JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (2000), p. 1-10

<http://www.numdam.org/item?id=JEDP_2000____A15_0>

Journées *Équations aux dérivées partielles*
Nantes, 5–9 juin 2000
GDR 1151 (CNRS)

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Abstract

We present some recent results, obtained jointly with Detlef Müller, on solvability of operators of the form

$$\sum_{j,k=1}^{2n} a_{jk} V_j V_k + i\alpha U$$

where the V_j are left-invariant vector fields on the Heisenberg group, such that $[V_j, V_{j+n}] = U$ $(1 \le j \le n)$ are the only nontrivial relations, and $A = (a_{jk})$ is a complex symmetric matrix with semi-definite real part.

The presentation also contains references on the work done in the past few years in this area.

1. Introduction

The Heisenberg group $\mathbb{H}_n$ can be viewed as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with coordinates $(x, y, u) = (x_1, \ldots, x_n, y_1, \ldots, y_n, u)$ and product

$$(x, y, u)(x', y', u') = \left(x + x', y + y', u + u' + \frac{1}{2}(x \cdot y' - y \cdot x')\right) . \qquad (1.1)$$

The vector fields

$$\begin{aligned} X_j &= \partial_{x_j} - \frac{1}{2} y_j \partial_u , \qquad j = 1, \cdots, n , \\ Y_j &= \partial_{y_j} + \frac{1}{2} x_j \partial_u , \qquad j = 1, \cdots, n , \\ U &= \partial_u , \end{aligned} \qquad (1.2)$$

span the Lie algebra $\mathfrak{h}_n$ of left-invariant vector fields, with the relations:

$$[X_j, Y_k] = \delta_{jk} U , \quad [X_j, X_k] = [Y_j, Y_k] = [X_j, U] = [Y_j, U] = 0 .$$

MSC 2000 : 35A05, 43A80

The analysis of differential operators on $\mathbb{H}_n$ involving quadratic expressions in the X_j and Y_j present many interesting aspects and applications to different problems in complex and harmonic analysis. The most relevant operators in this class are the *sublaplacian*

$$\mathcal{L} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) , \tag{1.3}$$

together with the *Folland-Stein operators*

$$\mathcal{L}_\alpha = \mathcal{L} + i\alpha U , \tag{1.4}$$

with $\alpha \in \mathbb{C}$. It is well-known that properties of $\mathcal{L}_\alpha$, such as hypoellipticity and local solvability, depend on the value of α. This is coherent with the fact that $\mathcal{L}$ is not principal type. The "singular values" of α, i.e. those for which $\mathcal{L}_\alpha$ is neither hypoelliptic nor locally solvable, are

$$\alpha = \pm n, \pm(n+2), \dots, \pm(n+2k), \dots \tag{1.5}$$

(see [FS]). Observe that the X_j and Y_j generate the full Lie algebra $\mathfrak{h}_n$, so that hypoellipticity for $\mathcal{L}$ itself, and for $\mathcal{L}_\alpha$ when α is purely imaginary, follows from Hörmander' classical theorem [Hö1].

Basically, the pattern is the same if $\mathcal{L}$ is replaced by any real positive definite quadratic combination of the X_j and Y_j (see [BG]). One can then suitably change coordinates and reduce to

$$\mathcal{L}' = \sum_{j=1}^{n} \lambda_j (X_j^2 + Y_j^2) , \tag{1.6}$$

with $\lambda_j > 0$. The singular values for $\mathcal{L}'_\alpha = \mathcal{L}' + i\alpha U$ now become

$$\alpha = \pm \sum_{j=1}^{n}(2k_j + 1)\lambda_j . \tag{1.7}$$

Some attention to indefinite quadratic expression was initially given in [P], where operators of the form (1.6) with $\lambda_j = \pm 1$ were considered.

A more systematic study of local solvability for second order left-invariant operators on the Heisenberg group was initiated by Detlef Müller and myself in [MR2]. A description of these results requires some changes in the notation, and some considerations on the symplectic invariance which is intrinsic to the structure of the Heisenberg group.

Hence we denote the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ (in this order) by $V_1, \dots, V_{2n}$. Consistently, we set $v = (x, y) \in \mathbb{R}^{2n}$. Given a $2n \times 2n$ symmetric matrix $A = (a_{jk})$, set

$$\mathcal{L}_A = \sum_{j,k=1}^{2n} a_{jk} V_j V_k , \tag{1.8}$$

and $\mathcal{L}_{A,\alpha} = \mathcal{L}_A + i\alpha U$. These operators can be characterized as the second order operators on $\mathbb{H}_n$ that are left-invariant and homogeneous of degree 2 under the dilations $(v, u) \mapsto (\delta v, \delta^2 u)$.

We recall that $\mathrm{Sp}(n, \mathbb{R})$, resp. $\mathrm{Sp}(n, \mathbb{C})$, is the group of $2n \times 2n$ real, resp. complex, matrices g such that $gJ\,{}^t g = J$, with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Lie algebras $\mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{sp}(n, \mathbb{C})$ consist of the -real or complex- matrices S such that $SJ + J\,{}^tS = 0$.

If $g \in \mathrm{Sp}(n, \mathbb{R})$, the linear transformation $(v, u) \mapsto (gv, u)$ preserves the product (1.1) (i.e. this transformation is a group isomorphism); hence the operators $\mathcal{L}_{A,\alpha}$ and $\mathcal{L}_{gA\,{}^tg,\alpha}$ must share the same properties.

When A is real, such properties are conveniently described in terms of the matrix $S = -AJ \in \mathfrak{sp}(n, \mathbb{R})$ defining the Hamilton map associated to A. A simplified, even if not complete, presentation of the main result of [MR2] can be given as follows. We refer to [MR2] for the precise statement, where the cases where solvability occurs are completely determined.

Theorem 1.1 *Suppose that A is real. The operator $\mathcal{L}_{A,\alpha} = \mathcal{L}_A + i\alpha U$ is locally solvable for all values of α, unless $S = -AJ$ is semisimple and with purely imaginary eigenvalues. In this case, we can assume, modulo a symplectic change of variables, that $\mathcal{L}_A$ is as in* (1.6)*, with $\lambda_j \in \mathbb{R}$. Then the values of α such that $\mathcal{L}_{A,\alpha}$ is not locally solvable are those described by* (1.7)*, plus, possibly, limits of sequences of such values.*

We point out what the situation is when A is real and positive semidefinite. Then the eigenvalues of $S = -AJ$ are always purely imaginary, but there are two cases: either S is semisimple, in which case $\mathcal{L}_A$ can be put in the form (1.6) with $\lambda_j \geq 0$, or S is not, and then $\mathcal{L}_A$ can be reduced to

$$\mathcal{L}_A = \sum_{j=1}^{k} \lambda_j (X_j^2 + Y_j^2) + \sum_{j=k+1}^{k'} X_j^2 \,, \tag{1.9}$$

with $0 \leq k < k' \leq n$ and $\lambda_j > 0$ for $j \leq k$.

In the first case, we have the singular values (1.7); in the second case there are no singular values.

The analysis of the operators $\mathcal{L}_{A,\alpha}$ for A complex is much more involved, at least for two reasons: one is that the classification of the conjugacy classes of matrices $S \in \mathfrak{sp}(n, \mathbb{C})$ under the action of the group $\mathrm{Sp}(n, \mathbb{R})$ (which determines the admissible changes of coordinates) is not as well understood as for real matrices, and apparently it is very complicated; the other reason is that one of the main tools that are used in the real case, i.e. the so called *oscillator semigroup*, is not available, unless one assumes that the real part of A is positive semidefinite.

More recently, Detlef Müller and I have attacked the case of a complex symmetric matrix A with positive semidefinite real part, using the oscillator semigroup. In the rest of these notes we give a description of these results, which are still unpublished.

2. Twisted convolution and the oscillator semigroup

If f is a reasonable function on $\mathbb{H}_n$ and $\mu \in \mathbb{R} \setminus \{0\}$, define

$$f^\mu(v) = \int_{-\infty}^{+\infty} f(v,u) e^{-2\pi i \mu u}\, du \ . \tag{2.1}$$

It is then easy to see that

$$\begin{aligned}(f * g)^\mu(v) &= \int_{\mathbb{R}^{2n}} f^\mu(v-v') g^\mu(v') e^{\pi i \mu\, {}^t v' J v}\, dv' \\ &\stackrel{\text{def}}{=} f^\mu \times_\mu g^\mu(v) \ ,\end{aligned} \tag{2.2}$$

the so-called *μ-twisted convolution* of f^μ and g^μ.

For the left-invariant vector fields on $\mathbb{H}_n$ we have the following formulas:

$$\begin{aligned}(X_j f)^\mu &= (\partial_{x_j} - \pi i \mu y_j) f^\mu = f \times_\mu (\partial_{x_j}\delta_0) \stackrel{\text{def}}{=} \tilde{X}_j^\mu f^\mu \\ (Y_j f)^\mu &= (\partial_{y_j} + \pi i \mu x_j) f^\mu = f \times_\mu (\partial_{y_j}\delta_0) \stackrel{\text{def}}{=} \tilde{Y}_j^\mu f^\mu \\ (Uf)^\mu &= 2\pi i \mu f^\mu \ .\end{aligned} \tag{2.3}$$

Hence, if $\mathcal{L}_A$ is the operator (1.8), we have $(\mathcal{L}_A f)^\mu = \tilde{\mathcal{L}}_A^\mu f^\mu$, where $\tilde{\mathcal{L}}_A^\mu$ is obtained by replacing each V_j with the corresponding $\tilde{V}_j^\mu$ in (1.8). If we also set

$$\Delta_A = \sum_{j,k=1}^{2n} a_{jk} \partial_{v_j} \partial_{v_k} \ ,$$

then

$$\tilde{\mathcal{L}}_A^\mu f^\mu = f^\mu \times_\mu \Delta_A \delta_0 \ .$$

We are interested in developing a functional calculus for $\mathcal{L}_A$, which will allow us to construct, whenever possible, fundamental solutions for $\mathcal{L}_{A,\alpha}$. In doing do, we initially assume that the real part of A is positive definite.

The first step is the explicit determination of the semigroup generated by $\tilde{\mathcal{L}}_A^\mu$. Modulo scaling and complex conjugation, we can assume that $\mu = 1$. In doing so, we drop the index μ at all places.

If we naively copy the formula of the "heat kernel" for Δ_A, and set

$$u(v,t) = f \times \left(\det(4\pi t A)^{-\frac{1}{2}} e^{-\frac{1}{4}\, {}^t v (tA)^{-1} v} \right) \ , \tag{2.4}$$

we have that $u(v,0) = f$ for any test function f, and $\partial_t u(v,0) = \tilde{\mathcal{L}}_A u(v,0)$. Unfortunately, the Gaussian functions in (2.4) do not form a semigroup under twisted convolution, so that the equation $\partial_t u = \tilde{\mathcal{L}}_A u$ is not satisfied for $t > 0$.

However we are not far from the correct solution, and (2.4) gives a first order approximation for small values of t. One simply has to understand the "algebra" underlying twisted convolution of Gaussian functions.

Denote by $\mathfrak{S}_n$ the *Siegel half-space* consisting of all $2n \times 2n$ complex symmetric matrix with positive definite real part.

Lemma 2.1 ([Hw]) *Given two matrices $A_1, A_2 \in \mathfrak{S}_n$, we have*

$$e^{-\pi\,{}^t\!vA_1v} \times e^{-\pi\,{}^t\!vA_2v} = \det(A_1 + A_2)^{-\frac{1}{2}} e^{-\pi\,{}^t\!vA_3v} \;,$$

where

$$A_3 + iJ/2 = (A_2 + iJ/2)(A_1 + A_2)^{-1}(A_1 + iJ/2) \in \mathfrak{S}_n \;. \tag{2.5}$$

So the scalar multiples of these Gaussians form a semigroup, called the *oscillator semigroup* (in one of its realizations, see [Hw, Fo]).

If we set $A_j = \frac{1}{2}JS_j$, with $S_j \in \mathfrak{sp}(n, \mathbb{C})$, (2.3) becomes

$$S_3 + iI = (S_2 + iI)(S_1 + S_2)^{-1}(S_1 + iI) \;.$$

Therefore, we look for one-parameter families S_t such that

$$S_{t+t'} + iI = (S_{t'} + iI)(S_t + S_{t'})^{-1}(S_t + iI) \;.$$

As this condition implies that the S_t commute among themselves, we obtain the addition formula for the cotangent:

$$S_{t+t'} = (S_tS_{t'} - I)(S_t + S_{t'})^{-1} \;.$$

The following statement is a modification of Theorem 3.1 in [MR1] (see also [MPR2]).

Lemma 2.2 *Given $A \in \mathfrak{S}_n$, write $A = SJ$ with $S \in \mathrm{Sp}(n, \mathbb{C})$. Then*

$$\Gamma_{t,S}(v) = \big(\det(2\sin 2\pi tS)\big)^{-\frac{1}{2}} e^{-\frac{\pi}{2}\,{}^t\!vJ\cot(2\pi tS)v} \;. \tag{2.6}$$

is a Schwartz function for every $t > 0$ (in particular $J\cot(2\pi tS) \in \mathfrak{S}_n$) and

$$f \times \Gamma_{t,S} = e^{t\tilde{\mathcal{L}}_A} f \;.$$

For generic $\mu \neq 0$ we then set

$$\Gamma^\mu_{t,S}(v) = |\mu|^n \Gamma_{t,S}\big(|\mu|^{\frac{1}{2}}v\big) \;, \tag{2.7}$$

in order to have

$$f \times_\mu \Gamma^\mu_{t,S} = e^{\frac{1}{|\mu|}t\tilde{\mathcal{L}}^\mu_A} f \;.$$

As the $\Gamma^\mu_{t,S}$ become singular for t tending to zero, it will be more convenient to do computations on their Fourier transforms,

$$\widehat{\Gamma^\mu_{t,S}}(\xi) = \int_{\mathbb{R}^{2n}} \Gamma^\mu_{t,S}(v) e^{-2\pi i v\cdot\xi}\, d\xi = \big(\det(\cos 2\pi tS)\big)^{-\frac{1}{2}} e^{-\frac{2\pi}{|\mu|}\,{}^t\xi(\tan 2\pi tS)J\xi} \;. \tag{2.8}$$

3. Construction of fundamental solutions when $\Re e A > 0$

Write the formal identity

$$
\begin{aligned}
(\mathcal{L}_A + i\alpha U)^{-1} &= \int_{-\infty}^{+\infty} (\tilde{\mathcal{L}}_A^{\mu} - 2\pi\alpha\mu)^{-1} e^{2\pi i u \mu}\, d\mu \\
&= -\int_{-\infty}^{+\infty}\int_0^{\infty} e^{t(\tilde{\mathcal{L}}_A^{\mu} - 2\pi\alpha\mu)} e^{2\pi i u\mu}\, dt\, d\mu \\
&= -\int_{-\infty}^{+\infty}\int_0^{\infty} e^{\frac{t}{|\mu|}\tilde{\mathcal{L}}_A^{\mu} - 2\pi\alpha t\,\mathrm{sgn}\,\mu} e^{2\pi i u\mu}\, dt\, \frac{d\mu}{|\mu|}\,.
\end{aligned}
$$

In order to give a meaning to it, we must discuss the convergence, in the sense of distributions, of the integral

$$
\int_{-\infty}^{+\infty}\int_0^{\infty} \widehat{\Gamma_{t,S}^{\mu}} e^{-2\pi\alpha t\,\mathrm{sgn}\,\mu} e^{2\pi i u\mu}\, dt\, \frac{d\mu}{|\mu|}\,.
$$

If f is a test function, passing to its Fourier transform, this amounts to requiring that

$$
\int_{-\infty}^{+\infty}\frac{d\mu}{|\mu|}\int_0^{\infty} \frac{e^{-2\pi\alpha t\,\mathrm{sgn}\,\mu}}{\big(\det(\cos 2\pi t S)\big)^{\frac{1}{2}}}\, dt \int_{\mathbb{R}^{2n}} e^{-\frac{2\pi}{|\mu|}\,{}^t\xi(\tan 2\pi t S)J\xi}\hat{f}(\xi,\mu)\, d\xi \tag{3.1}
$$

is convergent.

This problem involves a careful discussion of the spectral properties of S: location of its eigenvalues in the complex plane, symplectic properties of its generalized eigenspaces (as subspaces of $\mathbb{C}^{2n}$ with the symplectic form induced by J). This analysis is developed in detail in [Hö2] (see also [Sj]).

The first remark is that S does not have real eigenvalues. In fact, if C is the convex hull of the set $\{{}^t v A v : v \in \mathbb{R}^n\}$ (so that C is an angle intersecting the imaginary axis only at the origin) then the eigenvalues of S are contained in $\pm iC$.

Let $\pm\lambda_1, \dots, \pm\lambda_n$ be the eigenvalues of S, labeled so that $\Im m\lambda_j = \nu_j > 0$. Then the quantity $\big(\det(\cos 2\pi t S)\big)^{\frac{1}{2}}$ that appears in (3.1) grows exponentially, as t tends to infinity, like $e^{2\pi t\nu}$, with $\nu = \sum_j \nu_j$.

One can prove that the integral (3.1) converges for $|\Re e\alpha| < \nu$. The argument requires two different estimates of the integral in $d\mu$, to be applied for large and for small values of μ respectively.

For $|\mu| > 1$, the integral can be controlled, uniformly in t, by the decay of the Fourier transform of f.

For $|\mu| < 1$, Hölder's inequality gives a bound of the form $|\mu|^{\delta n}\big|\det(\tan 2\pi t S)\big|^{-\delta/2}$ for any $\delta < 1$. It is then convenient to choose $\delta < 1/n$ for t small and $\delta = 1/2$ for t large.

The integral (3.1) naturally splits into the sum of two integrals, obtained by restricting the integration in $d\mu$ to one of the two half-lines. Let us restrict our attention to $\mu > 0$. The argument above shows that the integral defines a distribution K_α^+ for $\Re e\alpha > -\nu$.

Clearly, K_α^+ depends analytically on α, so it makes sense to look for an analytic continuation to other values of α.

What is technically easier is to produce an analytic continuation of each derivative $U^\ell K_\alpha^+$ to a larger half-plane, apart from a finite number of isolated poles. As ℓ varies, these half-planes cover the full plane.

These continuations can be obtained by iterated integration by parts in dt. We simply sketch how the first step is performed on UK_α^+.

If we set $k(t) = e^{-2\pi\alpha t}\big(\det(\cos 2\pi tS)\big)^{-\frac{1}{2}}$, it is not hard to see that, for $\Re e\alpha > -\nu$, k has a primitive that vanishes at infinity and is asymptotic to a constant times $\frac{1}{\alpha-i\lambda}e^{-2\pi(\alpha-i\lambda)t}$, with $\lambda=\sum_j\lambda_j$.

This denominator will ultimately produce a pole at $\alpha = i\lambda$, which lies on the line $\Re e\alpha = -\nu$.

On the other hand, the differentiation of the Gaussian in dt produces a factor

$$\frac{1}{\mu}\,{}^t\xi(\cos 2\pi tS)^{-2}J\xi \ ,$$

which improves the exponential decay in t and allows to take α in a larger half-plane. The factor $1/\mu$ that has been introduced is compensated by the fact that the analogue of (3.1) for UK_α^+ has a $d\mu$ instead of $\frac{d\mu}{|\mu|}$.

A similar argument applied to the other half of the integral in (3.1), corresponding to $\mu < 0$, produces the continuation of UK_α^- to the symmetric half-plane, with a pole at $\alpha = -i\lambda$.

As $(\mathcal{L}_A + i\alpha U)(UK_\alpha^+ + UK_\alpha^-) = U\delta_0 = \partial_u\delta_0$, this implies, by standard arguments, that $\mathcal{L}_A + i\alpha U$ is locally solvable in the enlarged strip.

The final result reads as follows.

Theorem 3.1 *Let $A \in \mathfrak{S}_n$. If $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $S=-AJ$ with $\Im m\lambda_j > 0$, then $\mathcal{L}_{A,\alpha}$ is locally solvable if and only if $\alpha \neq \pm i\sum_j(2k_j+1)\lambda_j$.*

The proof of the last part of the statement uses some arguments from [Höl] and [Sj].

Even though the notation is different, this result matches with what has been said for the operator (1.6): for that operator the eigenvalues of S are purely imaginary and equal i times the coefficients λ_j (see also Theorem 1.1 and the remarks that follow).

Using [Höl, Theorem 1.1], or [G] one can show that for the non-singular values of α $\mathcal{L}_{A,\alpha}$ is hypoelliptic, and this gives a different proof of Theorem 3.1. Our argument is relatively more elementary, even though its scope is much more limited. As we shall see next, it also applies to quadratic forms with degenerate real part.

4. Fundamental solutions for $\Re eA$ semidefinite

Much of the analysis developed in the previous Section can be extended, with some modifications, to the case of a matrix A with a degenerate, but semidefinite, real part, provided that we assume what we can call the *cone condition*: if $A = A_1 + iA_2$, with A_j real, then $|A_2| \le cA_1$ for some scalar $c > 0$. This means that the set $\{{}^tvAv : v \in \mathbb{R}^n\}$ is containing in the angle C defined by the condition $|\Im m\zeta| \le c\Re e\zeta$.

If such a condition is satisfied, then the eigenvalues of S are again contained in $\pm iC$, but the eigenvalue 0 will also appear.

The structure of the generalized eigenspace relative to the eigenvalue 0 is, however, relatively simple (see [Hö2]).

Lemma 4.1 *Assume that the cone condition is satisfied. Then, if* $S = S_1 + iS_2$, *with* S_j *real, then* $(\ker S) \cap \mathbb{R}^{2n} = (\ker S_1) \cap \mathbb{R}^{2n}$, *and* $\ker S$ *is invariant under complex conjugation. In addition,* $(\ker S) \cap \mathbb{R}^{2n}$ *is the direct sum* $\mathcal{V}_0 \oplus \mathcal{V}_1$, *with* $\mathcal{V}_0$ *symplectic and* $\mathcal{V}_1$ *isotropic. The generalized 0-eigenspace of* S_1 *in* $\mathbb{C}^{2n}$ *is* $\mathcal{W} = (\mathcal{V}_0 \oplus \mathcal{V}_1 \oplus J\mathcal{V}_1)^{\mathbb{C}}$, *and* $S^2 = 0$ *on* $\mathcal{W}$.

This means that we can perform a linear symplectic change of coordinates in $\mathbb{R}^{2n}$ in such a way that the expression of $\mathcal{L}_A$ involves a smaller number of vector fields, say $V_1, \dots, V_m, V_{n+1}, \dots, V_{n+m'}$, with $m \le m' \le n$, and $m < n$. Among the missing vector fields, $V_{m'+1}, \dots, V_n, V_{n+m'+1}, \dots, V_{2n}$ span $J\mathcal{V}_0$ and $V_{m+1}, \dots, V_{m'}$ span $J\mathcal{V}_1$.

Then $\mathcal{L}_A$ is in fact a left-invariant operator on a lower-dimensional subgroup, and solvability can be studied on this subgroup. As the subgroup is isomorphic to $\mathbb{H}_m \times \mathbb{R}^{m'-m}$, we are so led to study the operator

$$\mathcal{D} = \sum_{j,k=1}^{2m} a_{jk} V_j V_k - 2i \sum_{j=1}^{2m} \sum_{k=1}^{m'-m} b_{jk} V_j \partial_{s_k} + \sum_{j,k=1}^{m'-m} c_{jk} \partial_{s_j} \partial_{s_k} , \tag{4.1}$$

on $\mathbb{H}_m \times \mathbb{R}^{m'-m}$, where the V_j are as above (passing from H_n to H_m some relabeling has come out naturally) and the s_j are the variables in $\mathbb{R}^{m'-m}$. The matrices $A = (a_{jk})$ and $C = (c_{jk})$ have positive definite real parts, and the b_{jk} are real.

If $m = m'$, we are back to the previous case. Assume therefore that $m < m'$.

We begin by taking partial Fourier transform in the central variable u from $\mathbb{H}_m$ and in the variables s_j. Calling μ and η the respective dual variables, we obtain the operators

$$\tilde{\mathcal{D}}^{\mu,\eta} = \sum_{j,k=1}^{2m} a_{jk} \tilde{V}_j^\mu \tilde{V}_k^\mu + 4\pi \sum_{j=1}^{2m} \sum_{k=1}^{m'-m} b_{jk} \eta_k \tilde{V}_j^\mu - 4\pi^2 \sum_{j,k=1}^{m'-m} c_{jk} \eta_j \eta_k .$$

The mixed terms can be eliminated by conjugating $\tilde{\mathcal{D}}^{\mu,\eta}$ with multiplication by an exponential function in v.

Ultimately the setting is rather similar to the positive definite case. The new feature is the presence of a non-trivial quadratic form $Q(\eta) = {}^t\eta C \eta$.

Without going into precise computations, just observe that if we repeat the kind of integrations by parts that in the positive case produced the factor $\frac{1}{\alpha - i\lambda}$, we obtain now $\frac{|\mu|}{(\alpha - i\lambda)|\mu| + 2\pi Q(\eta)}$.

This tells us that, for every α in a vertical strip containing $\pm i\lambda$ in its interior, it is possible to construct a distribution H_α such that

$$(D + i\alpha U) H_\alpha = \big(Q(\partial_s) + (\lambda + i\alpha)U\big)\big(Q(\partial_s) + (\lambda + i\alpha)U\big)\delta_0 .$$

So we no longer have a pole at $\alpha = \pm i\lambda$, and the same occurs at the other "singular values".

Theorem 4.2 *Let $A \in \partial\mathfrak{S}_n$, and assume that the cone condition is satisfied. If* $\ker A$ *is not a symplectic subspace of* $\mathbb{R}^{2n}$*, then* $\mathcal{L}_{A,\alpha}$ *is locally solvable for every* α.

If $\ker A$ *is symplectic, let* $\lambda_1, \ldots, \lambda_m$ *be the eigenvalues of* $S = -AJ$ *with* $\Im\mathrm{m}\lambda_j > 0$. *Then* $\mathcal{L}_{A,\alpha}$ *is locally solvable if and only if* $\alpha \neq \pm i \sum_j (2k_j + 1)\lambda_j$.

5. Other cases

If $\Re\mathrm{e}A \geq 0$, but the cone condition is not satisfied, many different situations may occur. Surely the case of a purely imaginary A is included in this discussion, so that Theorem 1.1 gives the answer (replacing A with iA and α with $i\alpha$).

There are cases where the set of singular values is not symmetric. We take the following example from [MPR1].

Theorem 5.1 *Let* $A = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}$. *Then* $\mathcal{L}_{A,\alpha}$ *is locally solvable on* $\mathbb{H}_1$ *if and only if* $\alpha \notin 2\mathbb{N} + 1$.

This example can be generalized to higher dimensions (see [MPR1, MPR2, MZ]).

In a series of articles, [DPR, MPR1, MPR2, KM], the problem of local solvability has been studied under the assumption that S^2 is scalar, but it seems that each case requires some *ad hoc* argument.

It would be desirable to have a classification of complex quadratic forms on $\mathbb{R}^{2n}$, i.e. complex Hamiltonians, modulo symplectic changes of variables, or, equivalently, a description of the orbits of the adjoint action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathfrak{sp}(n, \mathbb{C})$. Under the restriction $S^2 = cI$, such a classification was obtained by [MT].

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