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RECENT EXISTENCE AND REGULARITY RESULTS FOR
WAVE MAPS

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The setting. We consider maps $u$ from $(m+1)$-dimensional Minkowski space to a compact, $k$-dimensional Riemannian manifold $(N, g)$ with $\partial N = \emptyset$, the “target”. By Nash’s embedding theorem, we may assume that $N \subset \mathbb{R}^n$, isometrically, for some $n > k$. We denote as $T_pN \subset T_p\mathbb{R}^n \cong \mathbb{R}^n$ the tangent space of $N$ at a point $p$, and we denote as $T^\perp_pN$ the orthogonal complement of $T_pN$ with respect to the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$. $TN, T^\perp N$ will denote, respectively, the corresponding tangent and normal bundles.

The space-time coordinates will be denoted as $z = (t,x) = (x^\alpha)_{0 \leq \alpha \leq m}$ and we denote as $\frac{\partial}{\partial x^\alpha} u = \partial_\alpha u = u_{x^\alpha}$ the partial derivative of $u$ with respect to $x^\alpha$, $0 \leq \alpha \leq m$. Also let $D = \left( \frac{\partial}{\partial t}, \nabla \right) = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x^\alpha} \right)_{0 \leq \alpha \leq m}$ and let $\eta$ be the Minkowski metric $\eta = (\eta_{\alpha \beta}) = (\eta^{\alpha \beta})^{-1} = \text{diag}(-1,1,\ldots,1)$. We raise and lower indeces with the metric. By convention, we tacitly sum over repeated indeces. Thus, for example, $\partial^\alpha = \eta^{\alpha \beta} \partial_\beta$. Moreover,

$$\Box = -\partial^\alpha \partial_\alpha = \frac{\partial^2}{\partial t^2} - \Delta$$

is the wave operator and

$$\frac{1}{2} (\partial^\alpha u, \partial_\alpha u) = \frac{1}{2} (|\nabla u|^2 - |u|^2)$$

is the Lagrangean density of $u$.

Wave maps. A map $u$ is a wave map if $u$ is a stationary point for the action integral

$$A(u; Q) = \frac{1}{2} \int_Q \langle \partial^\alpha u, \partial_\alpha u \rangle \, dz$$

with respect to compactly supported variations $u_\varepsilon : \mathbb{R} \times \mathbb{R}^m \to N, |\varepsilon| < \varepsilon_0$, such that $u_\varepsilon = u$ outside a compact set in space-time and for $\varepsilon = 0$, in the sense that

$$\left. \frac{d}{d\varepsilon} A(u_\varepsilon; Q) \right|_{\varepsilon = 0} = 0$$

for any $Q \subset \mathbb{R} \times \mathbb{R}^m$ strictly containing the support of $u_\varepsilon - u$.

Wave maps then satisfy the relation

$$\Box u \perp T_p N.$$
\( p \in N \) for \( p \) near \( p_0 = u(z_0) \). Then we can find scalar functions \( \lambda^i : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \), \( k < l \leq n \), such that near \( z = 0 \) there holds

\[
\Box u = \lambda^i (\nu_k \cdot u);
\]
in fact,

\[
\lambda^i = \langle \Box u, \nu_k \cdot u \rangle \\
= -\delta^a (\partial_\alpha u, \nu_k \cdot u) + \langle \partial_\alpha u, \partial^a (\nu_k \cdot u) \rangle \\
= (\partial_\alpha u, \Box u)_{(\nu_k \cdot u)} = A^i(u)(\partial_\alpha u, \partial^a u)
\]
is given by the second fundamental form \( A^i \) of \( N \) with respect to \( \nu_k \). Thus, the wave map equation takes the form

\[
\Box u = A(u)(\partial_\alpha u, \partial^a u) \perp T_u N, \tag{0.1}
\]
where \( A = A^i \nu_k \) is the second fundamental form of \( N \).

**Examples.** i) For \( N = S^k \subset \mathbb{R}^{k+1} \) equation (0.1) translates into the particularly simple equation

\[
\Box u = (|\nabla u|^2 - |u_t|^2) u.
\]
Indeed, since \( u \perp T_u S^k \) it suffices to check that

\[
\langle \Box u, u \rangle = -\delta^a (\partial_\alpha u, u) + \langle \partial_\alpha u, \partial^a u \rangle = |\nabla u|^2 - |u_t|^2.
\]

ii) Suppose \( \gamma : \mathbb{R} \to N \) is a geodesic parametrized by arc-length and \( u = \gamma \circ v \) for some map \( v : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \). Compute

\[
-\Box u = \partial^a (\gamma'(v) \partial_\alpha v) = \gamma''(v) \partial_\alpha v \partial^a v - \gamma'(v) \Box v.
\]
Note that \( \gamma' \) is parallel along \( \gamma \); that is, \( \gamma''(s) \perp T_{\gamma(s)} N \) for all \( s \in \mathbb{R} \). Thus, \( u \) satisfies (0.1) if and only if \( v \) solves the linear, homogeneous wave equation \( \Box v = 0 \).

**Basic questions.** In view of the hyperbolic nature of equation (0.1), it is natural to ask whether the Cauchy problem for equation (0.1) for (sufficiently) smooth initial data

\[
(u, u_t) \mid_{t=0} = (u_0, u_1) : \mathbb{R}^m \to TN \tag{0.2}
\]
always admits a unique smooth solution for small time \( |t| < T \). That is, we consider data \( u_0 : \mathbb{R}^m \to N, u_1 : \mathbb{R}^m \to \mathbb{R}^n \) such that \( u_1(x) \in T_{u_0(x)} N \) for almost every \( x \in \mathbb{R}^m \).

The smoothness hypothesis on the solution and the data may be rather weak. In fact, for a function \( u \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m ; N) \) it is possible to interpret equation (0.1) in the sense of distributions provided \( Du \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m) \). More generally, we may consider initial data \( (u_0, u_1) \) in Sobolev spaces \( H^s \times H^{s-1}(\mathbb{R}^m ; TN) \), \( s \geq 1 \), and solutions \( u \) of class \( H^s \), that is, such that \( u, u_t \) is \( L^\infty(\mathbb{R} ; H^s \times H^{s-1}(\mathbb{R}^m ; TN)) \).

Then we may ask for which \( s \) the initial value problem (0.1), (0.2) with data \( (u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^m ; TN) \) admits a unique local solution of class \( H^s \) ("local well-posedness in \( H^s \)) and for which \( s \) this solution may be extended for all time and also preserves higher regularity properties of the data ("global well-posedness" and regularity).

A dimensional analysis tells us what we may hope for. Assigning scaling dimensions 1 to each coordinate \( x^\alpha \), 0 to the function \( u \), the \( H^2 \)-energy in \( m \) space dimensions has dimension \( m - 2s \); that is, if \( s > \frac{m}{2} \), no concentration discontinuities...
on length scales $L \to 0$ are possible if the $H^s$-energy of $u$ remains bounded. We refer to this case as sub-critical, in contrast to the critical and supercritical cases $s = \frac{m}{2}, s < \frac{m}{2}$, respectively.

By a fixed point argument, using only classical energy estimates (for $u$ and derivatives), for a general hyperbolic equation $\Box u = f(u, Du)$ with a smooth function $f$ it is not hard to establish local well-posedness of the Cauchy problem in $H^s$, if $s > \frac{m}{2} + 1$.

Using, however, the special geometric, analytic, and algebraic structure properties of the wave map system, this result can be improved drastically.

**Geometric structure.** Orthogonality $\Box u \perp T_u N$ immediately implies the conservation law
$$0 = \langle \Box u, u_t \rangle = \frac{1}{2} \frac{d}{dt} |Du|^2 - \text{div}(\nabla u, u_t).$$
Integrating over $\mathbb{R}^m$, if $Du(t)$ has spatially compact support, we obtain the energy identity
$$E(u(t)) = \frac{1}{2} ||Du(t)||_{L^2}^2 = \text{const.} \quad (0.3)$$

Similarly, we can argue for higher derivatives. Suppose $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; T N)$. Let $\partial$ be any first order spatial derivative. Differentiating equation (0.1), we obtain
$$\Box (\partial u) = \partial [A(u)(\partial \alpha u, \partial^\alpha u)] = dA(u)(\partial u, \partial \alpha u, \partial^\alpha u) + 2A(u)(\partial \partial u, \partial^\alpha u)$$
with data
$$(\partial u, \partial u_t) |_{t=0} = (\partial u_0, \partial u_1) \in H^1 \times L^2(\mathbb{R}^m; \mathbb{R}^n).$$
Note that, since $\langle u_t, A(u)(\partial \alpha u, \partial^\alpha u) \rangle = 0$ by orthogonality, we have
$$\langle \partial u_t, A(u)(\partial \alpha \partial u, \partial^\alpha u) \rangle = -\langle u_t, dA(u)(\partial \alpha \partial u, \partial^\alpha u) \rangle.$$ 

Hence we obtain
$$\frac{d}{dt} E(\partial u(t)) = \int_{\{t\} \times \mathbb{R}^m} \langle \Box (\partial u), \partial u_t \rangle \, dx \leq C ||dA(u)||_{L^\infty} \cdot \int_{\mathbb{R}^m} |Du(t)|^2 |D^2 u(t)| \, dx.$$
Since $N$ is compact, $dA$ is uniformly bounded on $N$. Moreover, by Sobolev's embedding, we can estimate
$$\int_{\mathbb{R}^m} |Du(t)|^2 |D^2 u(t)| \, dx \leq C ||Du(t)||_{L^2}^{4-\alpha} ||D^2 u(t)||_{L^2}^{\alpha},$$
where $\alpha = 2, 3,$ or $4$ if $m = 1, 2,$ or $3,$ respectively.

Thus, by (0.3) we arrive at a Gronwall type inequality
$$\frac{d}{dt} ||D^2 u(t)||_{L^2}^{\alpha} \leq C ||D^2 u(t)||_{L^2}^{\alpha}.$$
A local-in-time $H^2$-bound follows. If $m = 1$, we have $\alpha = 2$, and we even obtain global unique $H^2$-solutions. We summarize these facts in the following result.

**Theorem 0.1.** Suppose $m \leq 3$. Then for any data $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; T N)$ there exists a unique local solution $u$ of class $H^2$. If $m = 1$, the solution extends uniquely for all time. If $(u_0, u_1) \in H^s, s > 2$, then so is $u$. 

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For $m = 1$, the above result is due to Gu [11] and Ginibre-Velo [10]; in [17], Shatah gave a very elegant and concise proof. Finally, Yi Zhou [21] showed that the initial value problem is globally well-posed even in the energy space $H^1$.

For $m = 2, 3$ the above result also was obtained by Klainerman-Machedon [13] by a completely different technique. The above proof was first given in [20]; proof of Theorem 3.3. See also Choquet-Bruhat [2] for early results on wave maps.

**Analytic structure.** As illustrated best by the wave map system for maps to the sphere, equation (0.1) also exhibits the special analytic structure of “null forms” in the sense of Klainerman-Machedon [13].

As a simple model, consider solutions $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ of the equation

$$\Box u = |\nabla u|^2 - |u_t|^2 \text{ on } \mathbb{R} \times \mathbb{R}^m$$

(0.4)

with initial data $u|_{t=0} = 0, u_t|_{t=0} = u_1 \in H^{s-1} (\mathbb{R}^m)$.

Letting $v = e^u$, we compute

$$\Box v = e^u (|\nabla u|^2 + |u_t|^2) = 0$$

with $u|_{t=0} = 1, u_t|_{t=0} = u_1 \in H^{s-1} (\mathbb{R}^m)$.

By exact dependence of the solution $v$ on its data in $H^s \times H^{s-1} (\mathbb{R}^m)$, we have $v \in C^0 (\mathbb{R}; H^s (\mathbb{R}^m))$. On the other hand, a necessary condition for $v$ to arise as $v = e^u$ from a (local) solution $u$ to (0.4) is $v > 0$ (for short time), which requires $H^s (\mathbb{R}^m) \hookrightarrow L^\infty (\mathbb{R}^m)$, that is, $s > \frac{m}{2}$.

In remarkable agreement with this classical example, Klainerman-Machedon [14] establish the following result.

**Theorem 0.2.** The initial value problem (1), (2) is locally well-posed for data $(u_0, u_1) \in H^s \times H^{s-1} (\mathbb{R}^m; TN)$ with $s > \frac{m}{2}$.

This result underscores the importance of the critical case $s = \frac{m}{2}$, in particular, the case $s = 1$ in $m = 2$ space dimensions. Progress on this issue can be made by taking into account a third structure property of the wave map system.

**Algebraic structure.** As an illustration, first consider the case of a homogeneous target space $N = G/H$, where $G$ is a Lie group and $H$ is a discrete subgroup of $G$.

Then there exist Killing vector fields $Y_i$ spanning $T_p N$ at any point $p \in N$ and (0.1) is equivalent to the system of equations

$$0 = \langle \Box u, Y_i \circ u \rangle = -\partial^\alpha (\partial_\alpha u, Y_i \circ u) + \langle \partial_\alpha u, dY_i (u) \cdot \partial^\alpha u \rangle$$

for all $i$. Since $Y_i$ is Killing, the last term vanishes and we obtain the first order Hodge system

$$-\partial^\alpha (\partial_\alpha u, Y_i \circ u) = 0$$

(0.5)

for all $i$, equivalent to (0.1). This form of (0.1) immediately implies the following weak compactness result. Suppose $(u^k)$ is a sequence of wave maps such that $u^k \to u$ in $L^2, Du^k \to Du$ weakly in $L^2$, locally, as $L \to \infty$. Then $u$ again is a (weak) wave map.
Coupling this observation with a suitable scheme for obtaining approximate solutions to (0.1), Shatah [17] (for \( N = S^k \)), Yi Zhou [22] (for \( m = 2 \)), and Freire [7] (for the general case) then obtain the following result.

**Theorem 0.3.** Suppose \( N = G/H \) is homogeneous. Then for any \((u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m;TN)\) there exists a global weak solution \( u \) of (0.1), (0.2) of class \( H^1 \).

In the case of a general target manifold, the algebraic structure giving rise to a Hodge system analogous to (0.5) was uncovered independently by Christodoulou-Tahvildar-Zadeh [3] and Hélein [12]. With no loss of generality (as shown by these authors) we may assume that \( TN \) is parallelizable; that is, there exists a smooth orthonormal frame field \( e_1, \ldots, e_k \) for \( TN \). Given a (weak) wave map \( u: \mathbb{R} \times \mathbb{R}^m \to N \), we then obtain a frame for the pull-back bundle \( u^*TN \) by letting

\[
e_i(z) = R_{ij}(z)e_j(u(z)) \quad \text{for} \quad z = (t, x) \in \mathbb{R} \times \mathbb{R}^m,
\]

where

\[
R = (R_{ij}): \mathbb{R} \times \mathbb{R}^m \to SO(k).
\]

Denote \( \theta_i = \langle du, e_i \rangle = \theta_{i,\alpha}dx^\alpha, \omega_{ij} = \langle de_i, e_j \rangle = \omega_{ij,\alpha}dx^\alpha, 1 \leq i, j \leq k. \)

Then (0.1) is equivalent to the system of equations

\[
0 = (D\theta_i) = -\partial^\alpha(\partial_\alpha u, e_i) + (\partial_\alpha u, \partial^\alpha e_i) = -\partial^\alpha\theta_{i,\alpha} + \omega_{ij,\alpha}^j, \quad \theta_i = \delta_{ij} \theta_j, \quad \theta_j = \omega_{ij,\alpha}^j
\]

for \( 1 \leq i \leq k \). Note that (0.6) is a first order Hodge system analogous to (0.5); however, (0.6) differs from (0.5) by a quadratic expression.

Using the Hodge structure (0.6), in joint work with A. Freire and S. Müller [8], [9] we obtain weak compactness of wave maps in \( m = 2 \) space dimensions.

**Theorem 0.4.** Let \( m = 2. \) Suppose \( (u^L) \) is a sequence of wave maps such that \( u^L \to u \) in \( L^2 \) and \( Du^L \to Du \) weakly in \( L^2 \), locally on \( \mathbb{R} \times \mathbb{R}^m \), as \( L \to \infty \). Then \( u \) is a (weak) wave map.

The proof makes contact with the work of Evans [5] and Bethuel [1] on the partial regularity of stationary harmonic maps. In particular, we also use special compensation properties of Jacobians ([4]) and \( H^1 - BMO \) duality ([6]).

The crucial determinant structure for the nonlinear term in (0.6) is achieved by localizing the equation to a compact domain which we then regard as contained in the fundamental domain of a torus \( T^3 = \mathbb{R}^3/\mathbb{Z}^3 \).

On \( T^3 \) (following Hélein [12]) we then impose the Coulomb gauge condition (with respect to the Euclidean background metric) by choosing, for each \( L \), a "gauge" \( R^L \in H^1(T^3;SO(k)) \) such that

\[
\sum_i \int_{T^3} |De_i|^2 \, dz = \min_R \sum_i \int_{T^3} |D(R_{ij}(\bar{e}_j \cdot u^L))|^2 \, dz.
\]

In this gauge, we have

\[
\partial_\alpha \omega_{ij,\alpha} = \delta_{\text{euc}}\omega_{ij} = 0,
\]

and \( (e_i^L) \) is bounded in \( H^{1,2}(T^3) \) with

\[
\sum_i \int_Q |De_i|^2 \, dz \leq \sum_i \int_Q |D(\bar{e}_i \cdot u^L)|^2 \, dz \leq CE(u^L(0)) \leq C.
\]
Hence we may assume that $e^L_i \to e_i$ weakly in $H^{1,2}(T^3)$ and

$$
\theta^L_i = (du^L_i, e^L_i) = \theta^L_{i,\alpha} \ dx^\alpha \to \theta_i = (du, e_i),
$$

$$
\omega^L_{ij} = (de^L_i, e^L_j) = \omega^L_{i,j,\alpha} \ dx^\alpha \to \omega_{ij} = (de_i, e_j)
$$

weakly in $L^2$ as $L \to \infty$.

Using the Hodge $\ast$-operator (with respect to $\eta$), we may express

$$
\omega^L_{ij} \wedge \eta \theta^L_j \ dz = \omega^L_{ij} \wedge \ast \eta \theta^L_j.
$$

By Hodge decomposition (with respect to the Euclidean metric on $T^3$), moreover, we have

$$
\ast \eta \theta^L_j = da^L_j + \delta_{\text{eucl}} b^L_j + c^L_j,
$$

where $a^L_j \to a_j, b^L_j \to b_j, c^L_j \to c_j$ in $H^1(T^3)$ as $L \to \infty$. The harmonic forms $c^L_j$ are constant multiples of the basis vectors $dx^\alpha \wedge dx^\beta$; hence $c^L_j \to c_j$ smoothly, as $L \to \infty$, and $\omega^L_{ij} \eta c^L_j \to \omega_{ij} \eta c_j$ in $D'$. Using the Coulomb gauge condition, and letting $\beta^L_j = \ast b^L_j$, the second term may be re-written

$$
\omega^L_{ij} \wedge \delta_{\text{eucl}} b^L_j = \delta_{\text{eucl}}(\omega^L_{ij} \beta^L_j) \ dz,
$$

which tends to the desired distributional limit. Similarly, for the third term we have

$$
\omega^L_{ij} \wedge da^L_j = -d(\omega^L_{ij} \wedge a^L_j) + d\omega^L_{ij} \wedge a^L_j.
$$

Again, it is easy to pass to the limit $L \to \infty$ in the divergence term. The last term, finally, possesses a determinant structure

$$
d\omega^L_{ij} \wedge a^L_j = de^L_i \wedge de^L_j \wedge a^L_j.
$$

Using the Hardy space estimates for Jacobians of $[4]$ and $H^1 - BMO$ duality of $[6]$ we are able to show that, as $L \to \infty$,

$$
de^L_i \wedge de^L_j \wedge a^L_j \to de_i \wedge de_j \wedge a_j + \nu \text{ in } D',
$$

and to characterize the defect measure $\nu$ in a way analogous to P.L. Lions' $[15]$ concentration-compactness principle. In particular, from energy estimates we derive that the $H^1$-capacity of the support of $\nu$ vanishes. But, passing to the limit $L \to \infty$ in (0.6), on the other hand we have

$$
0 = \delta_{\eta} \theta^L_i + \omega^L_{ij} \cdot \eta \theta^L_j \to \delta_{\eta} \theta_i + \omega_{ij} \cdot \eta \theta_j + \nu \text{ in } D';
$$

that is,

$$
\nu = -\delta_{\eta} \theta_i - \omega_{ij} \cdot \eta \theta_j \in H^{-1} + L^1(T^3),
$$

and hence $\nu = 0$.

Finally, in joint work with S. Müller $[16]$ we couple the above weak compactness argument with the viscous approximation method suggested by Yi Zhou $[22]$ to obtain

**Theorem 0.5.** Let $m = 2$. Then for any $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m; TN)$ there exists a global weak solution to the Cauchy problem (0.1), (0.2).

It remains to question whether this solution is unique and regular for smooth data.
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