13) \quad \int \|u\|^2(t) a(t) \, dt \leq C T^2 \int \|P_0u\|^2(t) \, dt / a(t)
\end{equation}
if $u \in \mathcal{S}$ has support where $|t| \leq T$ is small enough. Introducing $s = \int_0^t a(t) \, dt$ as a new time variable and $P_0 = D_s + iB_t$, we find that it suffices to prove
\begin{equation}
(5.14) \quad \int \|u\|^2(s) \, ds \leq C T^2 \int \|P_0u\|^2(s) \, ds
\end{equation}
if $u \in \mathcal{S}$ has support where $|s| \leq CT$. In fact, we may then perturb the estimate with the $L^2$ bounded term $iR_t u$ for small $T$.

Now $[P_0^*, P_0] = 2\partial_s B_t \geq 0$, which implies
\begin{equation}
(5.15) \quad \|P_0 u\|^2 - \|P_0^* u\|^2 = \langle [P_0^*, P_0] u, u \rangle \geq 0.
\end{equation}
Since $\|D_s u\|^2 \leq 2(\|P_0 u\|^2 + \|P_0^* u\|^2)$, we find
\begin{equation}
(5.16) \quad \int \|u\|^2(s) \, ds \leq C_0 T^2 \int \|D_s u\|^2(s) \, ds \leq 4 C T^2 \int \|P_0 u\|^2(s) \, ds
\end{equation}
if $u \in \mathcal{S}$ has support where $|s| \leq CT$. This proves (5.13) in the case $a(t) \geq M^{-1}$.

Next we consider the case $a(t) \geq 0$. In order to reduce to the case $a \geq M^{-1}$ we conjugate with $E_t$ solving
\begin{equation}
(5.17) \quad \begin{cases}
\partial_t E_t = -E_t B_t / M \\
E_0 = \text{Id}.
\end{cases}
\end{equation}
This gives bounds on $\|E_t\|$ and $\|E_t^{-1}\|$ when $t$ is bounded (independently of $M$), and the conjugation transforms $P$ into
\begin{equation}
(5.18) \quad \tilde{P} = D_t + i(a(t) + M^{-1})B_t + a(t) \tilde{R}_t = D_t + i(a(t) + M^{-1})(B_t + S_t)
\end{equation}
where $\tilde{R}_t = i E_t^{-1}[B_t + R_t, E_t] + i R_t$ and $S_t = a(t) \tilde{R}_t / (a(t) + M^{-1})$ are uniformly bounded on $L^2(\mathbb{R}^n)$ for bounded $t$. In fact, if $F_r = [B_t, E_r]$, $\forall r$, then
\begin{equation}
(5.19) \quad \partial_t F_r = E_r [B_r, B_t] / M - F_r B_r / M
\end{equation}
and $F_0 \equiv 0$, thus $F_t = [B_t, E_t]$ is bounded on $L^2(\mathbb{R}^n)$ for bounded $t$ (independently of $M$). By using (5.13) with $\tilde{P}$ and $a(t) + M^{-1}$, we obtain (5.12).
REFERENCES


Department of Mathematics, University of Lund, Box 118, S-221 00 Lund, Sweden
E-mail address: dencker@maths.lth.se