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Global well-posedness and scattering for the mass-critical NLS

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Suppose $u(t,x)$ is a solution to the nonlinear partial differential equation

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^{4/d} u, \\ u(0,x) &= u_0 \in L^2(\mathbf{R}^d), \end{aligned} \tag{0.1}$$

$\mu = \pm 1$, $\mu = +1$ refers to the defocusing case and $\mu = -1$ refers to the focusing case.

Definition 0.1. (0.1) *is said to be globally well - posed if a solution* $u(t,x)$ *to* (0.1) *exists for all time,*

$$u(t,x) \in C_t^0(\mathbf{R}; L^2(\mathbf{R}^d)) \cap L_{t,loc}^{\frac{2(d+2)}{d}}(\mathbf{R}; L^{\frac{2(d+2)}{d}}(\mathbf{R}^d)), \tag{0.2}$$

and a solution to (0.1) *depends continuously on* u_0 *in the* $L^2(\mathbf{R}^d)$ *topology.*

Definition 0.2. *A global solution to* (0.1) *is said to scatter if there exist* $u_\pm \in L^2(\mathbf{R}^d)$ *such that*

$$\|u(t,x) - e^{it\Delta} u_+\|_{L^2(\mathbf{R}^d)} \to 0, \tag{0.3}$$

as $t \to +\infty$ *and*

$$\|u(t,x) - e^{it\Delta} u_-\|_{L^2(\mathbf{R}^d)} \to 0 \tag{0.4}$$

as $t \to -\infty$. *Additionally we say a solution to* (0.1) *scatters forward in time if it satisfies* (0.3) *and backward in time if it satisfies* (0.4).

The first progress toward proving well - posedness of (0.1) was

Theorem 0.1. (0.1) *is locally well - posed on* $[-T,T]$ *for some* $T(\|u_0\|_{H^1(\mathbf{R}^d)}) > 0$.

Proof: See [6]. □

Furthermore, it is possible to use conserved quantities of (0.1) to upgrade theorem 0.1 to global well - posedness. (0.1) has the conserved quantities mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{0.5}$$

and energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t,x)|^{\frac{2(d+2)}{d}} dx. \tag{0.6}$$

In the defocusing case (0.6) is positively definite, which implies $\|u(t)\|_{H^1(\mathbf{R}^d)}$ is uniformly bounded by $E(u(0))$ which is finite by the Sobolev embedding theorem. By (0.1) (0.1) is globally well - posed for $u_0 \in H^1(\mathbf{R}^d)$, $\mu = +1$.

In the focusing case (0.6) is not positive definite. Therefore having $E(u(0))$ finite is not enough to prove global well - posedness because $\|u(t)\|_{H^1(\mathbf{R}^d)}$ and $\|u(t)\|_{L^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}$ can and do blow up at the same rate, precisely canceling to maintain conservation of energy.

For $\|u(t)\|_{L^2(\mathbf{R}^d)}$ below a certain threshold it is still possible to prove global well - posedness and scattering in the case when $\mu = -1$ using the Gagliardo - Nirenberg inequality.

Theorem 0.2. *If Q is the positive solution to the elliptic partial differential equation*

$$\Delta Q + Q^{1+4/d} = Q, \tag{0.7}$$

the Sobolev embedding theorem has the best constant

$$\|u\|_{L_x^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}^{\frac{2(d+2)}{d}} \leq \frac{\|u\|_{L^2(\mathbf{R}^d)}^{4/d}}{\|Q\|_{L^2(\mathbf{R}^d)}^{4/d}} \|\nabla u\|_{L^2(\mathbf{R}^d)}^2. \tag{0.8}$$

Proof: See [30], [43], [44], and [5]. □

Combining theorem 0.2 with (0.5) proves (0.1) when $\|u_0\|_{L^2(\mathbf{R}^d)} < \|Q\|_{L^2(\mathbf{R}^d)}$. Furthermore, by (0.7)

$$u(t,x) = e^{it} Q(x) \tag{0.9}$$

is a solution to (0.1) when $\mu = -1$. This is a solution that certainly fails to scatter. Applying the conformal symmetry

Theorem 0.3. *u is a solution to* (0.1) *if and only if*

$$v(t,x) = \frac{1}{|t|^{d/2}} u(-\frac{1}{t}, \frac{x}{t}) e^{i|x|^2/t} \tag{0.10}$$

solves (0.1).

We obtain a solution to (0.1) that fails to be globally well - posed.

Furthermore, consider the variance

$$\int |x|^2 |u(t,x)|^2 dx. \tag{0.11}$$

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 16 E(u(t)). \tag{0.12}$$

If $u(0) \in H^1(\mathbf{R}^d)$, $0 > E(u(t)) > -\infty$, and $\int |x|^2 |u(0,x)|^2 dx < \infty$, then the variance is concave down in time, which implies that $\int |x|^2 |u(t,x)|^2 dx$ will cross the real axis twice. Since (0.11) is positive definite, this implies a solution to (0.1) can only exist in both directions for finite time. Such solutions are relatively straightforward to construct when $\|u_0\|_{L^2(\mathbf{R}^d)} > \|Q\|_{L^2(\mathbf{R}^d)}$.

The local well - posedness result in theorem 0.1 was substantially improved to

Theorem 0.4. (0.1) *is locally well - posed on* $[-T, T]$ *for* $u_0 \in L^2(\mathbf{R}^d)$, $T(u_0) > 0$, *where* T *depends on the profile of* u_0, *not just its size.*

Proof: See [6] and [7]. □

In this paper we sketch the proof of the natural extension of theorem 0.4,

Theorem 0.5. (0.1) *is globally well - posed and scattering for all* $u_0 \in L^2(\mathbf{R}^d)$, $\mu = +1$. (0.1) *is globally well - posed and scattering for* $\mu = -1$, $\|u_0\|_{L^2(\mathbf{R}^d)} < \|Q\|_{L^2(\mathbf{R}^d)}$.

Previous Results:

Theorem 0.6. (0.1) *is globally well - posed and scattering for* $\mu = +1$, $u_0 \in L^2(\mathbf{R}^d)$ *and for* $\mu = -1$, $\|u_0\|_{L^2(\mathbf{R}^d)} < \|Q\|_{L^2(\mathbf{R}^d)}$ *when* u_0 *is radial,* $d = 2$.

Proof: See [24]. □

Theorem 0.7. (0.1) *is globally well - posed and scattering for* $\mu = +1$, $u_0 \in L^2(\mathbf{R}^d)$ *when* u_0 *is radial,* $d \geq 3$.

Proof: See [37]. □

Theorem 0.8. (0.1) *is globally well - posed and scattering for* $\mu = -1$, $\|u_0\|_{L^2(\mathbf{R}^d)} < \|Q\|_{L^2(\mathbf{R}^d)}$ *when* u_0 *is radial,* $d \geq 3$.

Proof: See [26]. □

Conjecture: If (0.1) is not globally well - posed and scattering, and $\|u_0\|_{L^2(\mathbf{R}^d)} = \|Q\|_{L^2(\mathbf{R}^d)}$, then

$$u(t,x) = G \cdot e^{it} Q(x) \tag{0.13}$$

where $G = S^1 \times (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$ is a group of symmetries acting on solutions to (0.1), or a conformal symmetry of (0.13).

G is generated by four symmetries which act on solutions of (0.1), multiplication,

$$u(t,x) \mapsto e^{i\theta} u(t,x), \tag{0.14}$$

for $\theta \in \mathbf{R}$, scaling,

$$u(t,x) \mapsto \frac{1}{\lambda^{d/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \tag{0.15}$$

translation,

$$u(t,x) \mapsto u(t, x - x_0), \tag{0.16}$$

and Galilean invariance

$$u(t,x) \mapsto e^{-it|\xi_0|^2} e^{ix \cdot \xi_0} u(t, x - 2t\xi_0). \tag{0.17}$$

Let

$$A_\mu(m) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R}\times\mathbf{R}^d)} : \|u(t)\|_{L_x^2(\mathbf{R}^d)} = m, \text{ u solves } (0.1)\}. \tag{0.18}$$

To prove theorem 0.1 it suffices to prove $A_\mu(m) < \infty$ for all m in the defocusing case and for $m < \|Q\|_{L^2(\mathbf{R}^d)}$ in the focusing case.

Theorem 0.9. *$A_\mu(m)$ is a continuous function of m.*

Proof: See [35].
This already implies a small data result for $m < \epsilon(d,\mu)$ because $A_\mu(0) = 0$. Moreover,

$$\{m : A_\mu(m) = \infty\} \tag{0.19}$$

is a closed set so if (0.19) is nonempty then it possesses a least element m_0.

Theorem 0.10. *Suppose*

$$\|u_n(t)\|_{L_x^2(\mathbf{R}^d)} \nearrow m_0 \tag{0.20}$$

$$\|u_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(t\geq 0)} \nearrow \infty, \quad \|u_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(t\leq 0)} \nearrow \infty. \tag{0.21}$$

Then $u_n(t)$ has a subsequence that converges to $u(t)$ in $L^2(\mathbf{R}^d)/G$, $u(t) : I \subset \mathbf{R} \to \mathbf{C}$ is a solution to (0.1)*, I an open set.*

$$\|u(t)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(t\geq 0)} = \|u(t)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(t\leq 0)} = \infty. \tag{0.22}$$

Moreover, $\{u(t) : t \in I\}$ lies in a compact subset of $L^2(\mathbf{R}^d)/G$. By the Arzela - Ascoli theorem there exist

$$x(t), \xi(t) : I \to \mathbf{R}^d, \tag{0.23}$$

$$N(t) : I \to (0,\infty), \tag{0.24}$$

such that for all $\eta > 0$ there exists $C(\eta) < \infty$ such that

$$\int_{|x-x(t)|\geq \frac{C(\eta)}{N(t)}} |u(t,x)|^2 dx + \int_{|\xi-\xi(t)|\geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta. \tag{0.25}$$

Proof: See [36]. □

Because $u(t)$ lies in a precompact set we can take a limit of $u(t_n)$, $t_n \in I$, in $L^2(\mathbf{R}^d)/G$ and obtain an even more special solution to (0.1).

Theorem 0.11. *If theorem* 0.1 *fails then there exists a solution to* (0.1) *satisfying $N(0) = 1$, $u(t)$ exists on $[0,\infty)$, $N(t) \leq 1$ on $[0,\infty)$, $x(0) = \xi(0) = 0$,*

$$|\xi'(t)|, |N'(t)| \lesssim_{m_0,d} N(t)^3, \tag{0.26}$$

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\infty)\times\mathbf{R}^d)} = \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((\inf(I),0]\times\mathbf{R}^d)} = \infty. \tag{0.27}$$

Proof: See [25]. □

We consider two cases separately,

$$\int_0^\infty N(t)^3 dt < \infty, \tag{0.28}$$

and

$$\int_0^\infty N(t)^3 dt = \infty. \tag{0.29}$$

Theorem 0.12. *If* $\int_0^\infty N(t)^3 dt = K$*, then for all* $0 \leq s < 1 + \frac{4}{d}$*,*

$$\|u\|_{L_t^\infty \dot{H}_x^s([0,\infty)\times\mathbf{R}^d)} \lesssim_{m_0,d} K^s. \tag{0.30}$$

This is enough to exclude (0.28) in the defocusing case and in the focusing case for mass below the mass of the ground state. In both cases $E(u(0)) \geq \delta > 0$. (0.26) and (0.28) imply $N(t) \to 0$ as $t \to \infty$. So when $\xi(t) \equiv 0$ it is easy to see $N(t) \to 0$, Sobolev embedding, and (0.30) imply

$$E(u(t)) \to 0, \tag{0.31}$$

which contradicts conservation of energy. In the general case when $\xi(t)$ is free to move around, (0.26) implies that $|\xi(t)| \lesssim_{m_0,d} K$ for all $t \in [0,\infty)$. For T sufficiently large $N(T)$ is very small. After making a Galilean transformation sending $\xi(T)$ to 0, this implies $E(u(T))$ is very small. Because $|\xi(T)| \lesssim_{m_0,d} K$ this transformation preserves (0.30). On the other hand, because L^p is Galilean invariant, after any Galilean transformation

$$E(u(0)) \geq \delta > 0. \tag{0.32}$$

This again contradicts conservation of energy.

Having completely ruled out case (0.28) we turn to case (0.29) when $\mu = +1$. We use the interaction Morawetz estimate

Theorem 0.13. *If* $\mu = +1$*,*

$$\| |\nabla|^{\frac{3-d}{2}} |u(t,x)|^2 \|_{L_{t,x}^2([0,T]\times\mathbf{R}^d)}^2 \lesssim_{m_0,d} \int_0^T \partial_t M(t) dt, \tag{0.33}$$

where

$$M(t) = \int \frac{(x-y)_j}{|x-y|} Im[\bar{u}(t,x)\partial_j u(t,x)] |u(t,y)|^2 dx dy. \tag{0.34}$$

Because the solution $u(t)$ need not possess any additional regularity, we truncate in frequency.

Theorem 0.14. *Suppose* $\int_0^T N(t)^3 dt = K$*, choose* C *sufficiently large so that*

$$\int_0^T |\xi'(t)| dt << CK, \tag{0.35}$$

which is always possible by (0.26)*. Let*

$$M(t) = \int \frac{(x-y)_j}{|x-y|} Im[P_{\leq CK}\bar{u}(t,x)\partial_j P_{\leq CK}u(t,x)]|P_{\leq CK}u(t,y)|^2 dxdy. \tag{0.36}$$

Then

$$\int_0^T N(t)^3 dt \lesssim_{m_0,d} \int_0^T \partial_t M(t) dt, \tag{0.37}$$

and since $N(t) \leq 1$ *for* $t \in [0,\infty)$,

$$|M(t)| \lesssim_{m_0,d} o(K). \tag{0.38}$$

This rules out (0.29) in the case when $\mu = +1$ because K can be made arbitrarily large by taking T sufficiently large, giving the contradiction

$$K \lesssim o(K). \tag{0.39}$$

We now give a brief discussion of the proof of theorem 0.14. It is perhaps easiest to see that when $d \geq 4$, if u is a minimal mass blowup solution to (0.1),

$$N(t)^3 \lesssim_{m_0,d} \| |\nabla|^{\frac{3-d}{2}} |u(t,x)|^2 \|_{L_x^2(\mathbf{R}^d)}^2. \tag{0.40}$$

Indeed, when $d \geq 4$,

$$\| |\nabla|^{\frac{3-d}{2}} |u(t,x)|^2 \|_{L_x^2(\mathbf{R}^d)}^2 \sim_{m_0,d} \int \frac{1}{|x-y|^3} |u(t,x)|^2 |u(t,y)|^2 dxdy. \tag{0.41}$$

The spatial concentration in (0.25) implies (0.40). Because most of the mass is contained in $P_{\leq CK}$, we also have

$$\int_0^T N(t)^3 dt \lesssim_{m_0,d} \| |\nabla|^{\frac{3-d}{2}} |P_{\leq CK}u(t,x)|^2 \|_{L_{t,x}^2([0,T]\times\mathbf{R}^d)}^2. \tag{0.42}$$

(0.38) follows from (0.25), $N(t) \leq 1$, and the fact that the interaction Morawetz estimates are Galilean invariant.

$$i\partial_t(P_{\leq CK}u) + \Delta(P_{\leq CK}u) = \mu|P_{\leq CK}u|^{4/d}(P_{\leq CK}u) + \mu[P_{\leq CK}(|u|^{4/d}u) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)]. \tag{0.43}$$

If we were able to drop

$$\mu[P_{\leq CK}(|u|^{4/d}u) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)] \tag{0.44}$$

then the proof of (0.42) when $\mu = +1$ would be identical to the proof of (0.33). Therefore, most of the work in proving theorem 0.14 lies in showing that the error arising from (0.44) is bounded by $o(K)$. In fact, the error estimates are quite robust.

Theorem 0.15. *We can perform the same error estimates with* $\frac{(x-y)_j}{|x-y|}$ *replaced with* $a(t, x-y)_j$ *as long as*

$$|a(t,x)| \lesssim_{m_0,d} 1, \tag{0.45}$$

when $d = 2$,

$$|\nabla a(t,x)| \lesssim_{m_0,d} \frac{1}{|x|}, \tag{0.46}$$

and when $d = 1$,

$$\|\nabla a(t,x)\|_{L^1_x(\mathbf{R})} \lesssim_{m_0,d} 1, \tag{0.47}$$

$$a(t,x) = -a(t,-x), \tag{0.48}$$

and when $d = 2$,

$$\|\partial_t a(t,x)\|_{L^1(\mathbf{R}^2)} \lesssim_{m_0,d} 1. \tag{0.49}$$

Therefore it remains to construct an interaction Morawetz potential bounded below by $N(t)^3$ and which satisfies (0.45) - (0.49). We do this only in the case when $d = 1$ and u is radial. Suppose $\psi \in C^\infty(\mathbf{R})$, $\psi = \phi' \geq 0$, and

$$\begin{aligned} &= x, \quad |x| \leq 1, \\ \psi(x) &= \frac{3}{2}, \quad x > 2, \\ &= -\frac{3}{2}, \quad x < -2. \end{aligned} \tag{0.50}$$

Then let

$$M(t) = R \int \psi(\frac{x\tilde{N}(t)}{R}) Im[\bar{u}\partial_x u](t,x)dx, \tag{0.51}$$

such that for some $\delta > 0$

$$\delta N(t) \leq \tilde{N}(t) \leq N(t), \tag{0.52}$$

and

$$\int_0^T |\tilde{N}'(t)|dt \leq \delta_1 \int_0^T \tilde{N}(t)N(t)^2 dt. \tag{0.53}$$

Then

$$\partial_t M(t) = \tilde{N}(t) \int \phi(\frac{x\tilde{N}(t)}{R})[\frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{6}|u(t,x)|^6]dx \tag{0.54}$$

$$+ C\frac{\tilde{N}(t)^3}{R^2} \int \phi''(\frac{x\tilde{N}(t)}{R})|u(t,x)|^2 dx \tag{0.55}$$

$$+ R \int \phi(\frac{x\tilde{N}(t)}{R}) x\tilde{N}'(t) Im[\bar{u}\partial_x u](t,x)dx. \tag{0.56}$$

We choose $\tilde{N}(t)$ to be a sufficiently slowly varying (0.53) envelope for $N(t)$ which allows us to absorb (0.55) into (0.54) for $R(m_0)$ sufficiently large and for $\delta_1(R)$ sufficiently small we can absorb (0.56) into (0.54). This completes the proof of the focusing case.

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