JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (2000), p. 1-9
<http://www.numdam.org/item?id=JEDP_2000____A2_0>

Journées *Équations aux dérivées partielles*
Nantes, 5–9 juin 2000
GDR 1151 (CNRS)

Recovering Asymptotics at Infinity of Perturbations of Stratified Media

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Abstract

We consider perturbations of a stratified medium $\mathbf{R}_x^{n-1} \times \mathbf{R}_y$, where the operator studied is $c^2(x,y)\Delta$. The function c is a perturbation of $c_0(y)$, which is constant for sufficiently large $|y|$ and satisfies some other conditions. Under certain restrictions on the perturbation c, we give results on the Fourier integral operator structure of the scattering matrix. Moreover, we show that we can recover the asymptotic expansion at infinity of c from knowledge of c_0 and the singularities of the scattering matrix at fixed energy.

1. Introduction

In these lecture notes we describe the problem of recovering a sound speed c which is a perturbation of a stratified sound speed c_0. That is, c_0 depends on the variable $y \in \mathbf{R}$ but the main operator one studies is $c^2\Delta$, where $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$ and $z = (x,y) \in \mathbf{R}^{n-1} \times \mathbf{R}$. The wave equation $(-\frac{\partial^2}{\partial t^2} - c_0^2\Delta)v = 0$ models the propagation of acoustic waves in a layered, or stratified, fluid, and replacing c_0 by c allows perturbations. Under certain conditions on c and c_0 made more precise in Section 1.1, we show that when $c_0(y) = c_+$ when $|y| > y_M$, then the scattering matrix for $c^2\Delta$ is a Fourier integral operator and describe its singular set. Moreover, we show that the asymptotics of the perturbation can be recovered from the scattering matrix at fixed energy. We expect to show that very similar results hold when $c_0(y) = c_\pm$ for $\pm y > y_M$, and for this reason state intermediate results in the more general setting.

The inverse result given here is complementary to earlier results ([1, 7, 9, 18]), which showed that exponentially decaying perturbations can be recovered. Our results use techniques used by Joshi and Sá Barreto, [10, 11, 12] to study inverse problems in other settings.

As in those results, the fundamental idea here is to compute the symbol of the scattering matrix by solving transport equations along geodesics on the sphere

The first author was partially supported by a University of Missouri S.R.F. We also thank the London Mathematical Society for supporting this collaborative research through its small grants scheme.
MSC 2000 : 35P25, 58J40

at infinity. These equations express the propagation of growth at infinity. The fundamental difference here is that we have to consider the broken geodesic flow obtained by refraction and reflection at the equator (cf. [15]). This continues a chain of ideas initiated by Melrose and Zworski in [14] in order to prove that the scattering matrix on an asymptotically Euclidean manifold is a Fourier integral operator associated to geodesic flow at time π.

In this note we only state results and sketch some proofs, the details of which will appear elsewhere. We also do not attempt here to give our results in the greatest generality possible.

1.1. Assumptions and notation

Throughout, $z = (x, y) \in \mathbf{R}^{n-1} \times \mathbf{R}$.

Both sound speeds c and c_0 satisfy $0 < c_m < c, c_0 < c_M < \infty$. Moreover, $c_0(y)$ is piecewise smooth and there exists a finite y_M so that $c_0(y) = c_\pm$ when $\pm y > y_M$. We take $c_+ \leq c_-$. Moreover, all derivatives of c_0 are bounded except at finitely many values of y. This allows c_0 to be piecewise constant, for example.

We require that $c - c_0$ be smooth outside of a compact set. Moreover, we make requirements on the behaviour of $c - c_0$ at infinity. For any N and for any multiindex α,

$$D_z^\alpha(c(z) - c_0(y)) = D_z^\alpha \left(\sum_{j=J}^{N} \gamma_j\left(\frac{z}{|z|}\right)|z|^{-j} \right) + \mathcal{O}(|z|^{-N-|\alpha|-1}), \tag{1}$$

where $\gamma_j \in C^\infty(S^{n-1})$. For most of our results, $J \geq 2$ will suffice, and we take J to be at least two throughout.

We study the operators $c_0^2\Delta$ and $c^2\Delta$ acting on $L^2(\mathbf{R}^n, c_0^{-2}dz)$ and $L^2(\mathbf{R}^n, c^{-2}dz)$, respectively, so that the operators are symmetric and have self-adjoint extensions, which we denote in the same way.

2. The Spectrum of $c_0^2\Delta$ and the scattering matrix

The spectral and scattering theory of the operators $c_0^2\Delta$ and $c^2\Delta$ have been widely studied under assumptions much weaker than those we have made here (e.g. [2, 4, 5, 17, 19] and references). Below we summarize some of the results of other authors that are most relevant here.

2.1. Fine description of the spectrum

The spectrum of $c^2\Delta$ or of $c_0^2\Delta$ is $[0, \infty)$. However, in order to study the scattering theory of these operators, we need to have a better understanding of the nature of their spectra.

We begin with the operator $c_0^2\Delta$. Consider a generalized eigenfunction with eigenvalue λ^2 of the form $e^{ix\cdot\xi}\phi(y)$. Then

$$c_0^2(D_y^2 + |\xi|^2)\phi(y) = \lambda^2\phi(y).$$

This leads us to consider the family of one-dimensional differential operators $A_\tau = c_0^2(D_y^2 + \tau^2)$ on $L^2(\mathbf{R}, c_0^{-2}dy)$. It has continuous spectrum beginning at $c_+^2\tau^2$, of multiplicity two if $c_+ = c_-$ and of multiplicity one on $[c_+^2\tau^2, c_-^2\tau^2)$ and of multiplicity two on $[c_-^2\tau^2, \infty)$ if $c_+ < c_-$. In either case, the continuous spectrum of $A_{|\xi|}$, taken over ξ, means that there is continuous spectrum of $c_0^2\Delta$ which is parametrized by S_c^{n-1} and covers $[0,\infty)$. Here $S_c^{n-1} = \{\omega = (\overline{\omega}, \omega_n) \in S^{n-1} : \omega_n \neq 0, \omega_n \neq \sqrt{1 - c_+^2/c_-^2}\}$ and S^{n-1} is the $(n-1)$-dimensional unit sphere.

Additionally, A_τ may have discrete spectrum:

$$A_\tau f_j(y) = \lambda_j^2 f_j(y).$$

If this happens, for each fixed τ there are only finitely many, simple, eigenvalues:

$$\lambda_1^2(\tau) < \lambda_2^2(\tau) < \cdots < \lambda_{k(\tau)}^2(\tau) < c_+^2\tau^2.$$

For $\tau > 0$, λ_j is an increasing function of τ and the number of eigenvalues increases with τ. These eigenvalues and eigenfunctions produce guided waves for the corresponding wave equation $(D_t^2 - c_0^2\Delta)v = 0$.

Let

$$\tau_{j0} = \inf\{\tau > 0 : \text{ there are } j \text{ eigenvalues of } A_\tau\}.$$

The numbers $c_+^2\tau_{j0}^2$ are called thresholds – at each threshold, there begins another "branch" of continuous spectrum of $c_0^2\Delta$, parametrized by S^{n-2}. Let

$$T(\lambda) = \#\{c_+^2\tau_{j0}^2 \leq \lambda^2\}$$

be the counting function for the number of thresholds.

As a final piece of notation, let $\tau_j(\lambda)$ have the same sign as λ, and be the inverse function of λ_j, so that

$$c_0^2(D_y^2 + \tau_j^2(\lambda))f_j(y) = \lambda^2 f_j(y). \tag{2}$$

Notice that for $\overline{\omega} \in S^{n-2}$, $e^{i\tau_j(\lambda)x\cdot\overline{\omega}}f_j(y)$ is a generalized eigenfunction of $c_0^2\Delta$, with eigenvalue λ^2.

Although we have not given a proof, the two groups of generalized eigenfunctions described above form a full set of generalized eigenfunctions of $c_0^2\Delta$ – that is, together they can be used to give the spectral measure of $c_0^2\Delta$. At energy λ^2, the generalized eigenfunctions are parametrized by S_c^{n-1} and $T(\lambda)$ copies of S^{n-2}.

The generalized eigenfunctions of $c^2\Delta$ are quite similar and are parametrized by the same space. Under the assumptions that we have made, neither $c_0^2\Delta$ nor $c^2\Delta$ has any eigenvalues ([4]).

2.2. The scattering matrix

Because of the described parametrization of the continuous spectrum at fixed energy, the (absolute) scattering matrices $A_0(\lambda)$, $A(\lambda)$ of $c_0^2\Delta$ and $c^2\Delta$ are operators

$$A(\lambda),\ A_0(\lambda) : L^2(S_c^{n-1}) \oplus_{1\leq j\leq T(\lambda)} L^2(S^{n-2}) \to L^2(S_c^{n-1}) \oplus_{1\leq j\leq T(\lambda)} L^2(S^{n-2}).$$

Here we are using the notation $\oplus_{1\leq j\leq T(\lambda)} L^2(S^{n-2})$ to stand for the direct sum of $T(\lambda)$ copies of $L^2(S^{n-2})$.

There are several ways of defining a scattering matrix. One way is via a map from one set of generalized eigenfunctions to another– for example, from incoming to outgoing generalized eigenfunctions. In [3] a definition of the scattering matrix was given in terms of the generalized eigenfunctions. Here, however, it will be more useful to define the (absolute) scattering matrix using the Poisson operator. We begin by recalling a simpler example before describing the definition in this case.

Example: Potential Scattering on $\mathbf{R}^n$. On $\mathbf{R}^n$, let V be a real-valued function that has an asymptotic expansion at infinity of the form $V \sim \sum_{j=2}^{\infty} |z|^{-j} v_j(z/|z|)$. Then, for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, the *Poisson operator* $P_V(\lambda)$ is the operator defined by

$$P_V(\lambda) : C^\infty(S^{n-1}) \ni f \mapsto u_V \in (1+|z|)^{1/2+\epsilon} L^2(\mathbf{R}^n)$$

where u_V is determined by

$$(\Delta + V - \lambda^2) u_V = 0$$

and

$$u_V = |z|^{-(n-1)/2} \left(e^{i\lambda|z|} f\left(\frac{z}{|z|}\right) + e^{-i\lambda|z|} f'\left(\frac{z}{|z|}\right) + \mathcal{O}(|z|^{-1}) \right).$$

Note that f and f' are functions on S^{n-1}, which can be thought of as the "sphere at infinity." The (absolute) scattering matrix $A(\lambda)$ is determined by $A(\lambda)f = f'$ and extends by continuity to be a map on $L^2(S^{n-1})$.

As perhaps the simplest example of a Poisson operator, if $V \equiv 0$, then

$$P(\lambda, z, \omega) = C(\lambda) e^{i\lambda z \cdot \omega},$$

as is easily checked using the method of stationary phase. Then $A(\lambda)$ is a constant (depending on the dimension) times the antipodal map. See [10] for a construction of the Poisson operator for $\Delta + V$ in this setting and a discussion of the resulting scattering matrix.

To obtain the relative scattering matrix, which is usually just called the scattering matrix, one composes the absolute scattering matrix with the inverse of the absolute scattering matrix for the "model" problem. For potential scattering on $\mathbf{R}^n$, then, the relative scattering matrix is, up to a constant, obtained by composing the absolute scattering matrix with the antipodal map.

We now return to the case of the operator $c^2\Delta$. Note that while in our example $\Delta + V$ approaches a constant-coefficient differential operator when $|z| \to \infty$ in any direction, the same is not true for $c^2\Delta$. The consequences of this can be seen already in the asymptotic expansion used in defining the Poisson operator and will appear again in the construction of an approximate Poisson operator outlined in Section 3.1.

The Poisson operator is defined initially as an operator

$$P(\lambda) : C_c^\infty(S_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(S^{n-2}) \to (1+|z|)^{1/2+\epsilon} L^2(\mathbf{R}^n).$$

If $g = (g_0, g_1, ..., g_{T(\lambda)}) \in C_c^\infty(S_c^{n-1}) \oplus_{i=1}^{T(\lambda)} C^\infty(S^{n-2})$, then $P(\lambda)g = u$, $(c^2\Delta - \lambda^2)u = 0$, and u has asymptotics at infinity of the following type: for any $\epsilon > 0$, as $|z| \to \infty$

with $\pm y > y_M$, $|y|/|z| > \epsilon$,

$$u \sim |z|^{-(n-1)/2}\left(e^{i\lambda|z|/c_\pm} g_0\left(\tfrac{z}{|z|}\right) + e^{-i\lambda|z|/c_\pm} g_0'\left(\tfrac{z}{|z|}\right)\right) + o(|z|^{-(n-1)/2}). \tag{3}$$

Moreover, when $|y| < y_M$, we have

$$\begin{aligned} u \sim\ & |x|^{-(n-2)/2}\left(\sum_{j=1}^{T(\lambda)} e^{i\tau_j(\lambda)|x|} f_j(y,\lambda) g_j\left(\tfrac{x}{|x|}\right) + \sum_{j=1}^{T(\lambda)} e^{-i\tau_j(\lambda)|x|} f_j(y,\lambda) g_j'\left(\tfrac{x}{|x|}\right)\right) \\ & + e^{-i\lambda|x|/c_+} h\left(\tfrac{x}{|x|}, y\right) + o(|x|^{-(n-1)/2}). \end{aligned} \tag{4}$$

Here the f_j are eigenfunctions of the reduced operator as in (2), and we are assuming $J \geq 6+n$ or c, c_0 satisfy the assumptions of Theorem 3.1 to ensure a nice asymptotic expansion at infinity. In order to simplify the presentation, we avoid discussing the asymptotics in the transitional region between the areas $|y| < y_M$ and $|y|/|z| > \epsilon > 0$, but refer the interested reader to [3, Theorem 4.1] where the problem is treated for a more specialized case.

We will call the operator $A(\lambda)$ defined by $A(\lambda)g = g'$ the absolute scattering matrix. For fixed λ, $A(\lambda) = (A_{ij}(\lambda))$, $0 \leq i, j \leq T(\lambda)$. We call $A_{00}(\lambda)$ the "main part" of the scattering matrix; in case the operator A_τ has no eigenvalues it is the entire scattering matrix.

3. Inverse results

The first theorem, on the structure of the scattering matrix, extends some results of [3], which required $c - c_0$ to be rapidly decreasing. This theorem holds under more general conditions than those given here.

Theorem 3.1 *Suppose the asumptions of Section 1.1 are satisfied, $c_+ = c_-$, c, c_0 are smooth, and $J \geq 2$. Then the main part of the scattering matrix is a Fourier integral operator associated with broken geodesic flow at time π on S^{n-1}.*

Here the geodesic flow is broken at $\omega_n = 0$; that is, it is reflected and transmitted when it hits the region where $c_0 \neq c_+$ on the sphere at infinity. This phenomenon has also been observed in N-body scattering (e.g. [16]), which has many similarities.

Theorem 3.2 *Suppose the assumptions of Section 1.1 are satisfied, $c_+ = c_-$, c, c_0 are smooth, $J \geq 2$ and $n \geq 3$. Then the coefficients γ_j in the asymptotic expansion (1) of $c - c_0$ are determined by $c_0(y)$ and the transmitted singularities of the main part of the scattering matrix at a fixed nonzero energy.*

In fact, our proof gives more than this: if two sound speeds c_1, c_2 satisfy all the conditions of the theorem for the same c_0, and the difference of their scattering operators is of order $-j$, then $c_1 - c_2 = \mathcal{O}(|z|^{-j+1})$. We have intermediate and partial results in the cases $c_+ < c_-$ and relaxing the smoothness assumptions on c_0 and c, and hope to be able to extend them.

Theorem 3.2 is complementary to results of [1, 7, 9, 18], which concerned exponentially decaying perturbations. Below, we give an outline of the proof. We study the main part of the scattering matrix by first constructing an approximation of the (partial) Poisson operator. We use a modification of some results of [6, 13, 15] to show that all the singularities of the scattering matrix can be read off from the approximation of the Poisson operator we construct. Finally, we use some techniques similar to those of [11] to show that the coefficients in the expansion of $c - c_0$ can be recovered from the singularities of the scattering matrix.

3.1. The approximation to the Poisson operator

In order to study the "main part" of the scattering matrix, we construct an approximation to $P(\lambda)\Pi_0$, where $\Pi_0(g_0, g_1, ..., g_{T(\lambda)}) = (g_0, 0, ..., 0)$. We use some techniques developed in [10] as well as some additional techniques adapted to this situation.

Proposition 3.1 *For $\lambda \in \mathbf{R}\setminus\{0\}$, there exists an approximation of $P(\lambda)\Pi_0$, which we denote $\tilde{P}_0(\lambda)$, such that*

$$(c^2\Delta - \lambda^2)\tilde{P}_0(\lambda) \in (1+|z|)^{-\infty}L^2(\mathbf{R}^n).$$

Additionally, $\tilde{P}_0(\lambda)g$ has an asymptotic expansion at infinity as in (3) and (4), the expansions used in the definition of the Poisson operator.

We remark that this proposition and the proof we outline hold under considerably less restrictive conditions than the Theorems given above. For example, it is not necessary to have $c_+ = c_-$, and the smoothness conditions can be relaxed.

We outline the technique for construction of the Schwartz kernel $\tilde{P}_0(\lambda, z, \omega)$ of $\tilde{P}_0(\lambda)$ when $\omega = (\overline{\omega}, \omega_n)$, $\omega_n > 0$, $\omega_n \neq \sqrt{1 - c_+^2/c_-^2}$.

We begin with $\Phi_0(\lambda, z, \omega) = e^{i\lambda x\cdot\overline{\omega}/c_+}\phi_+(y)$, where

$$c_0^2(\lambda^2|\overline{\omega}|^2/c_+^2 + D_y^2)\phi_+(y) = \lambda^2\phi_+(y)$$

and

$$\phi_+ = \begin{cases} e^{i\lambda y\omega_n/c_+} + R_+e^{-i\lambda y\omega_n/c_+} & y > y_M \\ T_+e^{i\lambda y\sqrt{1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2}} & y < -y_M \end{cases}$$

where we take $\sqrt{1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2}$ to have negative imaginary part if $1/c_-^2 - 1/c_+^2 + \omega_n^2/c_+^2 < 0$. Note that for $\omega_n > 0$, $\Phi_0(\lambda, z, \omega)$ gives the Schwartz kernel of the Poisson operator composed with Π_0 for $c_0^2\Delta$, up to a multiple depending on λ and $c_\pm$.

Applying $c^2\Delta - \lambda^2$ to Φ_0, we obtain an error term which has a nice asymptotic expansion at infinity. We need to remove the error term by adding additional terms to Φ_0. How we remove the error depends on the region in which we are working. In each region described below, the error terms are removed by successively adding terms which result in an error that vanishes one order faster at infinity.

When $y > y_M$, there are two pieces of the error term: one of the form $e^{i\lambda z\cdot\omega/c_+}b_+$, and the other of the form $e^{i\lambda x\cdot\overline{\omega}/c_+ - i\lambda y\omega_n/c_+}b_-$, with $b_+, b_- \in S^{-J}_{\text{phg}}$, the classical polyhomogeneous symbols of order $-J$. To solve away an error of the form $e^{i\lambda z\cdot\omega/c_+}b_{+j}$, $b_{+j} \in S^{-j}_{\text{phg}}$ (modulo a term $e^{i\lambda z\cdot\omega/c_+}b_{+,j+1}$, with $b_{+,j+1} \in S^{-j-1}_{\text{phg}}$) we use

$$e^{i\lambda z\cdot\omega/c_+}|z|^{-j+1}a(z/|z|).$$

Here a solves a transport equation along geodesics through ω on S^{n-1}, with initial condition 0 at $z/|z| = \omega$, just as in [10]. We use this, at infinity, down to $y/|z| = 0$. In this way, for $y > y_M$ we can replace the error $e^{i\lambda z\cdot\omega/c_+}b_+$ with one of the same form, but with the symbol vanishing as rapidly as we like at infinity.

When $|y| < y_M$, we solve away the error terms by solving an ordinary differential equation with boundary conditions.

A similar technique as that described for the error of the form $e^{i\lambda z\cdot\omega/c_+}b$ in $y > y_M$ is used to solve away the other error in $y > y_M$ and the error in $y < -y_M$, but this time using initial conditions at $y/|z| = 0$, which come from the solutions to the ordinary differential equations in $|y| < y_M$. For example, the error term in $y > y_M$ of the form $e^{i\lambda x\cdot\overline{\omega}/c_+ - i\lambda y\omega_n/c_+}b_-$ is removed by adding terms where the transport equations are solved along geodesics through $z/|z| = (-\overline{\omega}, \omega_n)$. However, the solutions to the transport equations are badly behaved at $z/|z| = (-\overline{\omega}, \omega_n)$ in $y > y_M$ and at $(-\overline{\omega}c_-/c_+, -\sqrt{1 - c_-^2|\overline{\omega}|^2/c_+^2})$ (if $1 - c_-^2|\overline{\omega}|^2/c_+^2 > 0$) in $y < -y_M$, and the approximate Poisson operator takes a more complicated form near these points, of the type described in [10, Section 3]. The structure of the Poisson operator near these points contributes to the singularities of the scattering matrix.

If these pieces are all put together properly, we get an approximation of the Poisson operator as described in the Proposition.

3.2. Inversion

We have

$$P(\lambda)\Pi_0 = \tilde{P}_0(\lambda) - (c^2\Delta - (\lambda - i0)^2)^{-1}(c^2\Delta - \lambda^2)\tilde{P}_0(\lambda).$$

In order to fully understand the "main part" of the scattering matrix, it remains to understand the the asymptotic expansion at infinity of the second term.

In case $c_+ = c_-$, and c, c_0 are smooth, then a modification of results of [6, 13, 15] tells us that the coefficient of the leading order term is smooth away from $y/|z| = 0$; that is, all the singularities of the main part of the scattering matrix come from the construction of $\tilde{P}_0(\lambda)$. This finishes the proof of the first theorem. In other cases, we have partial results of this type.

For the inverse result, suppose we have two sound speeds c_1 and c_2 that satisfy all the requirements placed on c, and we know $c_1 - c_2 \sim \sum_{j\geq k}|z|^{-j}a_j(z/|z|)$. Then from the transmitted singularities of the scattering matrix we can recover, for $\omega_n \neq 0$,

$$T_+(\lambda, \omega_n)\int_0^\pi (\sin s)^{k-2}\tilde{a}_k(s, \theta; \omega)ds.$$

Here the integral is over a geodesic on S^{n-1} of length π starting at ω; as θ varies, we get all such geodesics. The function $\tilde{a}_k$ is a_k written in different coordinates. If $c_1^2\Delta$ and $c_2^2\Delta$ have the same transmitted singularities of the scattering matrix, this is 0 for all such ω, and the remaining step is to show that this implies $a_k = 0$. As in [11], by differentiating with respect to the endpoints, one can reduce this to an inversion problem for an integral transform with the exponent $k - 2 = 0$ or $k - 2 = 1$. If $k = 2$, then differentiating again shows that the function is even, and we get that the integral over closed geodesics of length 2π is 0. Using a result of Funk (e.g. [8]) on the kernel of the X-ray transform, this shows that $a_k = 0$. A similar technique handles the case where one reduces to $k = 3$.

Iterating over k, Theorem 3.2 follows.

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