Tuomo Kuusi & Giuseppe Mingione
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PARTIAL REGULARITY AND POTENTIALS

by Tuomo Kuusi & Giuseppe Mingione

Abstract. — We connect classical partial regularity theory for elliptic systems to Nonlinear Potential Theory of possibly degenerate equations. More precisely, we find a potential theoretic version of the classical ε-regularity criteria leading to regularity of solutions of elliptic systems. For non-homogenous systems of the type $-\operatorname{div} a(Du) = f$, the new ε-regularity criteria involve both the classical excess functional of Du and optimal Riesz type and Wolff potentials of the right hand side f. When applied to the homogenous case $-\operatorname{div} a(Du) = 0$ such criteria recover the classical ones in partial regularity. As a corollary, we find that the classical and sharp regularity results for solutions to scalar equations in terms of function spaces for f extend verbatim to general systems in the framework of partial regularity, i.e. optimal regularity of solutions outside a negligible, closed singular set. Finally, the new ε-regularity criteria still allow to provide estimates on the Hausdorff dimension of the singular sets.

Résumé (Régularité partielle et potentiels). — Nous relions la théorie classique de la régularité partielle des systèmes elliptiques à la théorie du potentiel non linéaire d'équations éventuellement dégénérées. Plus précisément, nous donnons une version en théorie du potentiel des critères classiques d'ε-régularité de solutions des systèmes elliptiques. Pour les systèmes non homogènes du type $-\operatorname{div} a(Du) = f$, les nouveaux critères d'ε-régularité font intervenir à la fois la fonctionnelle classique d'excès de Du et de type de Riesz optimal et les potentiels de Wolff du membre de droite f. Appliqués au cas homogène $-\operatorname{div} a(Du) = 0$, ces critères redonnent les critères classiques en théorie de la régularité partielle. Comme corollaire, nous montrons que les résultats classiques et précisés de régularité pour les solutions d'équations scalaires en terme d'espaces de fonctions pour f s'étendent mot pour mot aux systèmes généraux dans le cadre de la régularité partielle, à savoir la régularité partielle des solutions hors d'un ensemble singulier fermé négligeable. Enfin, ces nouveaux critères d'ε-régularité permettent encore d'obtenir des estimée sur la dimension de Hausdorff des ensembles singuliers.

Contents

1. Introduction and results

The purpose of this paper is to establish significant connections between the classical theory of partial regularity for general nonlinear elliptic systems and the Nonlinear Potential Theory of possibly degenerate quasilinear equations initiated in the fundamental papers of Mazya and Havin [36, 37]. The ultimate goal is to find, for solutions to non-homogeneous, potentially degenerate elliptic systems of the type

$$(1.1) \qquad\qquad -\operatorname{div} a(Du) = f \quad \text{in } \Omega \subset \mathbb{R}^n,$$

new local ε-regularity criteria guaranteeing the local continuity of Du. Such criteria are formulated both in terms of the classical excess type functionals of the gradient $E(\cdot)$ and in terms of Riesz type potentials $\boldsymbol{I}_{1,q}^f$ of the right hand side f. See (1.17) below, and (1.7) and (1.13) for the definitions of $E(\cdot)$ and $\boldsymbol{I}_{1,q}^f$, respectively. These potentials, that are actually Wolff potentials, are of the same type of the ones classically considered in Nonlinear Potential Theory for equations and elliptic systems with quasi diagonal structure [3, 4, 9, 23, 24, 25, 24, 26, 31, 33, 35, 30, 42, 43, 46]. As a corollary, we obtain local gradient regularity criteria which are sharp in terms of the function spaces the right hand side datum f is prescribed to belong to. Finally, the ε-regularity criteria obtained for solutions to (1.1) recover, in the case $f \equiv 0$, those available in the classical partial regularity theory for solutions to homogeneous elliptic systems of the type

$$(1.2) \qquad\qquad -\operatorname{div} a(Du) = 0.$$

For the notation used in this paper we immediately refer to Section 2 below.

1.1. Classical Partial Regularity theory. — In the following, the vector field $a \colon \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ appearing in (1.1)–(1.2) is assumed to be C^1-regular satisfying the following ellipticity and growth assumptions:

$$(1.3) \qquad \begin{cases} |a(z)| + |\partial a(z)||z| \leqslant L|z|^{p-1} \\[2mm] \nu|z|^{p-2}|\xi|^2 \leqslant \langle \partial a(z)\xi, \xi \rangle \\[2mm] |\partial a(z_2) - \partial a(z_1)| \leqslant L\mu\left(\dfrac{|z_2 - z_1|}{|z_1| + |z_2|}\right)(|z_1| + |z_2|)^{p-2}, \end{cases}$$

for every choice of $z, z_1, z_2, \xi \in \mathbb{R}^{N \times n}$, $|z_1| + |z_2| \neq 0$. Here $n, N \geqslant 2$, $0 < \nu \leqslant L$, and $\mu \colon \mathbb{R}^+ \to [0,1]$ is a modulus of continuity i.e. a bounded, concave, and non-decreasing function such that $\mu(0) = 0$. In the rest of the paper we shall always assume that

$$(1.4) \qquad\qquad p \geqslant 2.$$

As it is clear from the above assumptions, we are allowing the vector field $a(\cdot)$ to be degenerate elliptic at the origin. Specifically, we assume that $a(\cdot)$ is asymptotically close to the p-Laplacean operator at the origin in the sense that the limit

$$(1.5) \qquad\qquad \lim_{t \to 0} \frac{a(tz)}{t^{p-1}} = |z|^{p-2}z$$

holds locally uniformly with respect to $z \in \mathbb{R}^{N \times n}$. This means that there exists a function $\eta \colon (0, \infty) \to (0, \infty)$ with the property

$$(1.6) \qquad |z| \leqslant \eta(s) \implies \left| a(z) - |z|^{p-2} z \right| \leqslant s |z|^{p-1} \quad \text{for every } z \in \mathbb{R}^{N \times n} \text{ and } s > 0.$$

The systems in (1.1)–(1.2) do not have any special additional structure. For instance, they are not of quasi-diagonal structure i.e. $a(z)$ cannot be put in the form $a(z) \equiv g(|z|)z$ as required by Uhlenbeck theory of everywhere regularity [47]. Under such assumptions only partial regularity can be proved, that is, Hölder continuity of the gradient outside a closed negligible set. Let us briefly recall the basics of this theory for the situation we are considering. The regular set of the solutions u to (1.2) is naturally defined as

$$\mathrm{Reg}\,(u) \equiv \Omega_u := \Big\{ x \in \Omega : \text{ there exists an}$$
$$\text{open subset } \Omega_x \subset \Omega \text{ such that } Du \in C^0(\Omega_x; \mathbb{R}^{N \times n}) \Big\},$$

and it can be proved it has full measure $|\Omega \smallsetminus \Omega_u| = 0$. Obviously, this is an open subset of Ω and actually Du is locally Hölder continuous in Ω_u. The regular set Ω_u admits an integral characterization through the excess functional

$$(1.7) \qquad E(Du, B) := \left(\fint_B |(Du)_B|^{p-2} |Du - (Du)_B|^2 + |Du - (Du)_B|^p \, dx \right)^{1/p}.$$

The main point of partial regularity is that it relies on so called ε-regularity criteria. This means that there exists a universal threshold quantity $\varepsilon \equiv \varepsilon(n, N, p, \nu, L)$ such that $x_0 \in \mathrm{Reg}\,(u)$ iff there exists a ball $B_r(x_0) \subset \Omega$ such that

$$(1.8) \qquad\qquad\qquad E\left(Du, B_r(x_0)\right) < \varepsilon.$$

For results of this type under the structure assumptions (1.3) see for instance [12, 15]. Partial regularity is a classical topic, dating back to the pioneering contributions of De Giorgi [10], Giusti & Miranda [20] and Morrey [41]. It extends to integral functionals of the Calculus of Variations [1, 12, 17, 19, 27, 28, 44]. We refer to the survey paper [39] for an updated overview of the subject. We now want to put Partial Regularity in the context of Nonlinear Potential Theory, and vice-versa. For this we recall a few basic concepts in the next section.

1.2. Nonlinear Potential Theory. — The aim of Nonlinear Potential Theory is to provide a satisfying description of fine properties of solutions to nonlinear, possibly degenerate equations, resembling as much as possible the ones of harmonic functions. In particular, the classical pointwise gradient estimates for solutions to the Poisson equation $-\triangle u = \mu$ can be recast for solutions to quasilinear, possibly degenerate equations of the type

$$(1.9) \qquad\qquad\qquad -\operatorname{div} a(Du) = f.$$

Here f denotes in the most general case a Borel measure with finite total mass. In this case, when referring to the gradient of solutions, Riesz potentials come into the

play:

$$\boldsymbol{I}_1^f(x_0, r) := \int_0^r \frac{|f|(B_\varrho(x_0))}{\varrho^{n-1}} \frac{d\varrho}{\varrho}.$$

Pointwise estimates for solutions u can be instead produced by using the nonlinear Wolff potential $\boldsymbol{W}_{1,p}^\mu$ as first shown in [25], where

$$(1.10) \qquad \boldsymbol{W}_{\beta,p}^f(x_0, r) := \int_0^r \Big(\frac{|f|(B_\varrho(x_0))}{\varrho^{n-\beta p}} \Big)^{1/(p-1)} \frac{d\varrho}{\varrho}, \qquad \beta > 0.$$

Wolff potentials have been first introduced in [37]; important results are in [22]. Let us for instance recall two results proved in [32] (see also [40]). Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation in (1.9) under assumptions (1.3)–(1.4); there exists a constant c depending only on n, p, ν, L such that

$$(1.11) \qquad |Du(x_0)|^{p-1} \lesssim \boldsymbol{I}_1^f(x_0, r) + \Big(\fint_{B_r(x_0)} |Du| \, dx \Big)^{p-1}$$

holds whenever $B_r(x_0) \subset \Omega$ and the right hand side is finite. A crucial point here, which is bound to reproduce in the nonlinear setting a classical linear potential theory criterion, is that the finiteness of the right hand of (1.11) implies that Du admits a precise representative at x_0. This means that the limit in (1.18) below exists and thereby defines the pointwise value of Du at x_0. Moreover, if

$$\lim_{\varrho \to 0} \boldsymbol{I}_1^f(x, \varrho) = 0 \ \text{ locally uniformly in } \Omega \text{ w.r.t. } x,$$

then Du is continuous in Ω. When turning to equations different from the one in (1.9), it happens that a theory including right-hand side data being measures is not available. In this case modified Riesz type potentials $\boldsymbol{I}_{1,q}$ come into the play. These are nonlinear potentials of the type in (1.10) with the same *homogeneity and scaling, and ultimately, mapping properties* of the classical Riesz potential $\boldsymbol{I}_1$. The difference is that they are defined on functions having a certain degree of integrability rather than measures. Such degree of integrability relates to the admissible right-hand sides of the operator considered. An instance of their use is provided in [9], where fully nonlinear equations of the type

$$(1.12) \qquad\qquad\qquad F(D^2 u) = f$$

are considered. Such equations allow good regularity theory in the setting of viscosity solutions only in the case $f \in L^q$, for some q very close to n. Standard Riesz potentials $\boldsymbol{I}_1^f$ acting on measures are therefore ruled out. To overcome this problem the basic observation is that Hölder inequality allows to define by homogeneity a new related potential $\boldsymbol{I}_{1,q}^f$ as follows:

$$(1.13) \qquad \begin{aligned} \boldsymbol{I}_1^f(x_0, r) &\approx \int_0^r \fint_{B_\varrho(x_0)} |f| \, dx \, d\varrho \\ &\leqslant \int_0^r \Big(\varrho^q \fint_{B_\varrho(x_0)} |f|^q \, dx \Big)^{1/q} \frac{d\varrho}{\varrho} \\ &=: \boldsymbol{I}_{1,q}^f(x_0, r). \end{aligned}$$

The newly defined Riesz type potential $\boldsymbol{I}^f_{1,q}$ is actually a Wolff potential in the sense that the following identity holds:

$$(1.14) \qquad \boldsymbol{I}^f_{1,q}(x_0, r) = |B_1|^{-1/q} \boldsymbol{W}^{|f|^q}_{q/(q+1),q+1}(x_0, r).$$

A precise analog of estimate (1.11), that is,

$$(1.15) \qquad |Du(x_0)| \lesssim \boldsymbol{I}^f_{1,q}(x_0, r) + \left(\fint_{B_r(x_0)} |Du|^\gamma \, dx \right)^{1/\gamma},$$

now holds for solutions to (1.12), for some $\gamma > n$, together with the characterization of Lebesgue points of Du via finiteness of $\boldsymbol{I}^f_{1,q}$. Estimate (1.15) allows to control Du via $\boldsymbol{I}^f_{1,q}$ exactly as it happens for solutions to equations as in (1.9) via $\boldsymbol{I}^f_1$ and formula (1.11). Regularity in the relevant classes of rearrangement invariant functions spaces for Du then follows via the mapping properties of the operator $X \ni f \mapsto \boldsymbol{I}^f_{1,q} \in Y$. These properties can be reconstructed via (1.14) since those of the Wolff potential are known [2, 6, 37], and are similar to those of the classical Riesz potential $\boldsymbol{I}_1$. The only difference is the that largest space on which $\boldsymbol{I}_{1,q}$ is acting on is not any longer L^1 but, obviously, L^q. For more details we again refer to [9].

1.3. MAIN RESULTS. — In this section we show how to connect partial regularity and Nonlinear Potential Theory; this is done using modified Riesz potentials as defined in (1.13). The key idea, leading to optimal potential estimates for general elliptic systems, is to match smallness conditions of the type in (1.8) with similar smallness conditions on potentials. We shall therefore consider non-homogeneous elliptic systems of the type (1.1), where $a(\cdot)$ satisfies (1.3)–(1.5), $f\colon \Omega \to \mathbb{R}^N$. For such systems an existence theory of problems with (vector valued) measure data f is not available and therefore the initial assumption we make on f is

$$(1.16) \qquad f \in L^q(\Omega; \mathbb{R}^N), \quad \text{where} \ \ 2 > q > \begin{cases} \dfrac{2n}{n+2} & \text{if } n > 2 \\[2mm] 3/2 & \text{if } n = 2. \end{cases}$$

Eventually we consider $f \in L^q(\mathbb{R}^n; \mathbb{R}^N)$ by letting $f \equiv 0$ outside Ω. Notice that the above lower bound on q implies that $f \in (W^{1,p})'$ and therefore solutions to (1.1) can be found by standard monotonicity methods in $W^{1,p}$. We are therefore in the realm of traditional *energy solutions*. The first result is the following:

THEOREM 1.1. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16), and let $B_r(x_0) \subset \Omega$ be a ball. There exists a number $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)) > 0$ such that if*

$$(1.17) \qquad E\left(Du, B_r(x_0)\right) + \left[\boldsymbol{I}^f_{1,q}(x_0, r)\right]^{1/(p-1)} < \varepsilon,$$

then the limit

$$(1.18) \qquad \lim_{\varrho \to 0} (Du)_{B_\varrho(x_0)} =: Du(x_0)$$

exists and defines the precise representative of Du at x_0. Moreover, the local oscillation estimate

$$(1.19) \qquad |Du(x_0) - (Du)_{B_\varrho(x_0)}| \leqslant c \left[\boldsymbol{I}_{1,q}^f(x_0, \varrho) \right]^{1/(p-1)} + cE\left(Du, B_\varrho(x_0)\right)$$

holds for every concentric ball $B_\varrho(x_0) \subset B_r(x_0)$, where the constant c depends only on $n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)$.

Requiring condition (1.17) is very close to (1.8), because if $\boldsymbol{I}_{1,q}^f(x_0, r)$ is finite at x_0, then $\boldsymbol{I}_{1,q}^f(x_0, \varrho)$ can be made arbitrarily small by taking $\varrho \leqslant r$ small enough. This is in fact the key observation to prove the sharp partial regularity result of the next theorem, together with a characterization of the singular set, which is similar to the one for homogeneous systems (1.8).

Theorem 1.2. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16). If*

$$(1.20) \qquad \lim_{\varrho \to 0} \boldsymbol{I}_{1,q}^f(x, \varrho) = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t. } x,$$

then there exists an open subset $\Omega_u \subset \Omega$ such that

$$(1.21) \qquad |\Omega \smallsetminus \Omega_u| = 0 \qquad \text{and} \qquad Du \in C^0(\Omega_u; \mathbb{R}^{N \times n}).$$

Moreover, there exists a positive constant ε_s and a positive radius ϱ_s, such that

$$(1.22) \qquad \Omega_u = \left\{ x \in \Omega : \exists\, B_\varrho(x) \Subset \Omega \text{ with } \varrho \leqslant \varrho_s \, : \, E(Du, B_\varrho(x)) < \varepsilon_s \right\}.$$

The constant ε_s depends only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$, while ϱ_s depends on the same parameters and additionally on the rate of convergence in (1.20).

Notice that estimate (1.19) in particular implies the potential estimate

$$|Du(x_0)|^{p-1} \leqslant c \boldsymbol{I}_{1,q}^f(x_0, r) + c \left(\fint_{B_r(x_0)} |Du|^p \, dx \right)^{(p-1)/p},$$

which is analogous to the one in (1.11). More in general we see that Theorems 1.1–1.2 draw a precise partial regularity analog of the results in Section 1.2.

Theorem 1.2 allows to transport to the setting of partial regularity the classical Lorentz space borderline criterion for Lipschitz continuity of solutions and, eventually, for gradient continuity. This has been the object of intensive investigation over the last years and has been previously established for equations and for systems with quasi diagonal structure in [7, 8, 9, 13, 32, 35]. No result was available for general elliptic systems.

Theorem 1.3. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5). If*

$$(1.23) \qquad f \in L(n, 1) \quad \text{holds locally in } \Omega,$$

then (1.21) holds with the open subset Ω_u described in (1.22).

We just recall here that (1.23) means that

$$\int_0^\infty |\{x \in \widetilde{\Omega} : |f(x)| > \lambda\}|^{1/n} \, d\lambda < \infty$$

holds for every open subset $\widetilde{\Omega} \Subset \Omega$. Further estimates and corollaries in various function spaces can be then obtained by using mapping properties of potentials; see for instance [3, 6].

A preliminary step in the proof of the above results gives a VMO-criterion for the gradient, again leading to further partial regularity results.

THEOREM 1.4. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16). If*

$$(1.24) \qquad \lim_{\varrho \to 0} \varrho^q \fint_{B_\varrho(x)} |f|^q \, dy = 0$$

holds locally uniformly in Ω with respect to x, then there exists an open subset $\Omega_u \subset \Omega$ such that

$$(1.25) \qquad |\Omega \smallsetminus \Omega_u| = 0 \qquad and \qquad Du \in \mathrm{VMO}_{\mathrm{loc}}(\Omega_u; \mathbb{R}^{N \times n}).$$

A characterization of Ω_u as in (1.22) holds for different values ε_s and ϱ_s, with the dependence as in Theorem 1.2 and on the rate of convergence in (1.24).

The previous result reproduces the sharp criteria for VMO-regularity of the gradient available in the scalar case, both for quasilinear and fully-nonlinear equations; see for instance [9] and references therein. The parallel can be pushed further, mixing ε-regularity criteria and different borderline cases.

THEOREM 1.5. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16). There exists a number ε_*, depending only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$ such that, if*

$$(1.26) \qquad \sup_{B_\varrho \subset \Omega} \varrho^q \fint_{B_\varrho} |f|^q \, dy < \varepsilon_*,$$

then there exists an open subset $\Omega_u \subset \Omega$ such that

$$(1.27) \qquad |\Omega \smallsetminus \Omega_u| = 0 \qquad and \qquad Du \in \mathrm{BMO}_{\mathrm{loc}}(\Omega_u; \mathbb{R}^{N \times n}).$$

The set Ω_u can be again characterized as in (1.22). In particular, this holds if $f \in \mathscr{M}^n(\Omega) \equiv L(n, \infty)(\Omega)$ and

$$(1.28) \qquad \|f\|_{\mathscr{M}^n(\Omega)} := \sup_{\lambda > 0} \lambda |\{x \in \Omega : |f(x)| > \lambda\}|^{1/n} < \left(\frac{|B_1|}{4}\right)^{1/n} \varepsilon_*^{1/q}.$$

The so-called singular set $\Omega \smallsetminus \Omega_u$ is not only negligible, but actually an estimate on its Hausdorff dimension $\dim_{\mathscr{H}}(\Omega \smallsetminus \Omega)$ can be provided in the case of Theorems 1.2, 1.3, 1.4 and 1.5. This is possible thanks to the characterization of Ω_u given in (1.22). The result follows by a gradient differentiability argument in fractional Sobolev spaces. It is in the following:

Theorem 1.6. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16); assume in addition that $f \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. In the situation of Theorems 1.2, 1.3, 1.4 and 1.5 we then have that the Hausdorff dimension of the singular set can be estimated as follows:*

$$\dim_{\mathscr{H}} (\Omega \smallsetminus \Omega_u) \leqslant n - \frac{p}{p-1}. \tag{1.29}$$

Remark 1.1. — We observe that, when p is large enough, the integrability assumption $f \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ considered in the above theorem is not really an additional one with respect to the previously considered integrability in (1.16). Specifically, it is implied by (1.16) in the case $p \geqslant 2^* = 2n/(n+2)$, for $n > 2$, and $p \geqslant 3$ when $n = 2$. The last number is the Sobolev conjugate exponent of 2. We also notice that assuming some more regularity of f it is possible to get improved estimates for the singular set; see Remark 6.1 and (6.84) below.

We finally propose a different, more technical version of Theorem 1.1, which is anyway significant in the setting of classical partial regularity. Indeed, for systems as in (1.2) the local regularity properties of the solutions are usually expressed via a non-linear vector field of the gradient, which encodes the degeneracy properties of the system, that is,

$$V(Du) := |Du|^{(p-2)/2} Du. \tag{1.30}$$

As a matter of fact, the following equivalence between the excess functional defined in (1.7) and the natural one involving the quantity in (1.30)

$$[E(Du, B)]^{p/2} \approx \widetilde{E}(Du, B) \tag{1.31}$$

where

$$\widetilde{E}(Du, B) := \left(\fint_B |V(Du) - (V(Du))_B|^2 \, dx \right)^{1/2}. \tag{1.32}$$

For this fact see Section 2.3 below. We then have

Theorem 1.7. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a solution to (1.1) under assumptions (1.3)–(1.5) and (1.16), and let $B_r(x_0) \subset \Omega$ be a ball. There exists a number $\widetilde{\varepsilon} \equiv \widetilde{\varepsilon}(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)) > 0$ such that if*

$$\widetilde{E}(Du, B_r(x_0)) + \left[\boldsymbol{I}^f_{1,q}(x_0, r) \right]^{p/[2(p-1)]} < \widetilde{\varepsilon} \tag{1.33}$$

holds, then the following limit exists

$$\lim_{\varrho \to 0} (V(Du))_{B_\varrho(x_0)} =: V(Du)(x_0). \tag{1.34}$$

Moreover, the precise representative of the composition of $V(\cdot)$ and Du at x_0 defined in (1.34) coincides with the composition of $V(\cdot)$ with the precise representative of Du at x_0 defined in (1.18), i.e.

$$V(Du)(x_0) = V(Du(x_0)). \tag{1.35}$$

Finally, the local oscillation estimate

$$(1.36) \quad |V(Du(x_0)) - (V(Du))_{B_\varrho(x_0)}| \leqslant c \left[\boldsymbol{I}^f_{1,q}(x_0, \varrho) \right]^{p/[2(p-1)]} + c\widetilde{E}\left(Du, B_\varrho(x_0)\right)$$

holds for every concentric ball $B_\varrho(x_0) \subset B_r(x_0)$ where the constant c depends only on $n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)$.

We conclude this section by spending a few words on the techniques used and briefly describing the organization of the paper. In Section 2 we collect a few technical preliminaries. In Sections 3 and 4 we implement the local linearization procedures, which are necessary to get partial regularity results for solutions. Decay estimates for the excess functionals are obtained; we pay special attention in tracing back the dependence of the inequalities on the natural additional terms stemming from the right-hand side datum f in (1.1). At this stage a careful proof is needed in order to get estimates with a dependence of the constants that eventually allow to implement suitable iteration procedures. To achieve the final results we use a few compactness lemmas of harmonic approximation type (see Section 2.4 below). At this stage the combined ε-regularity condition involving both the excess functional $E(\cdot)$ and the potential $\boldsymbol{I}^f_{1,q}$ appearing in (1.17) comes into the play to guarantee the conditions for the excess decay. In Section 5 we start combining the excess decay lemmas in order to get pointwise BMO and VMO estimates for the gradient of solutions. These provide the proof of Theorem 1.4 and a preliminary step for the proof of Theorems 1.1–1.3. The final Section 6 contains the proof of the remaining results; these are based on a rather careful combination of the excess decay estimates obtained in Sections 3 and 4 and on the preliminary VMO-regularity criteria established in Section 5.

2. Notation, preliminaries

2.1. Notation. — We start fixing a few basic notation. Starting from (1.1), $\Omega \subset \mathbb{R}^n$ will always denote an open subset, and we shall always be dealing with the multi-dimensional case $n \geqslant 2$. In this paper constants are denoted by c; these are larger or equal than one and can change their precise value in different occurrences. Relevant dependence on parameters will be indicated using parentheses. For instance, a constant c depending only on other quantities indicated by n, N, p, ν, L is denoted by $c \equiv c(n, N, p, \nu, L)$; dependence on additional parameters can be indicated occasionally with no parameters. In the following $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ denotes the open ball with center x and radius $r > 0$. When not important, or it is clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x_0)$. We shall

often abbreviate $B_1 \equiv B_1(0)$. With $\mathscr{O} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $g\colon \mathscr{O} \to \mathbb{R}^k$, $k \geqslant 1$, being a measurable map, we shall denote by

$$(g)_{\mathscr{O}} \equiv \fint_{\mathscr{O}} g\,dx := \frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} g(x)\,dx$$

its integral average; here $|\mathscr{O}|$ denotes the Lebesgue measure of $\mathscr{O}$. In the rest of the paper we shall use several times the following elementary property of integral averages:

$$(2.1) \qquad \left(\fint_{\mathscr{O}} |g - (g)_{\mathscr{O}}|^{\gamma}\,dx \right)^{1/\gamma} \leqslant 2 \left(\fint_{\mathscr{O}} |g - z|^{\gamma}\,dx \right)^{1/\gamma},$$

whenever $z \in \mathbb{R}^{N \times n}$ and $\gamma \geqslant 1$; when $\gamma = 2$ the constant 2 can be replaced by 1. Finally, with $q > 1$, its conjugate will be defined as $q' = q/(q-1)$.

2.2. Various technical results. — The following is a standard algebraic lemma; see for instance [21] for the proof.

Lemma 2.1. — *For every $s > -1/2$ we have*

$$\frac{1}{c}(|z_1|^2 + |z_0|^2)^s \leqslant \int_0^1 |z_1 + t(z_1 - z_0)|^{2s}\,dt \leqslant c(|z_1|^2 + |z_0|^2)^s$$

for any $z_0, z_1 \in \mathbb{R}^{N \times n}$ and a constant $c \equiv c(s)$.

Assume now that $u \in L^2(B_{\varrho}(x_0); \mathbb{R}^N)$ and denote by $\ell_{x_0, \varrho}$ the unique affine function realizing

$$\ell_{x_0, \varrho} \mapsto \min_{\ell} \fint_{B_{\varrho}(x_0)} |u - \ell|^2\,dx$$

amongst all the affine functions ℓ. We have $\ell_{x_0, \varrho}(x) = u_{x_0, \varrho} + D\ell_{x_0, \varrho}(x - x_0)$ where

$$D\ell_{x_0, \varrho} = \frac{n+2}{\varrho^2} \fint_{B_{\varrho}(x_0)} u(x) \otimes (x - x_0)\,dx.$$

Then, the following properties hold (see for instance [29]):

Lemma 2.2. — *Let $p \geqslant 2$. There exists a constant $c \equiv c(n, p)$ such that the following assertions hold: for every $u \in L^p(B_{\varrho}(x_0); \mathbb{R}^N)$ it holds that*

$$(2.2) \qquad |D\ell_{x_0, \varrho} - D\ell_{x_0, \tau\varrho}|^p \leqslant \frac{c}{(\tau\varrho)^p} \fint_{B_{\tau\varrho}(x_0)} |u - \ell_{x_0, \varrho}|^p\,dx.$$

Moreover, for every $u \in W^{1,p}(B_{\varrho}(x_0); \mathbb{R}^N)$ it holds

$$(2.3) \qquad |D\ell_{x_0, \varrho} - (Du)_{B_{\varrho}}|^p \leqslant c \fint_{B_{\varrho}(x_0)} |Du - (Du)_{B_{\varrho}}|^p\,dx.$$

We still have the following lemma:

Lemma 2.3. — *Let $p \geqslant 2$; there exists a constant $c \equiv c(n, N, p)$ such that*

$$\fint_{B_{\varrho}(x_0)} |u - \ell_{x_0, \varrho}|^p\,dx \leqslant c \fint_{B_{\varrho}(x_0)} |u - \ell|^p\,dx,$$

where ℓ varies amongst affine functions.

Proof. — The result is classical and relies on the fact that on a finite vector spaces all norms are equivalent (in this case the space is given by the set of affine functions). We report the simple proof for completeness. We observe that by a standard scaling argument we can reduce ourselves to the case $B_\varrho(x_0) \equiv B_1$. Then, we have

$$\left(\fint_{B_1} |u - \ell_{0,1}|^p \, dx \right)^{1/p} \leqslant \left(\fint_{B_1} |u - \ell|^p \, dx \right)^{1/p} + \left(\fint_{B_1} |\ell_{0,1} - \ell|^p \, dx \right)^{1/p}.$$

In turn, we have, using the minimality property of $\ell_{0,1}$ and the finite dimension argument mentioned above we have

$$\left(\fint_{B_1} |\ell_{0,1} - \ell|^p \, dx \right)^{1/p} \leqslant c \left(\fint_{B_1} |\ell_{0,1} - \ell|^2 \, dx \right)^{1/2}$$

$$\leqslant c \left(\fint_{B_1} |u - \ell_{0,1}|^2 \, dx \right)^{1/2} + c \left(\fint_{B_1} |u - \ell|^2 \, dx \right)^{1/2}$$

$$\leqslant 2c \left(\fint_{B_1} |u - \ell|^2 \, dx \right)^{1/2}$$

$$\leqslant 2c \left(\fint_{B_1} |u - \ell|^p \, dx \right)^{1/p}.$$

The assertion of the lemma follows combining the content of the last two displays. □

2.3. EXCESS FUNCTIONALS. — In the following we shall several times use the excess functional given by

$$(2.4) \qquad E(Du, z_0, B) := \left(\fint_B \left[|z_0|^{p-2} |Du - z_0|^2 + |Du - z_0|^p \right] \, dx \right)^{1/p},$$

where $z_0 \in \mathbb{R}^{N \times n}$ and $B \subset \Omega$ is a ball. In the case $z_0 \equiv (Du)_B$ we shall simply denote $E(Du, (Du)_B, B) \equiv E(Du, B)$, in fact recovering the notation fixed in (1.7). We shall widely use the map $V \colon \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ defined by

$$(2.5) \qquad\qquad\qquad V(z) := |z|^{(p-2)/2} z.$$

The inequality

$$(2.6) \quad \frac{1}{c}(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2 \leqslant |V(z_1) - V(z_2)|^2 \leqslant c(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2,$$

holds whenever $z_1, z_2 \in \mathbb{R}^{N \times n}$ and for a constant c depending only on n, N, p. This in turn implies that the map $V(\cdot)$ is locally bi-Lipschitz regular. For the proof we for instance refer to [21]. We now collect a few consequences of these definitions and facts. Starting from the map $V(\cdot)$ introduced in (2.5), with $B \subset \Omega$ we define the new excess type functional

$$(2.7) \qquad\qquad \overline{E}(Du, B) := \left(\fint_B |V(Du) - V((Du)_B)|^2 \, dx \right)^{1/2},$$

which is obviously related to the one defined in (1.32) and to the original excess functional $E(\cdot)$ defined in (1.7). It turns out that all these functionals are equivalent,

thereby substantiating the claim made in (1.31). Indeed observe that as an immediate consequence of (2.6) we have that

$$(2.8) \qquad \frac{[E(Du,B)]^{p/2}}{c} \leqslant \overline{E}(Du,B) \leqslant c[E(Du,B)]^{p/2}$$

holds for a constant $c \equiv c(n,N,p)$. On the other hand the inequality

$$(2.9) \qquad \widetilde{E}(Du,B) \leqslant \overline{E}(Du,B)$$

is a direct consequence of (2.1) and the remark in the subsequent line. The last inequality is

$$(2.10) \qquad \overline{E}(Du,B) \leqslant c\widetilde{E}(Du,B)$$

and again holds for a constant $c \equiv c(n,N,p)$. For this we refer to [18, (2.6)].

2.4. Harmonic type approximation lemmas. — In this section we report a few basic properties of A-harmonic maps, i.e. solutions to elliptic systems with constants coefficients, and together with compactness lemmas yield approximation properties. We then do the same in the case of p-harmonic maps.

Let us consider a bilinear form $\mathscr{A}$ defined on $\mathbb{R}^{N \times n}$ satisfying conditions

$$(2.11) \qquad |\mathscr{A}| \leqslant L \quad \text{and} \quad \nu|\xi|^2 \leqslant \mathscr{A}(\xi,\xi) \quad \text{for every } \xi \in \mathbb{R}^{N \times n}.$$

We shall say that a map $h \in W^{1,2}(\Omega;\mathbb{R}^N)$ is an $\mathscr{A}$-harmonic map in Ω provided

$$\int_\Omega \mathscr{A}(Dh, D\varphi)\, dx = 0 \qquad \text{for every } \varphi \in C_0^\infty(\Omega;\mathbb{R}^N).$$

We then have the following different version of the A-harmonic approximation lemma; it can be easily obtained by the similar $\mathscr{A}$-caloric one build for the parabolic case in [14, Chap. 3].

Lemma 2.4 ($\mathscr{A}$-harmonic approximation). — *For each $\nu, L, \varepsilon > 0$, $p > 1$ and $d \in (0,1]$, there exists a positive number $\delta \in (0,1)$, depending on $n, N, \nu, L, \varepsilon$, but not on d, with the following property: Assume that $\mathscr{A}$ is a bilinear form on $\mathbb{R}^{N \times n}$ satisfying conditions (2.11) and moreover, assume that $\varrho > 0$, and $w \in W^{1,2}(B_\varrho;\mathbb{R}^N)$, with*

$$\fint_{B_\varrho} |Dw|^2\, dx + d^{p-2} \fint_{B_\varrho} |Dw|^p\, dx \leqslant 1$$

is approximatively $\mathscr{A}$-harmonic in the sense that

$$(2.12) \qquad \left| \fint_{B_\varrho} \mathscr{A}(Dw, D\varphi)\, dx \right| \leqslant \delta \|D\varphi\|_{L^\infty(B_\varrho)}$$

holds for every $\varphi \in C_0^\infty(B_\varrho, \mathbb{R}^N)$. Then there exists an $\mathscr{A}$-harmonic function $h \in W^{1,2}(B_\varrho;\mathbb{R}^N)$ such that

$$\fint_{B_{3\varrho/4}} |Dh|^2\, dx + d^{p-2} \fint_{B_{3\varrho/4}} |Dh|^p\, dx \leqslant 8^{2np}$$

and

$$\fint_{B_{3\varrho/4}} \left|\frac{w-h}{\varrho}\right|^2 + d^{p-2}\left|\frac{w-h}{\varrho}\right|^p dx \leqslant \varepsilon.$$

A fundamental result of Uraltseva [48] and Uhlenbeck [47] states that the gradient of p-harmonic maps h – that is, solutions to (2.13) below – are locally Hölder continuous. This can be quantified in a precise way as proved in [16, 18]. The result is in the following:

THEOREM 2.1. — *There exist constants $c \geqslant 1$ and $\alpha \in (0,1)$, depending only on n, N and $p > 1$ with the following property: Whenever $h \in W^{1,p}(\Omega;\mathbb{R}^N)$ solves*

$$(2.13) \qquad \int_\Omega |Dh|^{p-2}\langle Dh, D\varphi\rangle \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega;\mathbb{R}^N),$$

and $B_R(x_0) \subset \Omega$ then, for any $0 < r \leqslant R$ the following estimates hold:

$$(2.14) \qquad \sup_{B_{R/2}(x_0)} |Dh|^p \leqslant c \fint_{B_R(x_0)} |Dh|^p \, dx$$

and

$$(2.15) \qquad E(Dh, B_r(x_0)) \leqslant c \left(\frac{r}{R}\right)^\alpha E(Dh, B_R(x_0)),$$

where $E(\cdot)$ has been defined in (1.7).

The next results now tells that almost p-harmonic maps can be approximated via compactness methods by genuine p-harmonic maps exactly as almost harmonic maps can be approximated by genuine harmonic maps. In the case $p = 2$, the next lemma is classical and has been first used by De Giorgi in his proof of regularity of minimal surfaces [10].

LEMMA 2.5 (*p-harmonic approximation lemma [11]*). — *Let $p \geqslant 2$. For every $\varepsilon > 0$ and $p_1 < p$ there exists a positive constant $\delta \in (0,1]$, depending only on n, N, p, p_1 and ε, such that the following is true: Whenever $w \in W^{1,p}(B_\varrho;\mathbb{R}^N)$ with*

$$(2.16) \qquad \fint_{B_\varrho} |Dw|^p \, dx \leqslant 1$$

is approximatively p-harmonic in the sense that

$$\left|\fint_{B_\varrho} |Dw|^{p-2}\langle Dw, D\varphi\rangle \, dx\right| \leqslant \delta \, \|D\varphi\|_{L^\infty(B_\varrho)}$$

holds for every $\varphi \in C_0^\infty(B_\varrho;\mathbb{R}^N)$, then there exists a p-harmonic map $h \in W^{1,p}(B_\varrho;\mathbb{R}^N)$ such that

$$\fint_{B_\varrho} |Dh|^p \, dx \leqslant 1 \quad \text{and} \quad \fint_{B_\varrho} |Dw - Dh|^{p_1} \, dx \leqslant \varepsilon.$$

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3. Decay estimates in the non-degenerate case

We a start with a suitable variant of a classical energy estimate.

Proposition 3.1 (Caccioppoli inequality). — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). There exists a constant $c \equiv c(n, N, p, \nu, L)$ such that for every ball $B_\varrho(x_0) \subset \Omega$, $u_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N \times n} \smallsetminus \{0\}$, the inequality*

$$(3.1) \quad \fint_{B_{\varrho/2}} \left[|z_0|^{p-2}|Du - z_0|^2 + |Du - z_0|^p \right] dx$$
$$\leqslant c \fint_{B_\varrho} \left[|z_0|^{p-2}\left|\frac{u-\ell}{\varrho}\right|^2 + \left|\frac{u-\ell}{\varrho}\right|^p \right] dx + \frac{c}{|z_0|^{p-2}} \left(\varrho^q \fint_{B_\varrho} |f|^q\, dx \right)^{2/q}$$

holds for a constant $c \equiv c(n, p, \nu, L)$, where $\ell(x) = u_0 + \langle z_0, x - x_0 \rangle$ and q is the number defined in (1.16).

Proof. — We start re-writing the weak formulation of (1.1) as

$$(3.2) \quad \int_\Omega \int_0^1 \langle \partial a(z_0 + t(Du - z_0))(Du - z_0), D\varphi \rangle\, dt\, dx$$
$$= \int_\Omega \langle a(Du) - a(z_0), D\varphi \rangle\, dx = \int_\Omega \langle f, \varphi \rangle\, dx$$

for every $\varphi \in C_0^\infty(\Omega; \mathbb{R}^{N \times n})$. We choose a standard cut-off function $\phi \in C_0^\infty(B_\varrho)$ such that $0 \leqslant \phi \leqslant 1$, $\phi \equiv 1$ on $B_{\varrho/2}$ and $|D\phi| \leqslant 10/\varrho$. We take as a test function set $\varphi = \phi^2(u - \ell)$. By using the growth and ellipticity assumptions in $(1.3)_{1,2}$ together with standard manipulations we get

$$\int_\Omega \int_0^1 \phi^2 |z_0 + t(Du - z_0)|^{p-2}|Du - z_0|^2\, dt\, dx$$
$$\leqslant c \int_\Omega \int_0^1 \phi |z_0 + t(Du - z_0)|^{p-2}|Du - z_0|\, |u - \ell|\, |D\phi|\, dt\, dx$$
$$+ c \int_\Omega \phi^2 |f|\, |u - \ell|\, dx.$$

Thanks to Lemma 2.1 we have

$$\int_0^1 |z_0 + t(Du - z_0)|^{p-2}\, dt \approx (|z_0| + |Du - z_0|)^{p-2} \approx (|z_0| + |Du|)^{p-2},$$

where the implied constant only depends on n, N, p, ν, L. Therefore, by using Young's inequality in a standard way we get

$$(3.3) \quad \fint_{B_\varrho} \phi^2 \left[|z_0|^{p-2}|Du - z_0|^2 + |Du - z_0|^p \right] dx$$
$$\leqslant c \fint_{B_\varrho} \left[|z_0|^{p-2}\left|\frac{u-\ell}{\varrho}\right|^2 + \left|\frac{u-\ell}{\varrho}\right|^p \right] dx + c \fint_{B_\varrho} \phi |f|\, |u - \ell|\, dx.$$

Notice than in order to re-absorb the p-terms we have used the fact that $\phi^p \leqslant \phi^2$ since $0 \leqslant \phi \leqslant 1$. The main task is now to estimate the last integral in the above display. By using Hölder inequality we have

$$\fint_{B_\varrho} \phi|f||u - \ell|\,dx \leqslant \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx\right)^{1/q} \left(\fint_{B_\varrho} \left|\frac{\phi(u-\ell)}{\varrho}\right|^{q'}\,dx\right)^{1/q'}.$$

We now recall some standard notation. In the following 2^* denotes the usual Sobolev conjugate exponent (i.e. $2^* = 2n/(n-2)$ for $n > 2$ and $2^* =$ any number larger than one otherwise) and $(2^*)' = 2^*/(2^* - 1)$. In particular observe that in the case $n > 2$ we have $(2^*)' = 2n/(n + 2)$ and that in the case $n = 2$ we can pick 2^* large enough in order to have $(2^*)' \leqslant 3/2$. In any case we conclude that $(2^*)' \leqslant q$, where q has been defined in (1.16), and therefore we have that $2^* \geqslant q'$. Using this last fact, continuing from the last display we have, by Sobolev and Young inequalities, and for every $\sigma \in (0, 1)$, we have

$$(3.4) \quad \fint_{B_\varrho} \phi|f|\,|u - \ell|\,dx \leqslant \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx\right)^{1/q} \left(\fint_{B_\varrho} \left|\frac{\phi(u-\ell)}{\varrho}\right|^{2^*}\,dx\right)^{1/2^*}$$

$$\leqslant \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx\right)^{1/q} \left(\fint_{B_\varrho} \phi^2|Du - D\ell|^2\,dx + \fint_{B_\varrho} |D\phi|^2|u - \ell|^2\,dx\right)^{1/2}$$

$$\leqslant \frac{1}{\sigma|z_0|^{p-2}} \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx\right)^{2/q}$$

$$+ c\fint_{B_\varrho} |z_0|^{p-2} \left|\frac{u - \ell}{\varrho}\right|^2\,dx + \sigma \fint_{B_\varrho} \phi^2|z_0|^{p-2}|Du - z_0|^2\,dx.$$

Connecting this last inequality with the one in (3.3) and choosing $\sigma \equiv \sigma(n, N, p, \nu, L)$ small enough in order to re-absorb the last integral in (3.4) in the left hand side of (3.3), we obtain (3.1). The proof is complete. $\qquad\square$

The next result is concerned with a classical self-improving property of solutions to elliptic systems. The point here is to gain the right explicit dependence on the additional terms containing the assigned datum f.

PROPOSITION 3.2 (Reverse Hölder inequality). — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system* (1.1) *under assumptions* (1.3)–(1.5) *and* (1.16). *There exists a higher integrability exponent $p_2 \equiv p_2(n, N, p, \nu, L) > p$, such that $Du \in L^{p_2}_{\mathrm{loc}}(\Omega; \mathbb{R}^{N \times n})$. Moreover, the following reverse Hölder type inequality*

$$(3.5) \quad \left(\fint_{B_{\varrho/2}} |Du|^{p_2}\,dx\right)^{1/p_2} \leqslant c\left(\fint_{B_\varrho} |Du|^p\,dx\right)^{1/p} + c\left(\varrho^q \fint_{B_\varrho} |f|^q\,dx\right)^{1/[q(p-1)]}$$

holds for a constant $c \equiv c(n, N, p, \nu, L)$, whenever $B_\varrho \subset \Omega$ is a ball.

Proof. — First of all, let us observe that by a simple scaling argument we can reduce to the case we are proving (3.5) in the case $B_\varrho \equiv B_\varrho(x_0) \equiv B_1$. This follows by

considering the new maps

$$\widetilde{u}(x) := \frac{u(x_0 + \varrho x)}{\varrho} \quad \text{and} \quad \widetilde{f}(x) := \varrho f(x_0 + \varrho x), \qquad x \in B_1,$$

and solving $-\operatorname{div} a(D\widetilde{u}) = \widetilde{f}$ in B_1. Applying the proposition to $\widetilde{u}, \widetilde{f}$ and scaling back to u, f we get the general case. Therefore, from now on, we consider the case $B_\varrho \equiv B_1$ and the new maps $\widetilde{u}$ and $\widetilde{f}$. Let us now take a ball $B_\varrho \subset B_1$; we restart from (3.3), that we apply with the special choice, $u_0 \equiv (\widetilde{u})_{B_\varrho}$ and $z_0 \equiv 0$, that is, $\ell(x) = \widetilde{u}_0 + \langle z_0, x - x_0 \rangle \equiv (\widetilde{u})_{B_\varrho}$. This means that we are dealing with the inequality

$$(3.6) \qquad \fint_{B_\varrho} \phi^2 |D\widetilde{u}|^p \, dx \leqslant c \fint_{B_\varrho} \left| \frac{\widetilde{u} - (\widetilde{u})_{B_\varrho}}{\varrho} \right|^p dx + c \fint_{B_\varrho} |\widetilde{f}| \, |\widetilde{u} - (\widetilde{u})_{B_\varrho}| \, dx.$$

In the following we shall denote by p^* the usual Sobolev conjugate exponent of p, determined in the following sense. When $p < n$ we have $p^* = np/(n-p)$, while in the case $p \geqslant n$ we then fix p^* as a number larger than one and such that $(p^*)' = p^*/(p^*-1) \in (1, 4/3]$. In this way we have that $q > (p^*)'$ holds in any case, where q has been defined in (1.16). By fixing $\sigma \in (0,1)$ and using Hölder, Sobolev-Morrey and Young inequalities we have

$$\fint_{B_\varrho} |\widetilde{f}| \, |\widetilde{u} - (\widetilde{u})_{B_\varrho}| \, dx \leqslant \left(\varrho^{(p^*)'} \fint_{B_\varrho} |\widetilde{f}|^{(p^*)'} dx \right)^{1/(p^*)'} \left(\fint_{B_\varrho} \left| \frac{\widetilde{u} - (\widetilde{u})_{B_\varrho}}{\varrho} \right|^{p^*} dx \right)^{1/p^*}$$

$$\leqslant c \left(\varrho^{(p^*)'} \fint_{B_\varrho} |\widetilde{f}|^{(p^*)'} dx \right)^{1/(p^*)'} \left(\fint_{B_\varrho} |D\widetilde{u}|^p \, dx \right)^{1/p}$$

$$\leqslant \frac{c}{\sigma^{1/(p-1)}} \left(\varrho^{(p^*)'} \fint_{B_\varrho} |\widetilde{f}|^{(p^*)'} dx \right)^{p/[(p^*)'(p-1)]} + \sigma \fint_{B_\varrho} |D\widetilde{u}|^p \, dx.$$

We now observe that in any case we have

$$(3.7) \qquad\qquad \frac{p}{(p^*)'(p-1)} - 1 \leqslant \frac{p}{n(p-1)}.$$

Indeed, when $p < n$ the above relation follows with actually equality sign. Instead, in the case $p \geqslant n$ any value of $p^* > 1$ works. We then manipulate as follows:

$$\left(\varrho^{(p^*)'} \fint_{B_\varrho} |\widetilde{f}|^{(p^*)'} dx \right)^{p/[(p^*)'(p-1)]} \leqslant c(n) \fint_{B_\varrho} |K\widetilde{f}|^{(p^*)'} dx,$$

where

$$(3.8) \qquad\qquad K^{(p^*)'} := \left(\fint_{B_1} |\widetilde{f}|^{(p^*)'} dx \right)^{p/[(p^*)'(p-1)]-1},$$

and we have in fact used (3.7) and that $\varrho \leqslant 1$. By further denoting $p_* = np/(n+p) \in [1, p)$, using the last inequality in (3.6) we finally arrive, again using Sobolev inequality, at

$$\fint_{B_{\varrho/2}} |D\widetilde{u}|^p \, dx \leqslant \sigma \fint_{B_\varrho} |D\widetilde{u}|^p \, dx + c \left(\fint_{B_\varrho} |D\widetilde{u}|^{p_*} dx \right)^{p/p_*} + \frac{c}{\sigma^{1/(p-1)}} \fint_{B_\varrho} |K\widetilde{f}|^{(p^*)'} dx,$$

where the constant c depends only on n, N, p, ν, L. We are therefore able to apply a well-known variant of Gehring's lemma (see for instance [19, Cor. 6.1]) and find a higher integrability exponent $s \equiv s(n, N, p, \nu, L) > 1$ such that $s \leqslant q/(p^*)'$ and that the following inequality:

$$(3.9) \quad \left(\fint_{B_{\varrho/2}} |D\widetilde{u}|^{ps} \, dx \right)^{1/(ps)} \leqslant c \left(\fint_{B_\varrho} |D\widetilde{u}|^p \, dx \right)^{1/p} + cK^{(p^*)'/p} \left(\fint_{B_\varrho} |\widetilde{f}|^{(p^*)'s} \, dx \right)^{1/(ps)}$$

holds whenever $B_\varrho \subset B_1$; in particular, it holds for $B_\varrho \equiv B_1$. By Hölder inequality, from the definition of K in (3.8) we have

$$K^{(p^*)'/p} \leqslant \left(\fint_{B_1} |\widetilde{f}|^{(p^*)'s} \, dx \right)^{1/[(p^*)'s(p-1)]-1/(ps)}.$$

Inserting the last inequality in (3.9) with $B_\varrho \equiv B_1$, and using again Hölder inequality (recall that $(p^*)'s \leqslant q$) to estimate the last integral, we obtain

$$\left(\fint_{B_{1/2}} |D\widetilde{u}|^{ps} \, dx \right)^{1/(ps)} \leqslant c \left(\fint_{B_1} |D\widetilde{u}|^p \, dx \right)^{1/p} + c \left(\fint_{B_1} |\widetilde{f}|^q \, dx \right)^{1/[q(p-1)]}.$$

that holds for a constant $c \equiv c(n, N, p, \nu, L)$. Now (3.5) follows with $p_2 := ps$ and scaling back to u and f. $\qquad\square$

In the next Proposition we exploit a linearization procedure that eventually will allow us to apply the $\mathscr{A}$-harmonic approximation lemma. The final outcome is the distributional inequality (3.11) below; in the right-hand side we notice the appearance of a term involving f, which is also appearing in the right-hand side of (3.1). To ease the notation in the following we shall denote

$$(3.10) \qquad\qquad \omega(t) := [\mu(t)]^{1/p}.$$

PROPOSITION 3.1 (Linearization). — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). There exists a constant $c \equiv c(n, N, p, L)$ such that for every ball $B_\varrho \subset \Omega$ and every $z_0 \in \mathbb{R}^{N \times n} \smallsetminus \{0\}$ such that $E(Du, z_0, B_\varrho) \neq 0$, we have:*

$$(3.11) \quad \left| \fint_{B_\varrho} \frac{\partial a(z_0)}{|z_0|^{p-2}} \left(|z_0|^{(p-2)/2} \frac{Du - z_0}{[E(Du, z_0, B_\varrho)]^{p/2}}, D\varphi \right) dx \right|$$

$$\leqslant c \left[1 + \left(\frac{E(Du, z_0, B_\varrho)}{|z_0|} \right)^{(p-2)/2} \right] \omega \left(\frac{E(Du, z_0, B_\varrho)}{|z_0|} \right) \|D\varphi\|_{L^\infty}$$

$$+ \frac{c|z_0|^{(2-p)/2}}{[E(Du, z_0, B_\varrho)]^{p/2}} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/q} \|D\varphi\|_{L^\infty}$$

for all $\varphi \in C_0^\infty(B_\varrho; \mathbb{R}^N)$, where q is the number defined in (1.16).

Proof. — By a simple scaling argument there is no loss of generality in assuming that $|D\varphi| \leqslant 1$. We manipulate the identity in (3.2) in order to obtain

$$(3.12) \quad \left| \fint_{B_\varrho} \partial a(z_0)(Du - z_0, D\varphi)\, dx \right|$$

$$\leqslant \left| \fint_{B_\varrho} \int_0^1 \left(\partial a(z_0) - \partial a(z_0 + t(Du - z_0)) \right)(Du - z_0, D\varphi)\, dt\, dx \right|$$

$$+ \left| \fint_{B_\varrho} \langle f, \varphi \rangle\, dx \right| := \text{(I)} + \text{(II)},$$

which is valid whenever $\varphi \in C_0^\infty(B_\varrho)$. We use $(1.3)_3$ in order to estimate the first term appearing in the right hand side as follows:

$$|\text{(I)}| \leqslant c \fint_{B_\varrho} |Du - z_0|(|z_0| + |Du - z_0|)^{p-2} \mu\Big(\frac{|Du - z_0|}{|z_0|} \Big)\, dx$$

$$\leqslant c \fint_{B_\varrho} |Du - z_0|^{p-1} \mu\Big(\frac{|Du - z_0|}{|z_0|} \Big)\, dx$$

$$+ c \fint_{B_\varrho} |z_0|^{p-2} |Du - z_0|\, \mu\Big(\frac{|Du - z_0|}{|z_0|} \Big)\, dx$$

$$\leqslant c \Big(\fint_{B_\varrho} |Du - z_0|^p\, dx \Big)^{(p-1)/p} \Big(\fint_{B_\varrho} \mu^p\Big(\frac{|Du - z_0|}{|z_0|} \Big)\, dx \Big)^{1/p}$$

$$+ c |z_0|^{(p-2)/2} \Big(\fint_{B_\varrho} |z_0|^{p-2} |Du - z_0|^2\, dx \Big)^{1/2}$$

$$(3.13) \qquad\qquad\qquad\qquad \cdot \Big(\fint_{B_\varrho} \mu^2\Big(\frac{|Du - z_0|}{|z_0|} \Big)\, dx \Big)^{1/2}$$

$$\leqslant c\mu \Big(\fint_{B_\varrho} \frac{|Du - z_0|}{|z_0|}\, dx \Big)^{1/p} [E(Du, z_0, B_\varrho)]^{p-1}$$

$$+ c |z_0|^{(p-2)/2} \mu\Big(\fint_{B_\varrho} \frac{|Du - z_0|}{|z_0|}\, dx \Big)^{1/2} [E(Du, z_0, B_\varrho)]^{p/2}$$

$$\leqslant c\mu \Big(\frac{E(Du, z_0, B_\varrho)}{|z_0|} \Big)^{1/p} [E(Du, z_0, B_\varrho)]^{p-1}$$

$$+ c |z_0|^{(p-2)/2} \mu\Big(\frac{E(Du, z_0, B_\varrho)}{|z_0|} \Big)^{1/2} [E(Du, z_0, B_\varrho)]^{p/2}.$$

We observe that we have used Jensen's inequality and the concavity of $\mu(\cdot)$, and that $\mu(\cdot) \leqslant 1$ to estimate $[\mu(\cdot)]^2 \leqslant \mu(\cdot)$ and $[\mu(\cdot)]^p \leqslant \mu(\cdot)$. We then continue to estimate

$$(3.14) \quad \text{(II)} \leqslant \fint_{B_\varrho} |f|\, dx \|\varphi\|_{L^\infty(B_\varrho)} \leqslant c\varrho \fint_{B_\varrho} |f|\, dx\, \|D\varphi\|_{L^\infty(B_\varrho)}$$

$$\leqslant c\varrho \fint_{B_\varrho} |f|\, dx \leqslant c\Big(\varrho^q \fint_{B_\varrho} |f|^q\, dx \Big)^{1/q}.$$

Using (3.13) and (3.14) in (3.12), dividing the resulting inequality by the quantity $|z_0|^{(p-2)/2}[E(Du, z_0, B_\varrho)]^{p/2}$ and finally recalling (3.10), we finally obtain (3.11) after a few manipulations. $\qquad\square$

PROPOSITION 3.3. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). For every $\tau \in (0, 1/4]$ there exist positive numbers $\varepsilon_0 = \varepsilon_0(n, N, p, \nu, L, \tau, \mu(\cdot)) > 0$ and $\varepsilon_1 = \varepsilon_1(n, N, p, \nu, L, \tau) > 0$, such that if for a ball $B_\varrho \equiv B_\varrho(x_0) \subset \Omega$ the smallness conditions*

$$E(Du, B_\varrho) < \varepsilon_0 |(Du)_{B_\varrho}| \tag{3.15}$$

and

$$\left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/q} \leqslant \varepsilon_1 |(Du)_{B_\varrho}|^{(p-2)/2} [E(Du, B_\varrho)]^{p/2} \tag{3.16}$$

are satisfied, then the following inequality holds:

$$E(Du, B_{\tau\varrho}) \leqslant c_0 \tau^{2/p} E(Du, B_\varrho) \tag{3.17}$$

for $c_0 \equiv c_0(n, N, p, \nu, L)$.

Proof. — Without loss of generality, we assume that $E(Du, B_\varrho) > 0$, otherwise the assertion is trivial; for the same reason, in view of (3.15), we can assume that $|(Du)_{B_\varrho}| > 0$. We initially take $\varepsilon_0 \in (0, 1)$; moreover, again without loss of generality we assume that the ball B_ϱ is centered at the origin; all the other balls in this proof will be centered at the origin as well. We define the map

$$w(x) := |(Du)_{B_\varrho}|^{(p-2)/2} \frac{u(x) - \langle (Du)_{B_\varrho}, x \rangle}{[E(Du, B_\varrho)]^{p/2}} \tag{3.18}$$

for $x \in B_\varrho$, and

$$d := \left[\frac{E(Du, B_\varrho)}{|(Du)_{B_\varrho}|} \right]^{p/2}. \tag{3.19}$$

Notice that since we are initially assuming that $\varepsilon_0 \leqslant 1$, we have that $0 < d \leqslant 1$. With such definitions easy computations give

$$\fint_{B_\varrho} |Dw|^2 \, dx + d^{p-2} \fint_{B_\varrho} |Dw|^p \, dx$$
$$= \frac{\fint_{B_\varrho} |(Du)_{B_\varrho}|^{p-2} |Du - (Du)_{B_\varrho}|^2 \, dx + |Du - (Du)_{B_\varrho}|^p \, dx}{[E(Du, B_\varrho)]^p} \leqslant 1.$$

Moreover from Proposition 3.1 we have that (note that $|(Du)_{B_\varrho}| > 0$)

$$
\left| \fint_{B_\varrho} \frac{\partial a((Du)_{B_\varrho})}{|(Du)_{B_\varrho}|^{p-2}} (Dw, D\varphi) \, dx \right|
$$

(3.20)
$$
\leqslant c \left[1 + \left(\frac{E(Du, B_\varrho)}{|(Du)_{B_\varrho}|} \right)^{(p-2)/2} \right] \omega \left(\frac{E(Du, B_\varrho)}{|(Du)_{B_\varrho}|} \right) \|D\varphi\|_{L^\infty}
$$
$$
+ \frac{c|(Du)_{B_\varrho}|^{(2-p)/2}}{[E(Du, B_\varrho)]^{p/2}} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/q} \|D\varphi\|_{L^\infty}
$$
$$
\leqslant c \left[\omega(\varepsilon_0) + \varepsilon_1 \right] \|D\varphi\|_{L^\infty}
$$

for all $\varphi \in C_0^\infty(B_\varrho; \mathbb{R}^N)$ and where $c \equiv c(n, N, p, L)$. We have of course used inequalities (3.15)–(3.16) in the last estimation. With

$$
\mathscr{A} := \frac{\partial a((Du)_{B_\varrho})}{|(Du)_{B_\varrho}|^{p-2}},
$$

let now $\varepsilon > 0$ to be chosen in a few lines, and determine $\delta = \delta(n, N, \nu, L, \varepsilon) \in (0, 1]$ from Lemma 2.4.

Now we start determining positive numbers $\varepsilon_0 \equiv \varepsilon_0(n, N, p, \nu, L, \mu(\cdot), \varepsilon)$ and $\varepsilon_1 \equiv \varepsilon_1(n, N, p, \nu, L, \varepsilon)$ in such a way that

(3.21) $c \left[\omega(\varepsilon_0) + \varepsilon_1 \right] \leqslant \delta.$

Notice that later we shall put further restrictions on the size of $\varepsilon_0, \varepsilon_1$; they will anyway always be determined in dependence of the final parameters announced in the statement of the Proposition. With (3.20) and (3.21) we conclude that

$$
\left| \fint_{B_\varrho} \mathscr{A}(Dw, D\varphi) \, dx \right| \leqslant \delta \|D\varphi\|_{L^\infty}
$$

holds for all $\varphi \in C_0^\infty(B_\varrho; \mathbb{R}^N)$, as required in (2.12). We are therefore in position to apply Lemma 2.4, that gives the existence of an $\mathscr{A}$-harmonic map $h \in W^{1,2}(B_\varrho; \mathbb{R}^N)$, such that

(3.22) $\fint_{B_{3\varrho/4}} |Dh|^2 \, dx + d^{p-2} \fint_{B_{3\varrho/4}} |Dh|^p \, dx \leqslant 1$

and

(3.23) $\fint_{B_{3\varrho/4}} \left| \frac{w - h}{\varrho} \right|^2 dx + d^{p-2} \fint_{B_{3\varrho/4}} \left| \frac{w - h}{\varrho} \right|^p dx \leqslant \varepsilon.$

We shall also use a few basic properties of solutions to linear elliptic systems with constants coefficients to get estimates for h. Namely, the following inequality holds for every choice of $\gamma > 1$:

(3.24) $\varrho^\gamma \sup_{B_{\varrho/2}} |D^2 h|^\gamma \leqslant c(n, N, \nu, L, \gamma) \fint_{B_{3\varrho/4}} |Dh|^\gamma \, dx.$

See for instance [19, Chap. 10]. Taking $\tau \in (0, 1/4)$ to be chosen later, we now have

$$\fint_{B_{2\tau\varrho}} \left| \frac{w(x) - h(0) - Dh(0)x}{\tau\varrho} \right|^2 dx$$

$$\leqslant c \fint_{B_{2\tau\varrho}} \left| \frac{h(x) - h(0) - Dh(0)x}{\tau\varrho} \right|^2 dx + c \fint_{B_{2\tau\varrho}} \left| \frac{w - h}{\tau\varrho} \right|^2 dx$$

$$\stackrel{(3.23)}{\leqslant} c(\tau\varrho)^2 \sup_{B_{\varrho/2}} |D^2 h|^2 + \frac{c\varepsilon}{\tau^{n+2}}$$

(3.25)

$$\stackrel{(3.24)}{\leqslant} c\tau^2 \fint_{B_{3\varrho/4}} |Dh|^2 \, dx + \frac{c\varepsilon}{\tau^{n+2}}$$

$$\stackrel{(3.22)}{\leqslant} c\tau^2 + \frac{c\varepsilon}{\tau^{n+2}}.$$

As a consequence, the constant c in (3.25) depends only on n, N, p, ν, L. We now set

(3.26) $$\varepsilon = \tau^{n+2p}.$$

Observe that this has the effect of fixing $\varepsilon_0, \varepsilon_1$ in (3.21) such that

$$\varepsilon_0 \equiv \varepsilon_0(n, N, p, \nu, L, \mu(\cdot), \tau) \quad \text{and} \quad \varepsilon_1 \equiv \varepsilon_1(n, N, p, \nu, L, \tau).$$

This choice of ε in (3.25) and the definition of w in (3.18) yield

(3.27) $$\fint_{B_{2\tau\varrho}} \frac{\left| u(x) - (Du)_{B_\varrho} x - |(Du)_{B_\varrho}|^{(2-p)/2} [E(Du, B_\varrho)]^{p/2} (h(0) + Dh(0)x) \right|^2}{(\tau\varrho)^2} \, dx$$

$$\leqslant c\tau^2 |(Du)_{B_\varrho}|^{2-p} [E(Du, B_\varrho)]^p,$$

where the constant $c \equiv c(n, N, p, \nu, L)$. Similarly to (3.25) we have

$$d^{p-2} \fint_{B_{2\tau\varrho}} \left| \frac{w(x) - h(0) - Dh(0)x}{\tau\varrho} \right|^p dx$$

$$\leqslant cd^{p-2} \fint_{B_{2\tau\varrho}} \left| \frac{h(x) - h(0) - Dh(0)x}{\tau\varrho} \right|^p dx + cd^{p-2} \fint_{B_{2\tau\varrho}} \left| \frac{w - h}{\tau\varrho} \right|^p dx$$

$$\stackrel{(3.23)}{\leqslant} cd^{p-2}(\tau\varrho)^p \sup_{B_{\varrho/2}} |D^2 h|^p + \frac{c\varepsilon}{\tau^{n+p}}$$

$$\stackrel{(3.24)}{\leqslant} c\tau^p d^{p-2} \fint_{B_{3\varrho/4}} |Dh|^p \, dx + \frac{c\varepsilon}{\tau^{n+p}}$$

$$\stackrel{(3.22)}{\leqslant} c\tau^p + \frac{c\varepsilon}{\tau^{n+p}} \stackrel{(3.26)}{\leqslant} c\tau^p.$$

Scaling back to u we find

(3.28) $$\fint_{B_{2\tau\varrho}} \frac{\left| u(x) - (Du)_{B_\varrho} x - |(Du)_{B_\varrho}|^{(2-p)/2} [E(Du, B_\varrho)]^{p/2} (h(0) + Dh(0)x) \right|^p}{(\tau\varrho)^p} \, dx$$

$$\leqslant cd^{2-p} |(Du)_{B_\varrho}|^{(2-p)p/2} [E(Du, B_\varrho)]^{p^2/2} \tau^p$$

$$= c\tau^p [E(Du, B_\varrho)]^p \leqslant c\tau^2 [E(Du, B_\varrho)]^p,$$

where, in order to obtain the equality in the second-last line, we have used the very definition of d in (3.19). The constant c still depends only on n, N, p, ν, L

With (3.27) and (3.28) at disposal, and denoting by $\ell_{2\tau\varrho}$ the unique affine function such that

$$\ell_{2\tau\varrho} \mapsto \min_{\ell} \int_{B_{2\tau\varrho}} |u - \ell|^2 \, dx$$

amongst all the affine functions ℓ, we conclude with

$$(3.29) \qquad \fint_{B_{2\tau\varrho}} |(Du)_{B_\varrho}|^{p-2} \Big| \frac{u - \ell_{2\tau\varrho}}{2\tau\varrho} \Big|^2 + \Big| \frac{u - \ell_{2\tau\varrho}}{2\tau\varrho} \Big|^p \, dx \leqslant c\tau^2 [E\,(Du, B_\varrho)]^p,$$

where $c \equiv c(n, N, p, \nu, L)$. Notice that we have used also Lemma 2.3. Using also the definition of $E\,(Du, B_\varrho)$ we have

$$
\begin{aligned}
|D\ell_{2\tau\varrho} - (Du)_{B_\varrho}| &\leqslant |D\ell_{2\tau\varrho} - (Du)_{B_{2\tau\varrho}}| + |(Du)_{B_{2\tau\varrho}} - (Du)_{B_\varrho}| \\
&\overset{(2.3)}{\leqslant} c \Big(\fint_{B_{2\tau\varrho}} |Du - (Du)_{B_{2\tau\varrho}}|^2 \, dx \Big)^{1/2} + \Big(\fint_{B_{2\tau\varrho}} |Du - (Du)_{B_\varrho}|^2 \, dx \Big)^{1/2} \\
&\leqslant \frac{c}{\tau^{n/2}} \Big(\fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^2 \, dx \Big)^{1/2} \\
&= \frac{c}{\tau^{n/2}} \Big(\fint_{B_\varrho} |(Du)_{B_\varrho}|^{p-2} |Du - (Du)_{B_\varrho}|^2 \, dx \Big)^{1/2} |(Du)_{B_\varrho}|^{(2-p)/2} \\
&\leqslant \frac{c(n)}{\tau^{n/2}} \Big[\frac{E\,(Du, B_\varrho)}{|(Du)_{B_\varrho}|} \Big]^{p/2} |(Du)_{B_\varrho}|,
\end{aligned}
$$

so that, if ε_0 is chosen such that

$$(3.30) \qquad \Big[\frac{E\,(Du, B_\varrho)}{|(Du)_{B_\varrho}|} \Big]^{p/2} \leqslant \varepsilon_0^{p/2} \leqslant \frac{\tau^{n/2}}{8c(n)},$$

holds, then we have

$$(3.31) \qquad |D\ell_{2\tau\varrho} - (Du)_{B_\varrho}| \leqslant \frac{|(Du)_{B_\varrho}|}{8}.$$

This information and (3.29) allows to conclude with

$$(3.32) \qquad \fint_{B_{2\tau\varrho}} |D\ell_{2\tau\varrho}|^{p-2} \Big| \frac{u - \ell_{2\tau\varrho}}{2\tau\varrho} \Big|^2 + \Big| \frac{u - \ell_{2\tau\varrho}}{2\tau\varrho} \Big|^p \, dx \leqslant c\tau^2 [E\,(Du, B_\varrho)]^p,$$

yet with $c \equiv c(n, N, p, \nu, L)$. On the other hand we also observe that by (3.31) and triangle inequality it follows

$$(3.33) \qquad |D\ell_{2\tau\varrho}| \geqslant |(Du)_{B_\varrho}| - |D\ell_{2\tau\varrho} - (Du)_{B_\varrho}| \geqslant \frac{7|(Du)_{B_\varrho}|}{8}$$

so that, in particular, we have $|D\ell_{2\tau\varrho}| > 0$. It follows

$$\fint_{B_{\tau\varrho}} |D\ell_{2\tau\varrho}|^{p-2}|Du - D\ell_{2\tau\varrho}|^2 dx + \inf_{z\in\mathbb{R}^{N\times n}} \fint_{B_{\tau\varrho}} |Du - z|^p \, dx$$

$$\overset{(3.1)}{\leqslant} \fint_{B_{2\tau\varrho}} |D\ell_{2\tau\varrho}|^{p-2}\left|\frac{u-\ell_{2\tau\varrho}}{2\tau\varrho}\right|^2 + \left|\frac{u-\ell_{2\tau\varrho}}{2\tau\varrho}\right|^p \, dx$$

$$(3.34) \qquad\qquad + \frac{c}{|D\ell_{2\tau\varrho}|^{p-2}}\left(\tau^q\varrho^q \fint_{B_{2\tau\varrho}} |f|^q \, dx\right)^{2/q}$$

$$\overset{(3.32)}{\leqslant} c\tau^2[E(Du, B_\varrho)]^p + \frac{c}{|D\ell_{2\tau\varrho}|^{p-2}}\left(\tau^q\varrho^q \fint_{B_{2\tau\varrho}} |f|^q \, dx\right)^{2/q}$$

$$\overset{(3.33)}{\leqslant} c\tau^2[E(Du, B_\varrho)]^p + \frac{c\tau^{2-2n/q}}{|(Du)_{B_\varrho}|^{p-2}}\left(\varrho^q \fint_{B_\varrho} |f|^q \, dx\right)^{2/q}.$$

We proceed estimating

$$\fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2}|Du - (Du)_{B_{\tau\varrho}}|^2 \, dx$$

$$\leqslant 3^p \fint_{B_{\tau\varrho}} |D\ell_{\tau\varrho} - (Du)_{B_{\tau\varrho}}|^{p-2}|Du - (Du)_{B_{\tau\varrho}}|^2 \, dx \quad (=: \text{(III)})$$

$$+ 3^p \fint_{B_{\tau\varrho}} |D\ell_{2\tau\varrho} - D\ell_{\tau\varrho}|^{p-2}|Du - (Du)_{B_{\tau\varrho}}|^2 \, dx \quad (=: \text{(IV)})$$

$$+ 3^p \fint_{B_{\tau\varrho}} |D\ell_{2\tau\varrho}|^{p-2}|Du - (Du)_{B_{\tau\varrho}}|^2 dx \quad (=: \text{(V)}).$$

We start using Young's inequality and (2.3) to obtain

$$\text{(III)} \quad\leqslant\quad c|D\ell_{\tau\varrho} - (Du)_{B_{\tau\varrho}}|^p + c\fint_{B_{\tau\varrho}} |Du - (Du)_{B_{\tau\varrho}}|^p \, dx$$

$$\overset{(2.3)}{\leqslant} c\fint_{B_{\tau\varrho}} |Du - (Du)_{B_{\tau\varrho}}|^p \, dx$$

$$\overset{(2.1)}{\leqslant} c\inf_{z\in\mathbb{R}^{N\times n}} \fint_{B_{\tau\varrho}} |Du - z|^p \, dx$$

$$\overset{(3.34)}{\leqslant} c\tau^2[E(Du, B_\varrho)]^p + \frac{c\tau^{2-2n/q}}{|(Du)_{B_\varrho}|^{p-2}}\left(\varrho^q \fint_{B_\varrho} |f|^q \, dx\right)^{2/q}.$$

for $c \equiv c(n, N, p, \nu, L)$. Similarly, we have

$$\text{(IV)} \quad\leqslant\quad c|D\ell_{2\tau\varrho} - D\ell_{\tau\varrho}|^p + c\fint_{B_{\tau\varrho}} |Du - (Du)_{B_{\tau\varrho}}|^p \, dx$$

$$(3.35) \qquad \overset{(2.1),(2.2)}{\leqslant} c\fint_{B_{2\tau\varrho}} \left|\frac{u-\ell_{2\tau\varrho}}{2\tau\varrho}\right|^p \, dx + c\inf_{z\in\mathbb{R}^{N\times n}} \fint_{B_{\tau\varrho}} |Du - z|^p \, dx$$

$$\overset{(3.32),(3.34)}{\leqslant} c\tau^2[E(Du, B_\varrho)]^p + \frac{c\tau^{2-2n/q}}{|(Du)_{B_\varrho}|^{p-2}}\left(\varrho^q \fint_{B_\varrho} |f|^q \, dx\right)^{2/q},$$

where again it is $c \equiv c(n, N, p, \nu, L)$. Finally, by (2.1) and subsequent remark, we obviously have

$$(\mathrm{V}) \leqslant 3^p \fint_{B_{\tau\varrho}} |D\ell_{2\tau\varrho}|^{p-2}|Du - D\ell_{2\tau\varrho}|^2 dx,$$

and the term appearing on the right hand side has been already estimated in (3.34). Combining the content of the last five displays with (3.34) we conclude that

$$(3.36) \quad E(Du, B_{\tau\varrho}) \leqslant c\tau^{2/p}E(Du, B_\varrho) + \frac{c\tau^{2/p-2n/(pq)}}{|(Du)_{B_\varrho}|^{(p-2)/p}}\left(\varrho^q \fint_{B_\varrho} |f|^q\, dx\right)^{2/(pq)},$$

where $c \equiv c(n, N, p, \nu, L)$. We are now going to use the assumed condition (3.16) again, putting a further, final restriction on the size of ε_1. We have

$$\frac{c\tau^{2/p-2n/(pq)}}{|(Du)_{B_\varrho}|^{(p-2)/p}}\left(\varrho^q \fint_{B_\varrho} |f|^q\, dx\right)^{2/(pq)} \leqslant c\tau^{2/p-2n/(pq)}\varepsilon_1^{2/p}E(Du, B_\varrho).$$

Therefore, by choosing

$$(3.37) \qquad\qquad\qquad \varepsilon_1 \leqslant \frac{\tau^{2n/(pq)}}{c},$$

we finally get

$$\frac{c\tau^{2/p-2n/(pq)}}{|(Du)_{B_\varrho}|^{(p-2)/p}}\left(\varrho^q \fint_{B_\varrho} |f|^q\, dx\right)^{2/(pq)} \leqslant \tau^{2/p}E(Du, B_\varrho).$$

Combining the last estimate with the one in (3.36) yields (3.17). As for the exact dependence on the various parameters of ε_0 and ε_1, this can be reconstructed by looking at the choices made in (3.21), (3.30) and (3.37). In particular, as already noticed above, once ε is determined in (3.26), then this makes δ appearing in (3.21) a function of n, N, p, ν, L and τ. This then reflects on the subsequent dependence on $\varepsilon_0, \varepsilon_1$. The proof is complete. $\qquad\square$

We then want to deal with the situation when (3.16) is not verified; for this we premise a technical lemma that works for any map u.

Lemma 3.1. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ and $B_{\tau\varrho} \subset B_\varrho \subset \Omega$ be two concentric balls with $\tau \in (0, 1)$. It holds that*

$$(3.38) \qquad\qquad [E(Du, B_{\tau\varrho})]^{p/2} \leqslant \frac{8^p}{\tau^{n/2}}[E(Du, B_\varrho)]^{p/2}.$$

Proof. — We have

$$[E(Du, B_{\tau\varrho})]^{p/2} \overset{(2.1)}{\leqslant} \left(\fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2}|Du - (Du)_{B_\varrho}|^2\, dx\right)^{1/2}$$

$$+ 2^p\left(\fint_{B_{\tau\varrho}} |Du - (Du)_{B_\varrho}|^p\, dx\right)^{1/2}$$

$$\leqslant \frac{4^p}{\tau^{n/2}}[E(Du, B_\varrho)]^{p/2} + 2^p|(Du)_{B_\varrho} - (Du)_{B_{\tau\varrho}}|^{(p-2)/2}\left(\fint_{B_{\tau\varrho}} |Du - (Du)_{B_\varrho}|^2\, dx\right)^{1/2}.$$

On the other hand, we have

$$|(Du)_{B_\varrho} - (Du)_{B_{\tau\varrho}}|^{(p-2)/2} \left(\fint_{B_{\tau\varrho}} |Du - (Du)_{B_\varrho}|^2 \, dx \right)^{1/2}$$

$$\leqslant \left(\fint_{B_{\tau\varrho}} |Du - (Du)_{B_\varrho}|^p \, dx \right)^{(p-2)/(2p)} \left(\fint_{B_{\tau\varrho}} |Du - (Du)_{B_\varrho}|^p \, dx \right)^{1/p}$$

$$\leqslant \frac{1}{\tau^{n/2}} \left(\fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^p \, dx \right)^{1/2} \leqslant \frac{1}{\tau^{n/2}} [E(Du, B_\varrho)]^{p/2}.$$

Combining the content of the last two displays yields (3.38). □

PROPOSITION 3.4. — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16); let $B_\varrho \equiv B_\varrho(x_0) \subset \Omega$ be a ball and assume that (3.15) holds for some $\varepsilon_0 \in (0,1)$ together with*

$$(3.39) \qquad \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/q} > \varepsilon_1 |(Du)_{B_\varrho}|^{(p-2)/2} [E(Du, B_\varrho)]^{p/2}$$

for some other $\varepsilon_1 \in (0,1)$. Then the following inequality holds whenever $\tau \in (0, 1/4]$:

$$(3.40) \qquad E(Du, B_{\tau\varrho}) \leqslant \left(\frac{8}{\tau^{n/p}} \right) \frac{\varepsilon_0^{(p-2)/[2(p-1)]}}{\varepsilon_1^{1/(p-1)}} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/[q(p-1)]}.$$

Proof. — By (3.39) we estimate

$$\left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/q} \geqslant \varepsilon_1 \left[\frac{E(Du, B_\varrho)}{\varepsilon_0} \right]^{(p-2)/2} [E(Du, B_\varrho)]^{p/2}$$

$$= \frac{\varepsilon_1}{\varepsilon_0^{(p-2)/2}} [E(Du, B_\varrho)]^{p-1},$$

and we get

$$(3.41) \qquad E(Du, B_\varrho) \leqslant \frac{\varepsilon_0^{(p-2)/[2(p-1)]}}{\varepsilon_1^{1/(p-1)}} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/[q(p-1)]}.$$

At this stage (3.40) follows using (3.38). □

4. DECAY ESTIMATES IN THE DEGENERATE CASE

In this section we deal with another situation in which a condition as in (3.15) is not verified.

PROPOSITION 4.1 (*p*-linearization). — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). Assume that for a ball $B_\varrho \equiv B_\varrho(x_0) \subset \Omega$ the inequality*

$$(4.1) \qquad \qquad \chi |(Du)_{B_\varrho}| \leqslant E(Du, B_\varrho)$$

holds for some constant $\chi \in (0,1]$. It follows that

$$(4.2) \qquad \qquad \fint_{B_\varrho} |Du|^p \, dx \leqslant c(\chi) [E(Du, B_\varrho)]^p,$$

where

(4.3)
$$c(\chi) := 2^{p-1}\Big(1 + \frac{1}{\chi^p}\Big) > 1.$$

Moreover, for every $s > 0$ we have that

(4.4)
$$\Big|\fint_{B_\varrho} \langle |Du|^{p-2}Du, D\varphi\rangle \, dx\Big| \leqslant c\Big[s\{c(\chi)[E\,(Du, B_\varrho)]^p\}^{(p-1)/p}$$

$$+ \frac{c(\chi)[E\,(Du, B_\varrho)]^p}{\eta(s)}\Big]\|D\varphi\|_{L^\infty} + c\Big(\varrho^q \fint_{B_\varrho} |f|^q \, dx\Big)^{1/q}\|D\varphi\|_{L^\infty}$$

holds for all $\varphi \in C_0^\infty(B_\varrho; \mathbb{R}^N)$, where the constant c depends on n, N, p, L and the function $\eta(\cdot)$ has been defined in (1.6).

Proof. — By scaling we again assume that $\|D\varphi\|_{L^\infty} \leqslant 1$. We start observing that (4.1) implies

$$\fint_{B_\varrho} |Du|^p \, dx \leqslant 2^{p-1} \fint_{B_\varrho} |Du - (Du)_\varrho|^p \, dx + 2^{p-1}|(Du)_\varrho|^p$$

$$\leqslant 2^{p-1}\Big(1 + \frac{1}{\chi^p}\Big)[E\,(Du, B_\varrho)]^p.$$

Therefore, recalling (4.3), we conclude with (4.2). We next estimate

$$\Big|\fint_{B_\varrho} \langle |Du|^{p-2}Du, D\varphi\rangle \, dx\Big| \leqslant \Big|\fint_{B_\varrho} \langle a(Du) - |Du|^{p-2}Du, D\varphi\rangle \, dx\Big|$$

$$+ \Big|\fint_{B_\varrho} \langle f, \varphi\rangle \, dx\Big| =: (\mathrm{VI}) + (\mathrm{II})$$

for every $\varphi \in C_0^\infty(B_\varrho)$. The term (II) can be estimate as the analogous term in (3.14). As for (VI), we proceed by splitting the domain of integration; we have

$$\frac{1}{|B_\varrho|}\Big|\int_{B_\varrho \cap \{|Du| \leqslant \eta(s)\}} \langle a(Du) - |Du|^{p-2}Du, D\varphi\rangle \, dx\Big|$$

$$\leqslant s \fint_{B_\varrho} |Du|^{p-1} \, dx \leqslant s\Big(\fint_{B_\varrho} |Du|^p \, dx\Big)^{(p-1)/p}$$

$$\leqslant s\{c(\chi)[E\,(Du, B_\varrho)]^p\}^{(p-1)/p},$$

where we have used (1.6) and (4.2). For the remaining piece we have

$$\frac{1}{|B_\varrho|}\Big|\int_{B_\varrho \cap \{|Du| > \eta(s)\}} \langle a(Du) - |Du|^{p-2}Du, D\varphi\rangle \, dx\Big| \leqslant \frac{c}{|B_\varrho|}\int_{B_\varrho \cap \{|Du| > \eta(s)\}} |Du|^{p-1} \, dx$$

$$\leqslant \frac{c}{|B_\varrho|}\big|B_\varrho \cap \{|Du| > \eta(s)\}\big|^{1/p}\Big(\int_{B_\varrho} |Du|^p \, dx\Big)^{(p-1)/p}$$

$$\leqslant \frac{c}{\eta(s)} \fint_{B_\varrho} |Du|^p \, dx \leqslant \frac{cc(\chi)[E\,(Du, B_\varrho)]^p}{\eta(s)}.$$

Collecting the estimates in the last three displays, and recalling the estimate in (3.14) for (II), finally yields (4.4) and the proof is complete. □

We are now ready to prove the degenerate analog of Proposition 3.3, using a sort of p-linearization. Notice that a crucial point in the following statement, and that requires a delicate proof, is the fact that the constant c_1 appearing in inequality (4.6) below is independent of τ and χ, while only c_2 is allowed to have such a dependence. This is a crucial point when eventually combining Propositions 3.3–3.4 and 4.2 in Sections 5 and 6.1 below.

PROPOSITION 4.2. — *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). For every* $\tau \in (0, 1/4]$ *and* $\chi \in (0, 1]$ *there exists a positive number* $\varepsilon_2 \equiv \varepsilon_2(n, N, p, \nu, L, \tau, \chi, \eta(\cdot))$ *such that if for a ball* $B_\varrho \equiv B_\varrho(x_0) \subset \Omega$ *the inequality*

$$(4.5) \qquad \chi |(Du)_{B_\varrho}| \leqslant E\,(Du, B_\varrho)$$

holds together with the smallness condition

$$(4.6) \qquad E\,(Du, B_\varrho) < \varepsilon_2,$$

then

$$(4.7) \qquad E\,(Du, B_{\tau\varrho}) \leqslant c_1 \tau^\alpha E\,(Du, B_\varrho) + c_2 \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx \right)^{1/[q(p-1)]}$$

holds too, where $c_1 \equiv c_1(n, N, p, \nu, L)$ *and* $c_2 \equiv c_2(n, N, p, \nu, L, \tau, \chi)$, *and* $\alpha \equiv \alpha(n, N, p)$ *is the exponent appearing in Theorem* 2.1.

Proof. — Once again, without loss of generality we assume that $x_0 = 0$ and that $E\,(Du, B_\varrho) > 0$. Let us define the map

$$(4.8) \qquad w(x) := \frac{u(x)}{\lambda}$$

for $x \in B_\varrho$, and

$$(4.9) \qquad \lambda := \{c(\chi)[E\,(Du, B_\varrho)]^p\}^{1/p} + \frac{1}{\varepsilon_3} \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx \right)^{1/[q(p-1)]},$$

where $c(\chi)$ has been defined in (4.3). Here $\varepsilon_3 \in (0, 1)$ is a number to be determined in due course of the proof. By assumption (4.5) we are able to use (4.2)–(4.4), and therefore we have

$$(4.10) \qquad \fint_{B_\varrho} |Dw|^p\,dx \leqslant 1,$$

and

$$(4.11) \qquad \left| \fint_{B_\varrho} \langle |Dw|^{p-2} Dw, D\varphi \rangle\,dx \right| \leqslant \widetilde{c} \Big[s + \frac{\{c(\chi)[E\,(Du, B_\varrho)]^p\}^{1/p}}{\eta(s)} + \varepsilon_3^{p-1} \Big] \|D\varphi\|_{L^\infty}$$

whenever $\varphi \in C_0^\infty(B_\varrho)$ and where $\widetilde{c} \equiv \widetilde{c}(n, N, p, L)$. The reverse Hölder inequality (3.5) in terms of w reads as

$$\left(\fint_{B_{\varrho/2}} |Dw|^{p_2}\,dx \right)^{1/p_2} \leqslant c \left(\fint_{B_\varrho} |Dw|^p\,dx \right)^{1/p} + \frac{c}{\lambda} \left(\varrho^q \fint_{B_\varrho} |f|^q\,dx \right)^{1/[q(p-1)]}$$

for $p_2 \equiv p_2(n, N, p, \nu, L) > p$ and $c \equiv c(n, N, p, \nu, L)$, so that (4.2) and the definition of λ in (4.9) give

$$(4.12) \qquad \fint_{B_{\varrho/2}} |Dw|^{p_2} \, dx \leqslant c,$$

again with $c \equiv c(n, N, p, \nu, L)$. We now determine the exponent $\theta \in (0, 1)$ as

$$(4.13) \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}$$

where $p_1 \in (1, p)$ is an exponent for which we plan to apply the p-harmonic approximation Lemma 2.5. We keep $p_1 < p$ from now on fixed, and therefore we omit to specify the dependence of the next constants on p_1. Now let τ be as specified in the statement and define

$$(4.14) \qquad \varepsilon = \left(\frac{\tau^{np/2 + p^2\alpha/2}}{2^p [c(\chi)]^{p/2}} \right)^{p_1/\theta p}.$$

We choose such ε in Lemma 2.5 and determine the corresponding $\delta \equiv \delta(n, N, p, \varepsilon) \equiv \delta(n, N, p, \tau, \chi) \in (0, 1]$. We then fix $s > 0$ such that $\widetilde{c} s \leqslant \delta/3$, where $\widetilde{c}$ has been introduced in (4.11). Note the dependence $s \equiv s(n, N, p, L, \tau, \chi)$. This fixes $\eta(s)$ as clarified in (1.6). We then determine $\varepsilon_2 \equiv \varepsilon_2(n, N, p, L, \tau, \chi, \eta(\cdot))$ from (4.6) such that

$$\frac{\widetilde{c}\{c(\chi)[E(Du, B_{\varrho})]^p\}^{1/p}}{\eta(s)} \leqslant \frac{\widetilde{c}\{c(\chi)\varepsilon_2^p\}^{1/p}}{\eta(s)} \leqslant \frac{\delta}{3}.$$

Finally we pick $\varepsilon_3 \equiv \varepsilon_3(n, N, p, \nu, L, \tau, \chi) > 0$ from (4.9) such that

$$(4.15) \qquad \widetilde{c}\varepsilon_3^{p-1} \leqslant \frac{\delta}{3}.$$

Using the content of the last three displays in (4.11) yields

$$\left| \fint_{B_{\varrho}} \langle |Dw|^{p-2} Dw, D\varphi \rangle \, dx \right| \leqslant \delta \|D\varphi\|_{L^\infty}$$

for any $\varphi \in C_0^\infty(B_{\varrho}; \mathbb{R}^N)$. This and (4.10) allow to apply the p-harmonic approximation Lemma 2.5; we therefore infer the existence of a p-harmonic function $h \in W^{1,p}(B_{\varrho}; \mathbb{R}^N)$ such that

$$(4.16) \qquad \fint_{B_{\varrho}} |Dh|^p \, dx \leqslant 1$$

and

$$(4.17) \qquad \fint_{B_{\varrho}} |Dw - Dh|^{p_1} \, dx \leqslant \varepsilon \overset{(4.14)}{=} \left(\frac{\tau^{np/2 + p^2\alpha/2}}{2^p [c(\chi)]^{p/2}} \right)^{p_1/\theta p}.$$

Using (2.14) we deduce

$$(4.18) \qquad \fint_{B_{\varrho/2}} |Dh|^{p_2} \, dx \leqslant \|Dh\|_{L^\infty(B_{\varrho/2})}^{p_2} \leqslant c \left(\fint_{B_{\varrho}} |Dh|^p \, dx \right)^{p_2/p} \overset{(4.16)}{\leqslant} c(n, N, p).$$

By (4.13) we can interpolate as follows:

$$\fint_{B_{\varrho/2}} |Dw - Dh|^p \, dx \leqslant \left(\fint_{B_{\varrho/2}} |Dw - Dh|^{p_1} \, dx \right)^{\theta p / p_1} \left(\fint_{B_{\varrho/2}} |Dw - Dh|^{p_2} \, dx \right)^{(1-\theta) p / p_2}$$

$$\overset{(4.17)}{\leqslant} \frac{\tau^{np/2 + p^2 \alpha / 2}}{[c(\chi)]^{p/2}} \left(\fint_{B_{\varrho/2}} [|Dw|^{p_2} + |Dh|^{p_2}] \, dx \right)^{(1-\theta) p / p_2}.$$

Therefore, by using (4.12) and (4.18), recalling the definition of w in (4.8) and setting $\overline{h} := \lambda h$ (which is still a p-harmonic map), we conclude with

$$(4.19) \qquad \fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx \leqslant \frac{c \lambda^p \tau^{np/2 + p^2 \alpha / 2}}{[c(\chi)]^{p/2}}$$

with $c \equiv c(n, N, p, \nu, L)$. We now estimate as follows:

$$[E(Du, B_{\tau\varrho})]^p = \fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2} |Du - (Du)_{B_{\tau\varrho}}|^2 + |Du - (Du)_{B_{\tau\varrho}}|^p \, dx$$

$$\overset{(2.1)}{\leqslant} c \fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2} |Du - (D\overline{h})_{B_{\tau\varrho}}|^2 + |Du - (D\overline{h})_{B_{\tau\varrho}}|^p \, dx$$

$$\leqslant c \fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2} |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^2 + |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^p \, dx$$

$$\qquad + c \fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}}|^{p-2} |Du - D\overline{h}|^2 + |Du - D\overline{h}|^p \, dx \ (=: \mathscr{R}_1)$$

$$\leqslant c \fint_{B_{\tau\varrho}} |(D\overline{h})_{B_{\tau\varrho}}|^{p-2} |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^2 + |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^p \, dx + \mathscr{R}_1$$

$$\qquad + c \fint_{B_{\tau\varrho}} |(Du)_{B_{\tau\varrho}} - (D\overline{h})_{B_{\tau\varrho}}|^{p-2} |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^2 \, dx$$

$$\overset{\text{Young}}{\leqslant} c \fint_{B_{\tau\varrho}} |(D\overline{h})_{B_{\tau\varrho}}|^{p-2} |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^2 + |D\overline{h} - (D\overline{h})_{B_{\tau\varrho}}|^p \, dx + c\mathscr{R}_1.$$

By using (2.15) we continue estimating

$$[E(Du, B_{\tau\varrho})]^p \overset{(2.15)}{\leqslant} c\tau^{p\alpha} \fint_{B_{\varrho/2}} |(D\overline{h})_{B_{\varrho/2}}|^{p-2} |D\overline{h} - (D\overline{h})_{B_{\varrho/2}}|^2 + |D\overline{h} - (D\overline{h})_{B_{\varrho/2}}|^p \, dx$$

$$\qquad\qquad\qquad\qquad\qquad + c\mathscr{R}_1$$

$$\overset{(2.1)}{\leqslant} c\tau^{p\alpha} \fint_{B_{\varrho/2}} |(D\overline{h})_{B_{\varrho/2}}|^{p-2} |Du - (Du)_{B_{\varrho/2}}|^2 + |Du - (Du)_{B_{\varrho/2}}|^p \, dx$$

$$\qquad + c \fint_{B_{\varrho/2}} |(D\overline{h})_{B_{\varrho/2}}|^{p-2} |Du - D\overline{h}|^2 + |Du - D\overline{h}|^p \, dx \ (=: \mathscr{R}_2) + c\mathscr{R}_1$$

$$\leqslant c\tau^{p\alpha} [E(Du, B_{\varrho/2})]^p + c\mathscr{R}_1 + \mathscr{R}_2$$

$$\qquad + c\tau^{p\alpha} \fint_{B_{\varrho/2}} |(Du)_{B_{\varrho/2}} - (D\overline{h})_{B_{\varrho/2}}|^{p-2} |Du - (Du)_{B_{\varrho/2}}|^2 \, dx$$

$$\overset{\text{Young}}{\leqslant} c\tau^{p\alpha} [E(Du, B_{\varrho})]^p + c\tau^{p\alpha} \fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx + c\mathscr{R}_1 + \mathscr{R}_2.$$

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We then estimate the terms $\mathscr{R}_1, \mathscr{R}_2$. The constant c depends only n, N, p. By using Hölder's inequality, (4.2) and (4.19), and yet recalling that $\tau \in (0, 1/4)$, we have

$$
\begin{aligned}
\mathscr{R}_1 &\leqslant \left(\fint_{B_{\tau\varrho}} |Du|^p \, dx \right)^{(p-2)/p} \left(\fint_{B_{\tau\varrho/2}} |Du - D\overline{h}|^p \, dx \right)^{2/p} + \fint_{B_{\tau\varrho}} |Du - D\overline{h}|^p \, dx \\
&\leqslant \frac{c}{\tau^n} \left(\fint_{B_\varrho} |Du|^p \, dx \right)^{(p-2)/p} \left(\fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx \right)^{2/p} + \frac{c}{\tau^n} \fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx \\
&\leqslant \frac{c}{\tau^n} \{c(\chi)[E\,(Du, B_\varrho)]^p\}^{(p-2)/p} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right)^{2/p} + \frac{c}{\tau^n} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right)
\end{aligned}
$$

for a constant c depending only on n, N, p, ν, L. In order to estimate $\mathscr{R}_2$ we recall (4.18) and that $\overline{h} = \lambda h$; then we use (4.19) to have

$$
\begin{aligned}
\mathscr{R}_2 &\leqslant c\lambda^{p-2} \|Dh\|_{L^\infty(B_{\varrho/2})}^{p-2} \left(\fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx \right)^{2/p} + c \fint_{B_{\varrho/2}} |Du - D\overline{h}|^p \, dx \\
&\leqslant c\lambda^{p-2} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right)^{2/p} + \frac{c\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}}.
\end{aligned}
$$

Connecting the content of the last three inequalities, and again using (4.19), we then obtain

$$
\begin{aligned}
[E\,(Du, B_{\tau\varrho})]^p &\leqslant c\tau^{p\alpha} [E\,(Du, B_\varrho)]^p \\
&\quad + \frac{c}{\tau^n} \{c(\chi)[E\,(Du, B_\varrho)]^p\}^{(p-2)/p} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right)^{2/p} \\
&\quad + \frac{c}{\tau^n} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right) + c\lambda^{p-2} \left(\frac{\lambda^p \tau^{np/2 + p^2\alpha/2}}{[c(\chi)]^{p/2}} \right)^{2/p},
\end{aligned}
$$

with $c \equiv c(n, N, p, \nu, L)$. We now insert the value of λ from (4.9) in the above inequality, thereby getting, after a few lengthy but elementary manipulations and use of Young's inequality:

$$
\begin{aligned}
[E\,(Du, B_{\tau\varrho})]^p &\leqslant c_3 \tau^{p\alpha} [E\,(Du, B_\varrho)]^p + c_4 \tau^{p\alpha} [E\,(Du, B_\varrho)]^{p-2} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{2/[q(p-1)]} \\
&\quad + c_4 \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{p/[q(p-1)]} \\
&\leqslant (c_3 + 1) \tau^{p\alpha} [E\,(Du, B_\varrho)]^p + (c_4 + c_4^{p/2}) \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{p/[q(p-1)]}.
\end{aligned}
$$

Here it is $c_3 \equiv c_3(n, N, p, \nu, L)$ and $c_4 \equiv c_4(n, N, p, \nu, L, \tau, \chi)$. Notice that such a peculiar dependence of the constants occurs by the choice made in (4.15). The last inequality easily yields (4.7) and the proof is complete. $\qquad\square$

5. MEAN OSCILLATION ESTIMATES AND THEOREMS 1.4 AND 1.5

Estimates in BMO and VMO for Du follow combining and iterating Propositions 3.3–3.4 and 4.2. Let us fix an exponent β such that

$$(5.1) \qquad 0 < \beta < \min\{2/p, \alpha\} =: \alpha_m.$$

We then consider the constants c_0 and c_1 appearing in the statements of Propositions 3.3 and 4.2, respectively; they both depend only on n, N, p, ν, L. We then determine $\tau \equiv \tau(n, N, p, \nu, L, \beta)$ in such a way that

$$(5.2) \qquad (c_0 + c_1)\tau^{\alpha_m - \beta} \leqslant \frac{1}{4}.$$

With such a choice of τ we are able to determine the constants ε_0 and ε_1 from (3.15) and (3.16), respectively, that are now functions of n, N, p, ν, L, β and $\mu(\cdot)$ (only the first actually depends on this last parameter). We next proceed applying Proposition 4.2 with the choice $\chi \equiv \varepsilon_0$ and with the one of τ made here. This determines the constants ε_2 and c_2 again as functions of $n, N, p, \nu, L, \beta, \mu(\cdot)$ and $\eta(\cdot)$. We then consider a ball $B_r \subset \Omega$ such that

$$(5.3) \qquad E(Du, B_r) < \varepsilon_2$$

and

$$(5.4) \qquad \sup_{\varrho \leqslant r} c_5 \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx \right)^{1/[q(p-1)]} \leqslant \frac{\varepsilon_2}{4},$$

where

$$(5.5) \qquad c_5 := \left[c_2 + \left(\frac{8}{\tau^{n/p}} \right) \frac{\varepsilon_0^{(p-2)/[2(p-1)]}}{\varepsilon_1^{1/(p-1)}} \right].$$

We now turn our attention to Proposition 4.2; since (4.6) is now verified (actually being (5.3)) we check whether (4.5) is verified too, that is, if $\varepsilon_0 |(Du)_{B_r}|^p \leqslant E(Du, B_r)$ holds. If this is the case we then apply Proposition 4.2 thereby getting the validity of (4.7). If, on the other hand $\varepsilon_0 |(Du)_{B_r}|^p > E(Du, B_r)$ holds, then we look at Propositions 3.3–3.4 and we deduce that either (3.17) or (3.40) applies. In any case we have proved the validity of the following estimate

$$(5.6) \qquad E(Du, B_{\tau r}) \leqslant \frac{\tau^\beta}{4} E(Du, B_r) + c_5 \left(r^q \fint_{B_r} |f|^q \, dx \right)^{1/[q(p-1)]},$$

where we have indeed applied (5.2) and the definition in (5.1). By further using (5.3)–(5.4) in the above estimate we also deduce that $E(Du, B_{\tau r}) < \varepsilon_2$ and this means, in view of (5.4), that the above reasoning can be applied on the ball $B_{\tau r}$, so that (5.6) holds with B_r replaced by $B_{\tau r}$. Proceeding in this way on the next balls $B_{\tau^j r}$, by induction we conclude that

$$E(Du, B_{\tau^j r}) < \varepsilon_2 \quad \text{holds for every } j \geqslant 0$$

together with the estimate

$$E\left(Du, B_{\tau^{j+1}r}\right) \leqslant \frac{\tau^{\beta}}{4} E\left(Du, B_{\tau^{j}r}\right) + c_5\left((\tau^{j}r)^q \fint_{B_{\tau^{j}r}} |f|^q\, dx\right)^{1/[q(p-1)]}.$$

Iterating the above inequality yields

$$E\left(Du, B_{\tau^{k+1}r}\right) \leqslant \tau^{\beta(k+1)} E\left(Du, B_r\right) + c_5 \sum_{j=0}^{k} (\tau^{\beta})^{j-k}\left((\tau^{j}r)^q \fint_{B_{\tau^{j}r}} |f|^q\, dx\right)^{1/[q(p-1)]}$$

for every integer $k \geqslant 0$ and therefore

$$E\left(Du, B_{\tau^{k+1}r}\right) \leqslant \tau^{\beta(k+1)} E\left(Du, B_r\right) + c\sup_{\varrho \leqslant r}\left(\varrho^q \fint_{B_\varrho} |f|^q\, dx\right)^{1/[q(p-1)]}.$$

By a standard interpolation argument we can conclude that the following inequality holds whenever $t \leqslant r$:

$$(5.7) \qquad E\left(Du, B_t\right) \leqslant c_6\left(\frac{t}{r}\right)^{\beta} E\left(Du, B_r\right) + c_6 \sup_{\varrho \leqslant r}\left(\varrho^q \fint_{B_\varrho} |f|^q\, dx\right)^{1/[q(p-1)]},$$

again with $c_6 \equiv c_6(n, N, p, \nu, L, \beta, \mu(\cdot), \eta(\cdot))$. We have meanwhile proved the following fact:

Proposition 5.1 (Pointwise BMO-estimate). — *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be weak solution to the system* (1.1) *under assumptions* (1.3)–(1.5) *and* (1.16)*, and let $B_r(x_0) \subset \Omega$ be a ball. There exists a number $\varepsilon_* \equiv \varepsilon_*(n, N, p, \nu, L, \beta, \mu(\cdot), \eta(\cdot))$ such that if*

$$(5.8) \qquad E\left(Du, B_r(x_0)\right) + \sup_{\varrho \leqslant r}\left(\varrho^q \fint_{B_\varrho(x_0)} |f|^q\, dx\right)^{1/[q(p-1)]} < \varepsilon$$

holds for some ε such that $0 < \varepsilon \leqslant \varepsilon_*$, then it also follows that

$$\sup_{\varrho \leqslant r} E\left(Du, B_\varrho(x_0)\right) < c_6 \varepsilon$$

for a constant c_6 depending only on $n, N, p, \nu, L, \beta, \mu(\cdot)$ and $\eta(\cdot)$; moreover, inequality (5.7) *holds.*

We continue to the

Proof of Theorem 1.4. — We shall in this proof use inequality (5.7), therefore when dealing with the arguments developed before the statement of Proposition 5.1 we shall always take $\beta = \min\{1/p, \alpha/2\}$ in (5.1). Therefore the dependence of the various constants on β will disappear in all the constants involved and in particular in c_6, that will just depend only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$. To start with, let us notice that since we are proving local results, with no loss of generality we can assume that (1.24) holds uniformly in the whole Ω. We first determine a radius $\varrho_1 \equiv \varrho_1(n, N, p, \nu, L, \beta, \mu(\cdot), \eta(\cdot)) > 0$ such that

$$(5.9) \qquad \sup_{\varrho \leqslant \varrho_1} c_5\left(\varrho^q \fint_{B_\varrho(x)} |f|^q\, dy\right)^{1/[q(p-1)]} < \frac{\varepsilon_2}{4c_6}$$

holds for every $x \in \Omega$. We recall that ε_2 has been determined in (5.3), that c_5 has been determined in (5.5) and c_6 appears in (5.7); they all depend only on $n, N, p, \nu, L, \beta, \mu(\cdot)$ and $\eta(\cdot)$. We now want to show that the set Ω_u appearing in (1.25) can be characterized by

$$(5.10) \qquad \Omega_u = \Big\{ x \in \Omega : \exists\, B_\varrho(x) \Subset \Omega \text{ with } \varrho \leqslant \varrho_1 \ : \ E(Du, B_\varrho(x)) < \varepsilon_2/4c_6 \Big\},$$

thereby fixing $\varrho_s := \varrho_1$ and $\varepsilon_s := \varepsilon_2/(4c_6)$ in the statement of Theorem 1.4. We first observe that $|\Omega \setminus \Omega_u| = 0$. Indeed let us consider the set

$$(5.11) \qquad \mathscr{L}_u := \Big\{ x \in \Omega : \liminf_{\varrho \to 0} \ \fint_{B_\varrho(x)} |V(Du) - (V(Du))_{B_\varrho(x)}|^2 \, dy = 0 \Big\}$$

which is such that $|\Omega \setminus \mathscr{L}_u| = 0$ by standard Lebesgue theory. Then by (2.8)–(2.10) we have

$$(5.12) \qquad \mathscr{L}_u := \Big\{ x \in \Omega : \liminf_{\varrho \to 0} E(Du, B_\varrho(x)) = 0 \Big\},$$

so that finally $\mathscr{L}_u \subset \Omega_u$ and finally $|\Omega \setminus \Omega_u| = 0$ follows. Let us now consider $x_0 \in \Omega_u$. We can then find a radius $\varrho_{x_0} \leqslant \varrho_1$ such that

$$(5.13) \qquad E\big(Du, B_{\varrho_{x_0}}(x_0)\big) < \frac{\varepsilon_2}{4c_6}.$$

By the absolute continuity of the integral and (5.13) we can then find a neighborhood of x_0, say $\mathscr{O}(x_0)$, such that

$$(5.14) \qquad E\big(Du, B_{\varrho_{x_0}}(x)\big) < \frac{\varepsilon_2}{4c_6} \quad \text{and } B_{\varrho_{x_0}}(x) \Subset \Omega \quad \text{hold for every } x \in \mathscr{O}(x_0).$$

This proves that Ω_u is an open set. It remains to prove that Du is locally VMO-regular in Ω_u. For this we start observing that (5.9) and (5.14) imply the validity of (5.3)–(5.4) with $B_r \equiv B_{\varrho_{x_0}}(x)$. Therefore the arguments developed in the proof Proposition 5.1 apply, eventually leading to (5.7) that in this case reads

$$E(Du, B_t(x)) \leqslant c_6 \Big(\frac{t}{\varrho_{x_0}} \Big)^\beta E\big(Du, B_{\varrho_{x_0}}(x)\big) + c_6 \sup_{\varrho \leqslant \varrho_{x_0}} \Big(\varrho^q \fint_{B_\varrho(x)} |f|^q \, dy \Big)^{1/[q(p-1)]}$$

for every $x \in \mathscr{O}(x_0)$ and $t \leqslant \varrho_{x_0}$. Using (5.9) and (5.13) in the previous inequality we have

$$(5.15) \qquad E(Du, B_s(x)) < \varepsilon_2$$

for every $s \leqslant \varrho_{x_0}$ and every $x \in \mathscr{O}(x_0)$. Therefore, also recalling (5.9), the validity of (5.3)–(5.4) this time follows $B_r \equiv B_s(x)$ for every $s \leqslant \varrho_{x_0}$ and for every $x \in \mathscr{O}(x_0)$. The same arguments devised after (5.3) lead to (5.7) and therefore we conclude again that

$$(5.16) \qquad E(Du, B_t(x)) \leqslant c_6 \Big(\frac{t}{s} \Big)^\beta + c_6 \sup_{\varrho \leqslant s} \Big(\varrho^q \fint_{B_\varrho(x)} |f|^q \, dy \Big)^{1/[q(p-1)]}$$

holds for every $x \in \mathscr{O}(x_0)$ and whenever $t \leqslant s \leqslant \varrho_{x_0}$; notice that we have once again used (5.15) to estimate the right-hand side. We now concentrate on $\mathscr{O}(x_0)$ and we essentially prove that Du has vanishing mean oscillations in $\mathscr{O}(x_0)$ in the following

stronger sense. We prove that for every $\sigma \in (0,1)$ there exists a radius $r_\sigma < \varrho_{x_0}$ such that

$$(5.17) \qquad t \leqslant r_\sigma \implies E\left(Du, B_t(x)\right) \leqslant \sigma \qquad \text{for every } x \in \mathscr{O}(x_0).$$

This fact will finally imply that $Du \in \mathrm{VMO}_{\mathrm{loc}}(\Omega_u; \mathbb{R}^{N \times n})$ via a standard covering argument. We are therefore left to prove the validity of (5.17). We start taking a positive radius $\varrho_2 \leqslant \varrho_{x_0}$ still depending on $n, N, p, \nu, L, \beta, \mu(\cdot), \eta(\cdot)$ and σ, such that

$$c_6 \sup_{\varrho \leqslant \varrho_2} \left(\varrho^q \fint_{B_\varrho(x)} |f|^q \, dy \right)^{1/[q(p-1)]} \leqslant \frac{\sigma}{2}$$

holds whenever $x \in \Omega$. We then finally choose the radius $r_\sigma \leqslant \varrho_2$, again depending on $n, N, p, \nu, L, \beta, \mu(\cdot), \eta(\cdot)$ and σ, such that

$$c_6 \left(\frac{r_\sigma}{\varrho_2} \right)^\beta \leqslant \frac{\sigma}{2}.$$

Using the last two inequalities in (5.16) with $s = \varrho_2$ and $t \leqslant r_\sigma$ yields $E\left(Du, B_t(x)\right) \leqslant \sigma$. We have therefore checked (5.17) and the proof is complete. $\qquad\square$

By carefully examining the above proof it is not difficult to see that we have also proved the following pointwise version of Theorem 1.4:

PROPOSITION 5.2 (Pointwise VMO). — *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be weak solution to the system* (1.1) *under assumptions* (1.3)–(1.5) *and* (1.16)*, and let* $B_r(x_0) \subset \Omega$ *be a ball. There exists a number* $\overline{\varepsilon} \equiv \overline{\varepsilon}(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot))$ *such that if*

$$(5.18) \qquad E\left(Du, B_r(x_0)\right) + \left(\sup_{\varrho \leqslant r} \varrho^q \fint_{B_\varrho(x_0)} |f|^q \, dx \right)^{1/[q(p-1)]} < \overline{\varepsilon}$$

and

$$(5.19) \qquad \lim_{\varrho \to 0} \varrho^q \fint_{B_\varrho(x_0)} |f|^q \, dx = 0$$

hold, then

$$(5.20) \qquad \lim_{\varrho \to 0} E\left(Du, B_\varrho(x_0)\right) = 0.$$

Moreover, if (5.18)–(5.19) *hold uniformly in an open subset* $\mathscr{O}$*, then the convergence in* (5.20) *happens to be uniform in* $\mathscr{O}$*.*

Proof of Theorem 1.5. — Theorem 1.5 now follows from Proposition 5.1 together with the same localization argument used in the proof of Theorem 1.4 to prove that the regular Ω_u set is open and has full measure. Needless to say, condition (1.26) serves to verify condition (5.8) on the set where the excess is small, and this allows to obtain the characterization of the regular set Ω_u as in (5.10). It just remains to prove the criterion in (1.28). This follows recalling the well-known Hölder type inequality

$$(5.21) \qquad \|f\|_{L^q(B)} \leqslant \left(\frac{n}{n-q} \right)^{1/n} |B|^{1/q - 1/n} \|f\|_{\mathscr{M}^n(B)},$$

which holds for any ball $B \subset \Omega$ (here it is $q < n$); see Lemma 5.1 below for the proof. Therefore, recalling the definition of q in (1.16), we have

$$\varrho^q \fint_{B_\varrho(x)} |f|^q \, dy \leqslant \Big(\frac{4}{|B_1|}\Big)^{q/n} \|f\|^q_{\mathscr{M}^n(\Omega)} \overset{(1.28)}{<} \varepsilon_*,$$

so that (1.28) implies (1.26) and the proof is complete. $\qquad\square$

For the sake of the reader we report the simple proof of inequality (5.21).

Lemma 5.1. — *Let* $1 \leqslant q < t$; *if* $\mathscr{O}$ *is an open subset with positive measure, then the Jensen type inequality*

$$(5.22) \qquad \|f\|_{L^q(\mathscr{O})} \leqslant \Big(\frac{t}{t-q}\Big)^{1/t} |\mathscr{O}|^{1/q-1/t} \|f\|_{\mathscr{M}^t(\mathscr{O})}$$

holds.

Proof. — For $\lambda_0 > 0$ to be chosen later, we have

$$\|f\|^q_{L^q(\mathscr{O})} = q \int_0^{\lambda_0} \lambda^q |\{|f| > \lambda\}| \, \frac{d\lambda}{\lambda} + q \int_{\lambda_0}^{\infty} \lambda^q |\{|f| > \lambda\}| \, \frac{d\lambda}{\lambda}.$$

In turn we estimate

$$\int_0^{\lambda_0} \lambda^q |\{|f| > \lambda\}| \, \frac{d\lambda}{\lambda} \leqslant \frac{\lambda_0^q |\mathscr{O}|}{q}$$

and

$$\int_{\lambda_0}^{\infty} \lambda^q |\{|f| > \lambda\}| \, \frac{d\lambda}{\lambda} \leqslant \|f\|^q_{\mathscr{M}^t(\mathscr{O})} \int_{\lambda_0}^{\infty} \frac{d\lambda}{\lambda^{1+t-q}} = \frac{\|f\|^t_{\mathscr{M}^t(\mathscr{O})}}{(t-q)\lambda_0^{t-q}}.$$

Connecting the content of the last three displays yields

$$\|f\|^q_{L^q(\mathscr{O})} \leqslant \lambda_0^t |\mathscr{O}| + \frac{q\|f\|^t_{\mathscr{M}^t(\mathscr{O})}}{(t-q)\lambda_0^{t-q}}.$$

Recalling that we can assume $\|f\|_{\mathscr{M}^q(\mathscr{O})} > 0$ (otherwise there is nothing to prove) and minimizing with respect to λ_0 the quantity appearing in the right-hand side of the last display, we are led to the choice $\lambda_0 = \|f\|_{\mathscr{M}^t(\mathscr{O})}/|\mathscr{O}|^{1/t}$, that finally allows us to conclude with the Jensen type inequality in (5.22) and the proof is complete. $\quad\square$

6. Proof of the main results

In this Section we present the proof of the main results, i.e. Theorems 1.1–1.3. These involve several steps, which are distributed in the next sections. The conclusions are in the very last ones 6.8–6.10. For the following sections, we fix a ball $B_r(x_0)$ which is initially the one considered in the statements of Theorems 1.1–1.7. Unless otherwise stated all the balls appearing through Sections 6.1–6.7 (B^j, B_ϱ, B_r, $B_{\tau r}$ and so on) will be centered at x_0. Needless to say, in the rest of the section $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a weak solution to the system (1.1) under assumptions (1.3)–(1.5) and (1.16). Moreover, in order to avoid trivialities we shall always assume that $\boldsymbol{I}^f_{1,q}(x_0, r) < \infty$.

6.1. Set-up of the parameters and basic alternatives. — With $\tau \in (0, 1/4)$ to be chosen in a few lines, we set

$$r_j := \tau^{j+1} r \quad \text{and} \quad B^j := B_{r_j}(x_0), \quad \forall j \in \mathbb{N} \cup \{0\}, \quad r_{-1} := r,$$

thereby defining a sequence of balls shrinking to x_0:

$$\cdots B_{r_{j+1}}(x_0) \equiv B^{j+1} \subset B^j \equiv B_{r_j}(x_0) \cdots \subset B^0 \equiv B_{\tau r}(x_0) \subset B_r(x_0) \equiv B^{-1}.$$

We then abbreviate as follows:

$$E_j := E\left(Du, B^j\right), \qquad A_j := (Du)_{B^j} \qquad \text{for every } j \in \mathbb{N} \cup \{-1, 0\}.$$

In order to determine the shrinking rate τ, we again look at Propositions 3.3–3.4 and 4.2 and follow the arguments developed in Section 5. We fix $\tau \in (0, 1/4)$ such that

$$(6.1) \qquad\qquad c_0 \tau^{2/p} + c_1 \tau^\alpha \leqslant \frac{1}{2^7} \implies \tau \equiv \tau(n, N, p, \nu, L).$$

With this choice of τ we determine

$$(6.2) \qquad\qquad \varepsilon_0 \equiv \varepsilon_0(n, N, p, \nu, L, \mu(\cdot)) \quad \text{and} \quad \varepsilon_1 \equiv \varepsilon_1(n, N, p, \nu, L)$$

from Proposition 3.3. We then use $\chi \equiv \varepsilon_0$ in Proposition 4.2 and therefore determine

$$(6.3) \qquad \varepsilon_2 \equiv \varepsilon_2(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)) \quad \text{and} \quad c_2 \equiv c_2(n, N, p, \nu, L, \mu(\cdot)).$$

We now assume that

$$(6.4) \qquad\qquad\qquad E\left(Du, B_{\tau r}\right) < \frac{\tau^n \min\{\bar\varepsilon, \varepsilon_2\}}{8^{2p+1}}$$

and that

$$(6.5) \qquad\qquad c_5 \sup_{s \leqslant \tau r} \left(s^q \fint_{B_s} |f|^q \, dx\right)^{1/[q(p-1)]} \leqslant \frac{\tau^n \min\{\bar\varepsilon, \varepsilon_2\}}{8^{2p+2}}$$

hold, where the constant c_5 is defined in (5.5) with the current choice of $\varepsilon_0, \varepsilon_1$ and τ, and where $\bar\varepsilon \equiv \bar\varepsilon(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot))$ has been introduced in Proposition 5.2. In the next Section we verify conditions (6.4)–(6.5) so that for the rest of the proof we shall always work with (6.4)–(6.5) being in force.

6.2. Verification of the smallness conditions. — Here we show that conditions (6.4)–(6.5) can be verified by choosing the number ε appearing in (1.17) to be properly small. This is indeed the moment where the smallness assumption (1.17) comes into the play. Using the definition of the potential $\boldsymbol{I}_{1,q}^f(x_0, r)$, we have

$$
\begin{aligned}
\sum_{j=0}^\infty \left(r_j^q \fint_{B^j} |f|^q \, dx\right)^{1/q} &\leqslant \sum_{j=0}^\infty \frac{r_j^{1-n/q}}{-\log \tau} \int_{r_j}^{r_{j-1}} \left(\frac{1}{|B_1|} \int_{B_\varrho} |f|^q \, dx\right)^{1/q} \frac{d\varrho}{\varrho} \\
(6.6) \qquad &\leqslant \frac{\tau^{1-n/q}}{-\log \tau} \sum_{j=0}^\infty \int_{r_j}^{r_{j-1}} \left(\varrho^q \fint_{B_\varrho} |f|^q \, dx\right)^{1/q} \frac{d\varrho}{\varrho} \\
&\leqslant \frac{\boldsymbol{I}_{1,q}^f(x_0, r)}{\tau^{2n}}.
\end{aligned}
$$

The above inequality readily implies that

$$\sup_{0<\varrho\leqslant\tau r}\left(\varrho^q \fint_{B_\varrho}|f|^q\,dx\right)^{1/q} \leqslant \frac{\boldsymbol{I}^f_{1,q}(x_0,r)}{\tau^{2n+n/q-1}}$$

and that

$$\lim_{\varrho\to 0}\varrho^q \fint_{B_\varrho}|f|^q\,dx = 0. \tag{6.7}$$

On the other hand applying (3.38) we get

$$E(Du,B_{\tau r}) \leqslant \frac{8^2 E(Du,B_r)}{\tau^{n/p}}.$$

The inequalities in the last two displays suggest to take

$$\varepsilon := \min\left\{\left[\frac{\tau^{3n+n/q-1}\min\{\bar\varepsilon,\varepsilon_2\}}{8^{2p+4}c_5}\right]^{1/(p-1)}, \frac{\tau^{n+2n/p}\min\{\bar\varepsilon,\varepsilon_2\}}{8^{2p+4}}\right\} \tag{6.8}$$

in (1.17). This choice allows to guarantee both (6.4) and (6.5). Recalling that τ,ε_2 and $\bar\varepsilon$ have been introduced as numbers depending only on $n,N,p,\nu,L,\mu(\cdot)$ and $\eta(\cdot)$, we are able to find the number ε finally appearing in (1.17) with the dependence on the various parameters described in the statement of Theorem 1.1.

6.3. **Basic alternatives.** — A first consequence of (6.4)–(6.5), of (6.7) and of Proposition 5.2 (applied to $B_{\tau r}(x_0)$ instead of $B_r(x_0)$) is that

$$\lim_{s\to 0} E\left(Du,B_s\right) = 0. \tag{6.9}$$

We notice that (6.4) in particular implies that

$$E\left(Du,B_{\tau r}\right) < \varepsilon_2. \tag{6.10}$$

By (6.10), we are able to combine Propositions 3.3–3.4 and 4.2 with the choices of the parameters described above, eventually arriving at

$$E\left(Du,B_{\tau^2 r}\right) \leqslant \frac{1}{2^7} E\left(Du,B_{\tau r}\right) + c_5\left[(\tau r)^q \fint_{B_{\tau r}}|f|^q\,dx\right]^{1/[q(p-1)]}. \tag{6.11}$$

The constant c_5 is formally defined in (5.5) but now the values of $\tau,\varepsilon_0,\varepsilon_1,c_2$ are determined in (6.1)–(6.3). Notice that we have indeed used (6.1). Using (6.4)–(6.5) in (6.11) we get

$$E_1 = E\left(Du,B_{\tau^2 r}\right) < \varepsilon_2$$

so that we can re-apply the same argument on $B_{\tau^2 r}$ instead of $B_{\tau r}$. Iterating the same argument on the sequence of balls $\{B^j\}$ yields

$$E_j = E\left(Du,B_{\tau^{j+1}r}\right) < \varepsilon_2 \qquad \text{for every } j \geqslant 0. \tag{6.12}$$

Now, since (6.12) is verified on each ball B^j we can apply Propositions 3.3–3.4 and 4.2 on the same sequence of balls, thereby obtaining the following three alternatives:

(A1) If

(6.13) $E_j < \varepsilon_0 |A_j|$ and $\left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q} \leqslant \varepsilon_1 |A_j|^{(p-2)/2} E_j^{p/2}$

are satisfied, then

(6.14) $E_{j+1} \leqslant \dfrac{E_j}{2^7}$

holds.

(A2) If

(6.15) $E_j < \varepsilon_0 |A_j|$ and $\left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q} > \varepsilon_1 |A_j|^{(p-2)/2} E_j^{p/2}$

are satisfied, then

(6.16) $E_{j+1} \leqslant c_7 \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/[q(p-1)]}$

holds for a constant $c_7 \equiv c_7(n, N, p, \nu, L, \mu(\cdot))$.

(A3) If

(6.17) $E_j \geqslant \varepsilon_0 |A_j|$

is satisfied, then

(6.18) $E_{j+1} \leqslant \dfrac{E_j}{2^7} + c_8 \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/[q(p-1)]}$

holds for a constant $c_8 \equiv c_8(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot))$.

With $\varepsilon_0, \varepsilon_1$ and τ, and then the constants c_7, c_8, having been determined as quantities that are globally depending only on $n, N, p, \nu, L, \mu(\cdot), \eta(\cdot)$, we start defining H_1 as

(6.19) $H_1 := \max \left\{ \dfrac{8^{8p}}{\tau^{np}}, \dfrac{2^6}{\varepsilon_0^{p/2}} \right\}$

and then H_2 as

(6.20) $H_2 := \max \left\{ (2^8 c_8)^{p+1} H_1, (2^8 c_7)^{p+1} H_1, \left(\dfrac{2^{20p} H_1}{\varepsilon_1} \right)^{p/[2(p-1)]} \right\}$.

These choices again determine H_1 and H_2 with the following dependence on the various parameters:

(6.21) $H_1 \equiv H_1(n, N, p, \nu, L, \mu(\cdot))$ and $H_2 \equiv H_2(n, N, p, \nu, L, \mu(\cdot), \eta(\cdot))$.

With H_1, H_2 being fixed, we define the composite excess functional $C(B)$ for every ball $B \subset \Omega$ as

$$C(B) := |(Du)_B|^{p/2} + H_1 [E(Du, B)]^{p/2},$$

and finally the non-homogeneous excess functional as

(6.22) $F(x_0, r) := \dfrac{8^{2p} 64 H_1}{\tau^{2n}} [E(Du, B_r(x_0))]^{p/2} + \dfrac{H_2}{\tau^{2n}} \left[\boldsymbol{I}_{1,q}^f(x_0, r) \right]^{p/[2(p-1)]}$.

Also recalling (6.9), it follows that

$$(6.23) \qquad \lim_{\varrho \to 0} F(x_0, \varrho) = 0.$$

In the following we shall abbreviate

$$(6.24) \qquad C_j := C(B^j) = |A_j|^{p/2} + H_1 E_j^{p/2} \qquad \text{for every } j \in \mathbb{N} \cup \{-1, 0\}.$$

6.4. INDUCTIVE LEMMA. — This is the following:

LEMMA 6.1. — *With the notation established in Sections* 6.1–6.3, *suppose that λ is a positive number and that for integers $k \geqslant m \geqslant 0$ the following inequalities*

$$(6.25) \qquad C_j \leqslant \lambda, \quad C_{j+1} \geqslant \frac{\lambda}{16} \qquad \forall\, j \in \{m, \dots, k\}, \qquad C_m \leqslant \frac{\lambda}{4},$$

and

$$(6.26) \qquad \left[\sum_{j=m}^{k} \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q} \right]^{p/[2(p-1)]} < \frac{2\lambda}{H_2}$$

hold. Then the inequalities

$$(6.27) \qquad C_{k+1} \leqslant \lambda,$$

$$(6.28) \qquad \sum_{j=m}^{k+1} E_j^{p/2} \leqslant \frac{\lambda}{2H_1}$$

and

$$(6.29) \qquad \sum_{j=m}^{k+1} E_j^{p/2} \leqslant 2 E_m^{p/2} + \frac{2^{7p} \lambda^{(2-p)/p}}{\varepsilon_1} \sum_{j=m}^{k} \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q}$$

hold true, where $\varepsilon_1 \equiv \varepsilon_1(n, N, p, \nu, L)$ has been introduced in (6.2).

To prove the above lemma we need the following:

LEMMA 6.2. — *With the notation established in Section* 6.1 *the estimates*

$$(6.30) \qquad \left| |A_{j+1}|^{p/2} - |A_j|^{p/2} \right| \leqslant \frac{8^p E_j^{p/2}}{\tau^{n/2}}$$

and

$$(6.31) \qquad E_{j+1}^{p/2} \leqslant \frac{8^p E_j^{p/2}}{\tau^n}$$

hold for every integer $j \geqslant -1$.

Proof. — Estimate (6.31) is a direct consequence of (3.38). In order to prove (6.30) we estimate

$$\left| |A_{j+1}|^{p/2} - |A_j|^{p/2} \right| \leqslant p \left(|A_{j+1}| + |A_j| \right)^{(p-2)/2} |A_{j+1} - A_j|$$

$$\leqslant 4^p |A_j|^{(p-2)/2} |A_{j+1} - A_j| + 4^p |A_{j+1} - A_j|^{p/2}$$

$$\leqslant 4^p |A_j|^{(p-2)/2} \left(\fint_{B^{j+1}} |Du - (Du)_{B^j}|^2 \, dx \right)^{1/2}$$

$$+ 4^p \left(\fint_{B^{j+1}} |Du - (Du)_{B^j}|^p \, dx \right)^{1/2}$$

$$\leqslant \frac{4^p}{\tau^{n/2}} \left(\fint_{B^j} |(Du)_{B^j}|^{p-2} |Du - (Du)_{B^j}|^2 \, dx \right)^{1/2}$$

$$+ \frac{4^p}{\tau^{n/2}} \left(\fint_{B^j} |Du - (Du)_{B^j}|^p \, dx \right)^{1/2}$$

$$\leqslant \frac{8^p E_j^{p/2}}{\tau^{n/2}},$$

and the proof is finished. □

Proof of Lemma 6.1. — The proof goes in several steps; with $m \leqslant k$ fixed as in the statement of the Lemma, the following arguments hold for an index $j \in \{m, \ldots, k\}$. We consider the alternatives (A1)–(A3) and their consequences.

Step 1: A preliminary estimate. — Using (6.30) and the definition of C_j gives

$$\left| |A_{j+1}|^{p/2} - |A_j|^{p/2} \right| \leqslant \frac{8^p C_j}{\tau^{n/2} H_1}.$$

Using the first inequality in (6.25) and recalling the definition of H_1 in (6.19), we then conclude with

$$(6.32) \qquad \left| |A_{j+1}|^{p/2} - |A_j|^{p/2} \right| \leqslant \frac{\lambda}{2^6}.$$

Step 2: (A3) *cannot hold.* — Indeed, we assume by contradiction that (6.17) holds. This means that (6.18) holds too and therefore, using the elementary inequality

$$(x + y)^{p/2} \leqslant 2^{\frac{p-2}{2}} x^{p/2} + 2^{\frac{p-2}{2}} y^{p/2} \qquad x, y \geqslant 0,$$

we can estimate

$$H_1 E_{j+1}^{p/2} \overset{(6.18)}{\leqslant} \frac{2^{\frac{p-2}{2}} H_1 E_j^{p/2}}{2^{7p/2}} + 2^{\frac{p-2}{2}} c_8^{p/2} H_1 \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{p/[2q(p-1)]}$$

$$\leqslant \frac{H_1 E_j^{p/2}}{2^7} + \frac{(2c_8)^{p/2}}{2} H_1 \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{p/[2q(p-1)]}$$

$$\overset{(6.26)}{\leqslant} \frac{H_1 E_j^{p/2}}{2^7} + \frac{(2c_8)^{p/2} H_1}{H_2} \lambda$$

$$\overset{(6.20),(6.24)}{\leqslant} \frac{C_j}{2^7} + \frac{\lambda}{2^7} \overset{(6.25)}{\leqslant} \frac{\lambda}{2^6}.$$

We again use the contradiction assumption (6.17) to infer

$$A_j \overset{(6.17)}{\leqslant} \frac{E_j}{\varepsilon_0} \leqslant \frac{1}{\varepsilon_0}\left(\frac{C_j}{H_1}\right)^{2/p} \overset{(6.25)}{\leqslant} \frac{1}{\varepsilon_0}\left(\frac{\lambda}{H_1}\right)^{2/p} \overset{(6.19)}{\leqslant} \left(\frac{\lambda}{2^6}\right)^{2/p}.$$

Therefore, by using the inequalities in the last two displays, we have

$$C_{j+1} \leqslant \left|\,|A_{j+1}|^{p/2} - |A_j|^{p/2}\right| + A_j^{p/2} + H_1 E_{j+1}^{p/2} \leqslant \frac{\lambda}{2^6} + \frac{\lambda}{2^6} + \frac{\lambda}{2^6} < \frac{\lambda}{16},$$

and this is impossible by the second inequality in (6.25). This means that (6.17) cannot hold.

Step 3: Consequences of (A1)–(A2). — Here we prove that

$$(6.33) \qquad E_{j+1}^{p/2} \leqslant \frac{E_j^{p/2}}{4} + \frac{2^{6p}\lambda^{(2-p)/p}}{\varepsilon_1}\left(r_j^q \fint_{B^j} |f|^q\,dx\right)^{1/q}$$

holds for every $j \in \{m, \dots, k\}$. By the result of Step 1 and by the fact that the index j considered was arbitrary, we conclude that under the assumption (6.25)–(6.26) we have that $E_j < \varepsilon_0 A_j$ holds for every $j \in \{m, \dots, k\}$ and this means that for each one of such indexes j one of the alternatives (A1)–(A2) holds. In the case (A1) holds, that is, the second inequality in (6.13) is verified, then we simply conclude with (6.14), that obviously implies (6.33). We are therefore left with the occurrence of (A2), and ultimately with the validity of the second inequality in (6.15). We can therefore estimate as follows:

$$H_1 E_{j+1}^{p/2} \overset{(6.16)}{\leqslant} c_7^{p/2} H_1 \left(r_j^q \fint_{B^j} |f|^q\,dx\right)^{p/[2q(p-1)]}$$

$$\overset{(6.26)}{\leqslant} \frac{2\lambda c_7^{p/2} H_1}{H_2} \overset{(6.20)}{\leqslant} \frac{\lambda}{2^6}.$$

This last inequality and (6.32) finally gives

$$C_{j+1} \leqslant \left|\,|A_{j+1}|^{p/2} - |A_j|^{p/2}\right| + A_j^{p/2} + H_1 E_{j+1}^{p/2} \leqslant A_j^{p/2} + \frac{\lambda}{2^5}.$$

Recalling that in (6.25) we are assuming that $C_{j+1} \geqslant \lambda/16$ we immediately find

$$(6.34) \qquad A_j^{p/2} \geqslant \frac{\lambda}{2^5},$$

so that

$$E_{j+1}^{p/2} \overset{(6.31)}{\leqslant} \frac{8^p E_j^{p/2}}{\tau^n} \overset{(6.15)}{\leqslant} \frac{8^p A_j^{(2-p)/2}}{\varepsilon_1}\left(r_j^q \fint_{B^j} |f|^q\,dx\right)^{1/q}$$

$$\overset{(6.34)}{\leqslant} \frac{2^{6p}\lambda^{(2-p)/p}}{\varepsilon_1}\left(r_j^q \fint_{B^j} |f|^q\,dx\right)^{1/q}$$

holds for every $j \in \{m, \dots, k\}$. This one again proves (6.33), that remains fully established in any occurrence of the alternatives (A1)–(A3).

Step 4: Proof of (6.27)–(6.29). — Summing inequalities (6.33) leads to

$$\sum_{j=m+1}^{k+1} E_j^{p/2} \leqslant \frac{1}{4}\sum_{j=m}^{k} E_j^{p/2} + \frac{2^{6p}\lambda^{(2-p)/p}}{\varepsilon_1}\sum_{j=m}^{k}\left(r_j^q \fint_{B^j}|f|^q\,dx\right)^{1/q}.$$

Reabsorbing terms and adding up $E_m^{p/2}$ to both sides of the resulting inequality yields

$$\sum_{j=m}^{k+1} E_j^{p/2} \leqslant \frac{4E_m^{p/2}}{3} + \frac{2^{6p+2}\lambda^{(2-p)/p}}{3\varepsilon_1}\sum_{j=m}^{k}\left(r_j^q \fint_{B^j}|f|^q\,dx\right)^{1/q},$$

which in turn implies (6.29). To proceed with the proof of (6.28) we have

$$\sum_{j=m}^{k+1} E_j^{p/2} \overset{(6.26)}{\leqslant} \frac{4C_m}{3H_1} + \frac{2^{6p+2}\lambda^{(2-p)/p}}{3\varepsilon_1}\frac{2^{2(p-1)/p}\lambda^{2(p-1)/p}}{H_2^{2(p-1)/p}}$$
$$\overset{(6.25)}{\leqslant} \frac{\lambda}{3H_1} + \frac{2^{6p+4}\lambda}{3\varepsilon_1 H_2^{2(p-1)/p}},$$

so that, recalling the choice of H_2 in (6.20), we conclude with

$$(6.35)\qquad\qquad \sum_{j=m}^{k+1} E_j^{p/2} \leqslant \frac{5\lambda}{12H_1}$$

which is particular implies (6.28). It remains to prove (6.27). For this we have

$$(6.36)\quad\begin{aligned}|A_{k+1}|^{p/2} &\leqslant |A_m|^{p/2} + \left||A_{k+1}|^{p/2}-|A_m|^{p/2}\right|\\ &\leqslant |A_m|^{p/2} + \sum_{j=m}^{k}\left||A_{j+1}|^{p/2}-|A_j|^{p/2}\right|\\ &\overset{(6.30)}{\leqslant} C_m + \frac{8^p}{\tau^{n/2}}\sum_{j=m}^{k}E_j^{p/2}\\ &\overset{(6.25),(6.35)}{\leqslant} \frac{\lambda}{4} + \frac{8^p 5\lambda}{12H_1\tau^{n/2}} \overset{(6.19)}{\leqslant} \frac{\lambda}{2}.\end{aligned}$$

Using this last inequality and (6.35) we finally have that

$$C_{k+1} = |A_{k+1}|^{p/2} + H_1 E_{k+1}^{p/2} \leqslant \lambda,$$

that is, (6.27), and the proof is complete. $\qquad\square$

6.5. **Non-zero gradient.** — Here we proceed with the proof of Theorem 1.1. We first treat the case when $Du(x_0)$, which is still to be proved to exist as precise representative of Du at x_0, is non-zero, in a quantitative defined sense. This goes via two technical lemmata; the first is the following:

Lemma 6.3. — *Assume that*

$$(6.37)\qquad\qquad \frac{\lambda}{8} := |A_0|^{p/2} > \frac{F(x_0,r)}{16}$$

holds. Then the estimate

$$(6.38) \qquad \sum_{j=0}^{\infty} E_j^{p/2} \leqslant \frac{F(x_0, r)}{H_1}$$

is true and

$$(6.39) \qquad \frac{\lambda}{16} \leqslant |A_j|^{p/2} \leqslant \lambda \qquad \text{holds for every } j \geqslant 0.$$

Proof. Step 1: A uniform lower bound. — We start proving that

$$(6.40) \qquad |A_j|^{p/2} \geqslant \frac{|A_0|^{p/2}}{2} \qquad \text{holds for every } j \geqslant 0,$$

which in fact implies the left-hand side inequality in (6.39) in view of (6.37). We preliminary observe that

$$
\begin{aligned}
|A_1|^{p/2} \quad \geqslant \quad & |A_0|^{p/2} - \left| |A_1|^{p/2} - |A_0|^{p/2} \right| \\
& \overset{(6.30)}{\geqslant} |A_0|^{p/2} - \frac{8^p E_0^{p/2}}{\tau^{n/2}} \\
& \overset{(6.31)}{\geqslant} |A_0|^{p/2} - \frac{8^{2p}[E(Du, B_r)]^{p/2}}{\tau^{2n}} \\
& \overset{(6.22)}{\geqslant} |A_0|^{p/2} - \frac{F(x_0, r)}{64 H_1} \overset{(6.37)}{\geqslant} \frac{|A_0|^{p/2}}{2}.
\end{aligned}
$$

In order to prove (6.40) we argue by contradiction, therefore assuming the existence of a finite exit time index $J \geqslant 2$ such that

$$(6.41) \qquad |A_J|^{p/2} < \frac{|A_0|^{p/2}}{2} \qquad \text{and} \qquad |A_j|^{p/2} \geqslant \frac{|A_0|^{p/2}}{2} \qquad \forall\, j \in \{0, \ldots, J-1\}.$$

Step 2: An upper bound. — Here we prove the implication

$$(6.42) \qquad |A_j|^{p/2} \geqslant \frac{|A_0|^{p/2}}{2} \quad \text{for } 0 \leqslant j \leqslant J-1 \Longrightarrow C_j \leqslant \lambda \quad \text{for } 0 \leqslant j \leqslant J-1.$$

We indeed prove by induction that

$$(6.43) \qquad C_j \leqslant \lambda \qquad \text{holds for every } j \in \{0, \ldots, J-1\}.$$

The starting inequality $C_0 \leqslant \lambda$ (induction basis) is proved as follows:

$$
\begin{aligned}
C_0 = \; & |A_0|^{p/2} + H_1 E_0^{p/2} \\
& \overset{(3.38)}{\leqslant} |A_0|^{p/2} + \frac{8^p H_1 [E(Du, B_r)]^{p/2}}{\tau^{n/2}} \\
& \overset{(6.22),(6.37)}{\leqslant} |A_0|^{p/2} + \frac{F(x_0, r)}{64} \overset{(6.37)}{\leqslant} 2|A_0|^{p/2} \overset{(6.37)}{=} \lambda/4.
\end{aligned}
$$

(6.44)

Then we assume by induction that

$$C_j \leqslant \lambda \qquad \text{holds for every } j \in \{0, \ldots, k\},$$

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and for a certain index $k \leqslant J - 2$, and prove that $C_{k+1} \leqslant \lambda$. In this respect, first observe that

$$
(6.45) \quad \left[\sum_{j=0}^{\infty} \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q} \right]^{p/[2(p-1)]} \overset{(6.6)}{\leqslant} \frac{\left[\boldsymbol{I}_{1,q}^f(x_0, r) \right]^{p/[2(p-1)]}}{\tau^{pn/(p-1)}}
$$

$$
\overset{(6.22), p \geqslant 2}{\leqslant} \frac{F(x_0, r)}{H_2}
$$

$$
\overset{(6.37)}{\leqslant} \frac{16|A_0|^{p/2}}{H_2}
$$

$$
(6.46) \quad \overset{(6.37)}{=} \frac{2\lambda}{H_2}.
$$

Next, by the very definition of C_j and the inductive assumption (6.41) it follows

$$
(6.47) \quad C_j \geqslant |A_j|^{p/2} \geqslant \frac{|A_0|^{p/2}}{2} = \frac{\lambda}{16} \quad \text{for every } j \in \{0, \ldots, J-1\}.
$$

Recalling (6.44), (6.46) and that $k + 1 \leqslant J - 1$, we apply Lemma 6.1 with $m = 0$ and k being the number used here; this yields $C_{k+1} \leqslant \lambda$. This in turn completes the proof of (6.43) by induction.

Step 3: Proof of (6.40) *completed.* — By (6.43), (6.44), (6.46) and (6.47) we can apply Lemma 6.1 with $m = 0$ and $k = J - 2$. We then have

$$
(6.48) \quad \sum_{j=0}^{J-1} E_j^{p/2} \overset{(6.28)}{\leqslant} \frac{\lambda}{2H_1} \overset{(6.37)}{\leqslant} \frac{4|A_0|^{p/2}}{H_1},
$$

so that

$$
\left| |A_J|^{p/2} - |A_0|^{p/2} \right| \leqslant \sum_{j=0}^{J-1} \left| |A_{j+1}|^{p/2} - |A_j|^{p/2} \right|
$$

$$
\overset{(6.30)}{\leqslant} \frac{8^p}{\tau^{n/2}} \sum_{j=0}^{J-1} E_j^{p/2}
$$

$$
\overset{(6.48)}{\leqslant} \frac{8^{p+1}|A_0|^{p/2}}{\tau^{n/2} H_1} \overset{(6.19)}{\leqslant} \frac{|A_0|^{p/2}}{4},
$$

and we can finally conclude with

$$
|A_J|^{p/2} \geqslant |A_0|^{p/2} - \left| |A_J|^{p/2} - |A_0|^{p/2} \right| \geqslant |A_0|^{p/2} - \frac{|A_0|^{p/2}}{4} = \frac{3|A_0|^{p/2}}{4}.
$$

This is a contradiction to (6.41) and therefore (6.40) is proved.

Step 4: Completion of the proof. — With (6.40) being now verified, we can apply (6.42) for every J and this leads to

$$
C_j \leqslant \lambda \quad \text{holds for every } j \in \mathbb{N} \cup \{0\}.
$$

By the very definition of the numbers $\{C_j\}$ this implies the right-hand side inequality in (6.39) so that the validity of (6.39) is completely established. To complete the proof

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it remains to check (6.38). For this, observe that inequality (6.40) also implies that $C_j \geqslant \lambda/16$ holds for every $j \in \mathbb{N} \cup \{0\}$, and this together with (6.44), allows to apply Lemma 6.1 with $m = 0$ and for every integer k. This gives

$$\sum_{j=0}^{\infty} E_j^{p/2} \overset{(6.29)}{\leqslant} 2E_0^{p/2} + \frac{2^{7p}\lambda^{(2-p)/p}}{\varepsilon_1} \sum_{j=0}^{\infty} \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q}$$

$$\overset{(6.37)}{\leqslant} 2E_0^{p/2} + \frac{2^{10p}[F(x_0,r)]^{(2-p)/p}}{\varepsilon_1} \sum_{j=0}^{\infty} \left(r_j^q \fint_{B^j} |f|^q \, dx \right)^{1/q}$$

$$\overset{(6.45)}{\leqslant} 2E_0^{p/2} + \frac{2^{10p}F(x_0,r)}{\varepsilon_1 H_2^{2(p-1)/p}}.$$

In turn, we continue estimating

$$E_0^{p/2} \overset{(6.31)}{\leqslant} \frac{8^p[E(Du, B_r)]^{p/2}}{\tau^n} \overset{(6.22)}{\leqslant} \frac{\tau^n F(x_0,r)}{8^{p+2}H_1} \leqslant \frac{F(x_0,r)}{4H_1}$$

and

$$\frac{2^{10p}F(x_0,r)}{\varepsilon_1 H_2^{2(p-1)/p}} \overset{(6.20)}{\leqslant} \frac{F(x_0,r)}{2H_1}.$$

Connecting the inequalities in the last three displays yields (6.38) and the proof is complete. $\qquad\square$

We are now ready for the next

LEMMA 6.4. — *Assume that* (6.37) *holds. Then the limits in* (1.18) *and* (1.34) *exist and* (1.35) *holds. Moreover, the inequalities*

$$(6.49) \qquad |Du(x_0) - (Du)_{B_r(x_0)}| \leqslant c \left[\boldsymbol{I}_{1,q}^f(x_0, r) \right]^{1/(p-1)} + cE\left(Du, B_r(x_0)\right)$$

and

$$(6.50) \quad |V(Du(x_0)) - (V(Du))_{B_r(x_0)}| \leqslant c \left[\boldsymbol{I}_{1,q}^f(x_0, r) \right]^{p/[2(p-1)]} + c\widetilde{E}\left(Du, B_r(x_0)\right)$$

hold for a constant c depending only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$, where $\widetilde{E}(\cdot)$ has been defined in (1.32).

Step 1: Proof of (6.49) *and assertions on Du.* — Recalling (2.7), we define

$$\overline{E}_j := \left(\fint_{B^j} |V(Du) - V((Du)_{B^j})|^2 \, dx \right)^{1/2}$$

for every integer $j \geqslant -1$, so that the inequalities in (2.8)–(2.10) imply that

$$(6.51) \quad \left(\fint_{B^j} |V(Du) - (V(Du))_{B^j}|^2 \, dx \right)^{1/2} \leqslant \overline{E}_j \leqslant cE_j^{p/2} \qquad \text{for every } j \geqslant -1.$$

We now have, for every integer $j \geqslant -1$

$$|V((Du)_{B^{j+1}}) - V((Du)_{B^j})| \leqslant |V((Du)_{B^{j+1}}) - (V(Du))_{B^{j+1}}|$$
$$+ |(V(Du))_{B^{j+1}} - (V(Du))_{B^j}| + |V((Du)_{B^j}) - (V(Du))_{B^j}|.$$

Then, by means of Jensen inequality we find

$$
\begin{aligned}
|V((Du)_{B^{j+1}}) - (V(Du))_{B^{j+1}}| \ &\leqslant\ \fint_{B^{j+1}} |V(Du) - V((Du)_{B^{j+1}})|\, dx \\
&\leqslant\ \overline{E}_{j+1} \\
&\overset{(6.51)}{\leqslant}\ cE_{j+1}^{p/2} \overset{(6.31)}{\leqslant}\ \frac{c8^p E_j^{p/2}}{\tau^n} = cE_j^{p/2}.
\end{aligned}
$$

The constant c depends only on n, N, p, ν, L and this follows by (6.1). Similarly

$$
(6.52) \qquad\qquad |V((Du)_{B^j}) - (V(Du))_{B^j}| \leqslant cE_j^{p/2}.
$$

and

$$
|(V(Du))_{B^{j+1}} - (V(Du))_{B^j}| \leqslant \fint_{B^{j+1}} |V(Du) - (V(Du))_{B^j}|\, dx \leqslant cE_j^{p/2}.
$$

Connecting the content of the last four displays yields

$$
(6.53) \qquad\qquad |V((Du)_{B^{j+1}}) - V((Du)_{B^j})| \leqslant cE_j^{p/2},
$$

for every $j \geqslant -1$ and for a constant c depending only on n, N, p, ν, L. On the other hand, by (6.39), we infer that

$$
|A_j|^{2-p} = |(Du)_{B^j}|^{2-p} \leqslant \left(\frac{\lambda}{16}\right)^{2(2-p)/p}
$$

holds for every $j \geqslant 0$. Therefore, with integers $0 \leqslant m \leqslant k-1$, we estimate as follows:

$$
\begin{aligned}
|(Du)_{B^k} - (Du)_{B^m}| \ &\leqslant\ \sum_{j=m}^{k-1} |(Du)_{B^{j+1}} - (Du)_{B^j}| \\
&\overset{(2.6)}{\leqslant}\ c\sum_{j=m}^{k-1} \frac{|V((Du)_{B^{j+1}}) - V((Du)_{B^j})|}{(|(Du)_{B^{j+1}}| + |(Du)_{B^j}|)^{(p-2)/2}} \\
(6.54) \qquad &\overset{(6.53)}{\leqslant}\ c\lambda^{(2-p)/p} \sum_{j=m}^{\infty} E_j^{p/2} \\
&\overset{(6.38)}{\leqslant}\ \frac{c\lambda^{(2-p)/p} F(x_0, r)}{H_1} \\
(6.55) \qquad &\overset{(6.21)}{\equiv}\ c\lambda^{(2-p)/p} F(x_0, r),
\end{aligned}
$$

where $c \equiv c(n, N, p, \nu, L, \mu(\cdot))$. This implies that $\{(Du)_{B^j}\}$ is a Cauchy sequence, which in fact gives that the limit

$$
\lim_{j\to\infty} (Du)_{B^j(x_0)} =: l \in \mathbb{R}^{N\times n}
$$

exists. It is then easy to see that this defines the whole limit in (1.18) and therefore the precise representative of Du at x_0, i.e. $l = Du(x_0)$. Indeed, for any positive $\varrho \leqslant \tau r$ we get the integer $j_\varrho \geqslant 1$ which is such that $\tau^{j_\varrho+1} r < \varrho \leqslant \tau^{j_\varrho} r$; thus, recalling that the series in (6.54) converges, we have

$$
(6.56) \qquad \lim_{\varrho\to 0} |l - (Du)_{B_\varrho(x_0)}| \leqslant \lim_{j_\varrho\to\infty} \left(|l - (Du)_{B^{j_\varrho}}| + \tau^{-n/p} E_{j_\varrho}\right) = 0,
$$

and (1.18) follows. We now let $k \to \infty$ in (6.55) with $m = 0$, thereby getting

$$(6.57) \qquad |Du(x_0) - (Du)_{B_{\tau r}(x_0)}| \leqslant c\lambda^{(2-p)/p} F(x_0, r)$$

and again

$$
\begin{aligned}
|Du(x_0) - (Du)_{B_r(x_0)}| &\leqslant |Du(x_0) - (Du)_{B_{\tau r}(x_0)}| + |(Du)_{B_r(x_0)} - (Du)_{B_{\tau r}(x_0)}| \\
&\stackrel{(6.57)}{\leqslant} c\lambda^{(2-p)/p} F(x_0, r) + \frac{E(Du, B_r(x_0))}{\tau^{n/p}} \\
&\stackrel{(6.37)}{\leqslant} c[F(x_0, r)]^{2/p} + cE(Du, B_r(x_0)) \\
&\stackrel{(6.22)}{\leqslant} c[F(x_0, r)]^{2/p}.
\end{aligned}
$$

In the above display the constant c depends only on n, N, p, ν, L and $\mu(\cdot)$. The inequality above is nothing but (6.49) once we recall again the definition in (6.22) and the dependence on the constants in (6.1) and (6.21).

Step 2: Proof of (6.50) *and assertions on* $V(Du)$. — The proof is similar to the one in Step 1. Indeed, with integers $0 \leqslant m \leqslant k - 1$ we have

$$
\begin{aligned}
|(V(Du))_{B^k} - (V(Du))_{B^m}| &\leqslant \sum_{j=m}^{k-1} |(V(Du))_{B^{j+1}} - (V(Du))_{B^j}| \\
&\stackrel{(6.53)}{\leqslant} c \sum_{j=m}^{k-1} E_j^{p/2} \leqslant c \sum_{j=m}^{\infty} E_j^{p/2} \stackrel{(6.38)}{\leqslant} cF(x_0, r),
\end{aligned}
$$

(6.58)

with c depending only on n, N, p, ν, L and $\mu(\cdot)$. We conclude that $\{(V(Du))_{B^j}\}$ is a Cauchy sequence, which in fact gives that the limit

$$\lim_{j \to \infty} (V(Du))_{B^j(x_0)} := l_V \in \mathbb{R}^{N \times n}$$

exists. We can now prove that the whole limit in (1.34) exists and coincides with l_V exactly as in Step 1. Indeed, observe that

$$(6.59) \qquad
\begin{aligned}
|(V(Du))_{B_r(x_0)} - (V(Du))_{B_{\tau r}(x_0)}| &\leqslant \frac{\widetilde{E}(Du, B_r(x_0))}{\tau^{n/2}} \\
&\leqslant c[E(Du, B_r(x_0))]^{p/2} \leqslant cF(x_0, r),
\end{aligned}
$$

while taking $m = 0$ and letting $k \to \infty$ in (6.58) we get the following analog of (6.57):

$$(6.60) \qquad |l_V - (V(Du))_{B_{\tau r}(x_0)}| \leqslant cF(x_0, r).$$

Then, with the same notation used for (6.56), and using a computation similar to the one in (6.59), we have

$$\lim_{\varrho \to 0} |l_V - (V(Du))_{B_\varrho(x_0)}| \leqslant \lim_{j_\varrho \to \infty} |l_V - (V(Du))_{B^{j_\varrho}}| + c \lim_{j_\varrho \to \infty} F(x_0, r_{j_\varrho}) = 0$$

and this proves that the limit in (1.34) equals l_V; recall (6.23). Moreover, (6.50) follows by connecting (6.59) to (6.60) and recalling (2.10). Finally, the identity in (1.35) follows directly from (6.52). $\qquad \square$

6.6. **Reiteration on small scales.** — We now want to see that the content of Lemma 6.4 can be actually replicated on smaller scales. We indeed have

Lemma 6.5. — *Assume that*

$$(6.61) \qquad |(Du)_{B_{\tau\varrho}}|^{p/2} > \frac{F(x_0, \varrho)}{16}$$

holds for some $\varrho \leqslant r$. Then the limits in (1.18) and (1.34) exist and (1.35) holds. Moreover, the inequalities

$$(6.62) \qquad |Du(x_0) - (Du)_{B_\varrho(x_0)}| \leqslant c \left[\boldsymbol{I}^f_{1,q}(x_0, \varrho) \right]^{1/(p-1)} + cE\left(Du, B_\varrho(x_0)\right)$$

and

$$(6.63) \qquad |V(Du(x_0)) - (V(Du))_{B_\varrho(x_0)}| \leqslant c \left[\boldsymbol{I}^f_{1,q}(x_0, \varrho) \right]^{p/[2(p-1)]} + c\widetilde{E}\left(Du, B_\varrho(x_0)\right)$$

hold for a constant c depending only on n, N, p, ν, L and $\eta(\cdot)$.

Proof. — To prove the lemma we basically have to show that the arguments developed for Lemma 6.4 can be applied by replacing $B_r(x_0)$ with $B_\varrho(x_0)$; then (6.62)–(6.63) just follow from (6.49)–(6.50). For this we have to go back to the previous proofs and single out the crucial conditions used to start the whole argument. These are given by the convergence in (6.9), which is obviously satisfied independently of the starting ball $B_\varrho(x_0)$, and by the initial smallness conditions (6.5) and (6.10), that we want to be satisfied on the smaller scale $B_{\tau\varrho}(x_0)$ once they are satisfied on $B_{\tau r}(x_0)$. This amounts to check that

$$(6.64) \qquad c_5 \sup_{s \leqslant \tau\varrho} \left(s^q \fint_{B_s} |f|^q \, dx \right)^{1/[q(p-1)]} \leqslant \frac{\tau^n \min\{\overline{\varepsilon}, \varepsilon_2\}}{8^{2p+2}}$$

and

$$(6.65) \qquad E\left(Du, B_{\tau\varrho}\right) < \varepsilon_2$$

are verified. Condition (6.64) trivially follows from (6.5) since $\varrho \leqslant r$. As for (6.65), iterated application of (6.4)–(6.5) in (6.11) gives

$$(6.66) \qquad E_{j-1} \equiv E\left(Du, B_{\tau^j r}\right) \leqslant \frac{\tau^n \min\{\overline{\varepsilon}, \varepsilon_2\}}{8^{2p+1}}$$

for every integer $j \geqslant 1$ (recall that $B_1 \equiv B_r$), in the same way as we already got (6.12). Now, let us consider a number $\varrho \leqslant r$ as in the statement of the Lemma and determine the integer $j \geqslant 0$ such that $\tau^{j+2} r < \tau\varrho \leqslant \tau^{j+1} r$; we have

$$[E(Du, B_{\tau\varrho})]^{p/2} \overset{(3.38)}{\leqslant} \frac{8^p E_j^{p/2}}{\tau^n} \overset{(6.66)}{<} \varepsilon_2.$$

Notice that the application of (3.38) has been made with values of τ and ϱ which are obviously different from here. The above inequality, together with (6.66), means that we have proved (6.65) for every $\varrho \leqslant r$ and the proof is complete. $\qquad\square$

6.7. Small gradients. — Here we treat the remaining case when (6.37) is not satis-fied, and the system is potentially degenerate.

Lemma 6.6. — *Assume that*

$$(6.67) \qquad |A_0|^{p/2} \leqslant \frac{F(x_0, r)}{16} =: \frac{\lambda}{8}$$

holds. Then the limits in (1.18) *and* (1.34) *exist and* (1.35) *holds. Moreover, the inequalities in* (6.49) *and* (6.50) *hold for a constant c depending only on* n, N, p, ν, L *and* $\eta(\cdot)$.

Proof. — With no loss of generality we consider the case when $\lambda > 0$, since otherwise Du is constant in $B_r(x_0)$ and there is nothing to prove. The rest of the proof goes in three different steps.

Step 1: Existence of the limit in (1.18). — We can assume that

$$(6.68) \qquad 0 < s \leqslant r \Longrightarrow |(Du)_{B_{\tau s}}|^{p/2} \leqslant \frac{F(x_0, s)}{16}$$

holds, otherwise, if for some $s \leqslant r$ the opposite inequality, i.e., (6.61), is verified, then by Lemma 6.5 the limit in (1.18) exists and we are done. Now, observe that since (6.68) holds for every $s \leqslant r$, by (6.9) it then follows

$$(6.69) \qquad \lim_{s \to 0} (Du)_{B_{\tau s}} = 0 \in \mathbb{R}^{N \times n},$$

and once again the existence of the limit in (1.18) is proved. Notice that (6.69) and (6.9) imply that $(|Du|^p)_{B_{\tau s}} \to 0$ and $s \to 0$ and in turn this immediately gives that

$$\lim_{s \to 0} (V(Du))_{B_{\tau s}} = 0 \in \mathbb{R}^{N \times n},$$

so that also (1.34) is proved together with (1.35). The rest of the proof is now dedicated to establish estimates (6.49)–(6.50).

Step 2: Uniform bound. — Here we prove that

$$(6.70) \qquad C_j \leqslant \lambda \qquad \text{holds for every } j \in \mathbb{N}$$

and argue by contradiction. By noting that

$$\begin{aligned}
C_0 &= |A_0|^{p/2} + H_1 E_0 \\
(6.71) \qquad &\overset{(6.68)}{\leqslant} \frac{F(x_0, r)}{16} + \frac{8^p [E(Du, B_r)]^{p/2}}{\tau^n} \\
&< \frac{F(x_0, r)}{8} \overset{(6.67)}{=} \frac{\lambda}{4},
\end{aligned}$$

we then define $k := \min\{s \in \mathbb{N} \cup \{0\} : C_{s+1} > \lambda\}$ as the smallest integer (minus one) for which (6.70) fails. Let us consider the set

$$\mathscr{J}_k := \{j \in \mathbb{N} \cup \{0\} : C_j \leqslant \lambda/4, \ j < k+1\} \quad \text{and} \quad m := \max \mathscr{J}_k.$$

Notice that $\mathscr{J}_k$ is non empty by (6.71). By the very definition of m we have $C_m \leqslant \lambda/4$. Also, notice that $j \in \{m+1, \dots, k+1\}$ implies that $C_j \geqslant \lambda/4 > \lambda/16$ (recall also the definition of k). Finally, by the same definition of k, we have that $j \in \{m, \dots, k\}$

implies that $C_j \leqslant \lambda$. Conditions in (6.25) are therefore satisfied in the set of indexes $\{m, \ldots, k\}$. In view of an application of Lemma 6.1 it remains to check that (6.26) is satisfied with the current choice of λ in (6.67). But this can be done exactly as in (6.45). Lemma 6.1 therefore applies and yields $C_{k+1} \leqslant \lambda$, which contradicts the fact that $C_{k+1} > \lambda$. This finally implies (6.70).

Step 3: Decay estimate and completion. — We first prove the inequality

$$(6.72) \qquad \sup_{s \leqslant r} |(Du)_{B_s}| \leqslant 2[F(x_0, r)]^{2/p}.$$

For this, we take $s \in (0, \tau r]$ and determine $k \in \mathbb{N} \cup \{0\}$ such that $\tau^{k+2} r < s \leqslant \tau^{k+1} r =: r_k$. We get

$$
\begin{aligned}
|(Du)_{B_s}| &\leqslant |A_k| + |(Du)_{B_s(x_0)} - (Du)_{B^k}| \\
&\leqslant |A_k| + \frac{1}{\tau^n} \fint_{B^k} |Du - (Du)_{B^k}| \, dx \\
&\leqslant |A_k| + \frac{1}{\tau^n} \left(\fint_{B^k} |Du - (Du)_{B^k}|^p \, dx \right)^{1/p} \\
&\overset{(6.19)}{\leqslant} |A_k| + H_1^{2/p} E_k \\
&\leqslant 2 \left(|A_k|^{p/2} + H_1 E_k^{p/2} \right)^{2/p} \\
&= 2 C_k^{2/p} \overset{(6.70)}{\leqslant} 2\lambda^{2/p} \overset{(6.67)}{\leqslant} 2^{1-2/p} [F(x_0, r)]^{2/p}.
\end{aligned}
$$

In the remaining case $s \in (\tau r, r]$ we similarly have

$$
\begin{aligned}
|(Du)_{B_s}| &\leqslant |A_0| + |(Du)_{B_s} - (Du)_{B^0}| \\
&\leqslant |A_0| + \frac{2}{\tau^n} \fint_{B_r} |Du - (Du)_{B_r}| \, dx \\
&\leqslant |A_0| + \frac{2 E(Du, B_r)}{\tau^{2n}} \\
&\leqslant \frac{[F(x_0, r)]^{2/p}}{2^{8/p}} + H_1^{2/p} E(Du, B_r) \\
&\leqslant 2 [F(x_0, r)]^{2/p}.
\end{aligned}
$$

Connecting the content of the last two displays yields the proof of (6.72). In turn, (6.72) trivially implies

$$|(Du)_{B_s} - (Du)_{B_r}| \leqslant 4 [F(x_0, r)]^{2/p}.$$

and (6.49) follows letting $s \to 0$ in the above inequality, also recalling the definition in (6.22) and that in Step 1 we have proved that the limit in (1.18) exists. The proof is complete.   □

We finally can reiterate the above lemma on smaller scales as done in Section 6.6 for Lemmas 6.4–6.5.

Lemma 6.7. — *Assume that*

$$(6.73) \qquad\qquad |(Du)_{B_{\tau_\varrho}}|^{p/2} \leqslant \frac{F(x_0, \varrho)}{16}$$

holds for some $\varrho \leqslant r$. Then the limits in (1.18) and (1.34) exist and (1.35) holds. Moreover, the inequalities in (6.49) and (6.50) hold for a constant c depending only on n, N, p, ν, L and $\eta(\cdot)$.

Proof. — The proof goes exactly as the one of Lemma 6.5. By the arguments developed there, and in particular observing that (6.65) holds, we can replicate Lemma 6.6 replacing $B_r(x_0)$ by $B_\varrho(x_0)$ and the proof follows. $\qquad\square$

6.8. Proof of Theorems 1.1 and 1.7. — The proof now follows from the content of Sections 6.1–6.7. In particular, after determining the various quantities $\tau, \varepsilon_0, \varepsilon_1, \varepsilon_2, \overline{\overline{\varepsilon}}$ as in (6.1)–(6.3) and in Proposition 5.2, we determine the final smallness threshold number ε appearing in (1.17) via the choice in (6.8). The result then follows by matching the content of Lemmas 6.5 and 6.7. Notice that according to the above scheme both the Theorems are proved using condition (1.17); on the other hand this is in fact equivalent to a condition of the type in (1.33) via (2.8)–(2.9) so that assumptions (1.17) and (1.33) are actually interchangeable.

6.9. Proof of Theorem 1.2. — We start considering the number ε that has been determined in Theorem 1.1 and and the number $\overline{\varepsilon}$ determined in Proposition 5.2; both of them depend only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$. We then let

$$(6.74) \qquad\qquad \varepsilon_s := \min\{\varepsilon, \overline{\varepsilon}\}/2,$$

which inherits the same dependence on the parameters and fixes the choice of ε_s in the statement of Theorem 1.2. Thanks to assumption (1.20), we determine a radius ϱ_s such that

$$(6.75) \qquad \left(\sup_{\varrho \leqslant \varrho_s} \varrho^q \fint_{B_\varrho(x)} |f|^q \, dx \right)^{1/[q(p-1)]} + \left[\boldsymbol{I}_{1,q}^f(x, r) \right]^{1/(p-1)} < \varepsilon_s$$

for every $x \in \Omega$ and $r \in (0, \varrho_s)$; this fixes the choice of ϱ_s in the statement. Moreover, we have that

$$(6.76) \qquad\qquad \lim_{\varrho \to 0} \varrho^q \fint_{B_\varrho(x)} |f|^q \, dy = 0$$

holds locally uniformly in Ω_u. Notice that this is possible since, as seen using (6.6) with for instance $r_j \equiv r/2^j$, assumption (1.20) implies that (1.24) holds uniformly with respect to x. As a consequence of this choice notice that the radius ϱ_s depends only on $n, N, p, \nu, L, \mu(\cdot)$ and $\eta(\cdot)$ and on the rate of convergence in (1.20). We then define Ω_u exactly as in (1.22), with the current choice of ε_s in (6.74) and (6.75), respectively. This set is larger then $\mathscr{L}_u$ defined in (5.12) and has therefore full measure. Consider now $x_0 \in \Omega_u$; we can find a radius $r_{x_0} \leqslant \varrho_s$ such

that $B_{r_{x_0}}(x_0) \Subset \Omega$ and $E\left(Du, B_{r_{x_0}}(x_0)\right) < \varepsilon_s$. Then, by the continuity of the function $x \mapsto E\left(Du, B_{r_{x_0}}(x)\right)$, we have that there exists a small neighborhood of x_0, call it $\mathscr{O}(x_0)$, such that

$$(6.77) \qquad E\left(Du, B_{r_{x_0}}(x)\right) < \varepsilon_s \quad \text{and} \quad B_{\varrho_{x_0}}(x) \Subset \Omega \quad \text{hold for every } x \in \mathscr{O}(x_0).$$

This shows that Ω_u is an open subset. Let us prove that $Du \in C^0(\mathscr{O}(x_0); \mathbb{R}^{N \times n})$; then the continuity of Du in Ω_u follows by covering. By (6.75) and (6.77) and the very definition of ε_s in (6.74), we find that

$$(6.78) \qquad \begin{cases} E\left(Du, B_{r_{x_0}}(x)\right) + \left[\boldsymbol{I}_{1,q}^f(x, r_2)\right]^{1/(p-1)} < \varepsilon \\[3mm] E\left(Du, B_{r_{x_0}}(x)\right) + \left(\sup_{\varrho \leqslant r_2} \varrho^q \fint_{B_\varrho(x)} |f|^q \, dy\right)^{1/[q(p-1)]} < \overline{\varepsilon} \end{cases}$$

hold for every $x \in \mathscr{O}(x_0)$. Those in (6.78), together with the one in (6.76), are exactly the conditions allowing to apply Theorem 1.1 and Proposition 5.2. Theorem 1.1 in particular implies the fact that the limit in (1.18) exists for every point $x \in \mathscr{O}(x_0)$, thereby defining the almost precise representative $Du(x)$ for every $x \in \mathscr{O}(x_0)$. The idea is now to prove that the limit in (1.18) is uniform. Specifically, we prove that the continuous maps $x \mapsto (Du)_{B_r(x)}$, defined for every $x \in \mathscr{O}(x_0)$ and $r \in (0, r_2)$, are uniformly converging to $Du(x)$ in $\mathscr{O}(x_0)$ as $r \to 0$; this immediately implies the continuity of Du in $\mathscr{O}(x_0)$. At this point this follows directly by estimate (1.19) since both terms on the right hand side uniformly converges to zero. Indeed, the first terms on the right hand side directly by assumption (1.20), while the second – the excess term – is converging to zero uniformly with respect to x by the last assertion in Proposition 5.2. The proof is complete.

6.10. Proof of Theorem 1.3. — This is a direct consequence of Theorem 1.2, since assumption (1.23) implies the one in (1.20). For this we refer for instance to [9, 34].

6.11. Proof of Theorem 1.6. — We prove the estimate for the singular set $\Omega \setminus \Omega_u$ in the case of Theorem 1.2, the proof for the remaining ones is the same. We adopt the point of view of [38, 27], where singular sets estimates have been obtained via fractional differentiability of gradients of solutions. For this we recall that a map $g \colon \Omega \to \mathbb{R}^{N \times n}$ belongs to the fractional Sobolev space $W_{\mathrm{loc}}^{\sigma, q}(\Omega; \mathbb{R}^{N \times n})$ for $\sigma \in (0, 1)$ and $q \in [1, \infty)$ if and only if

$$\int_{\widetilde{\Omega}} |g|^q \, dx + \int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} \frac{|g(x) - g(y)|^q}{|x - y|^{n + \sigma q}} \, dx \, dy < \infty$$

holds for every open subset $\widetilde{\Omega} \Subset \Omega$. We next recall a classical fact from potential theory (see for instance [38] for a short proof), that is,

$$(6.79) \qquad g \in W_{\mathrm{loc}}^{\sigma, q}(\Omega; \mathbb{R}^{N \times n}) \implies \dim_{\mathscr{H}}(\mathscr{B}_1(g) \cup \mathscr{B}_2(g)) \leqslant n - \sigma q$$

provided $\sigma q < n$ and where

$$(6.80) \qquad \mathscr{B}_1(g) := \left\{ x \in \Omega : \limsup_{\varrho \to 0} |(g)_{B_\varrho(x)}| < \infty \right\}$$

and

$$(6.81) \qquad \mathscr{B}_2(g) := \left\{ x \in \Omega : \ \liminf_{\varrho \to 0} \fint_{B_\varrho(x)} |g - (g)_{B_\varrho(x)}|^q \, dy > 0 \right\}.$$

We next recall a classical estimate from [45], stating that

$$(6.82) \qquad Du \in W_{\mathrm{loc}}^{\frac{1-\varepsilon}{p-1},p}(\Omega; \mathbb{R}^{N \times n}) \qquad \text{for every } \varepsilon \in (0,1).$$

This result is proved in [45] under the assumption that $f \in L_{\mathrm{loc}}^{p/(p-1)}(\Omega; \mathbb{R}^N)$, which is in fact also considered in Theorem 1.6. We just observe that, although the results in [45] are stated for equations, the proofs still work in the case of the system (1.1) considered under the only assumptions $(1.3)_{1,2}$. This is a consequence of the fact that the techniques are essentially based on a number of monotonicity properties that are indeed implied by $(1.3)_{1,2}$. To proceed, we have that (6.79) and (6.82), together with the definitions in (6.80)–(6.81), give

$$(6.83) \qquad \dim_{\mathscr{H}}(\mathscr{B}_1(Du) \cup \mathscr{B}_2(Du)) \leqslant n - \frac{p}{p-1}.$$

Again by the very definition of the sets $B_1(Du)$ and $B_2(Du)$, and the one of the excess functional $E(\cdot)$ in (2.4), we have the inclusion $(\Omega \smallsetminus B_1(Du)) \cap (\Omega \smallsetminus B_2(Du)) \subset \Omega_u$, where Ω_u has been defined in (1.22). Therefore we also have $\Omega \smallsetminus \Omega_u \subset B_1(Du) \cup B_2(Du)$ so that (1.29) follows from this last inclusion and (6.83), and the proof is complete.

REMARK 6.1. — Assuming more regularity on f allows to get singular sets estimates that are better than the one in (1.29). Specifically, a recent result of Brasco & Santambrogio [5] asserts that if $f \in W_{\mathrm{loc}}^{\sigma,p/(p-1)}(\Omega; \mathbb{R}^N)$ with $1 - 2/p < \sigma \leqslant 1$, then $V(Du) \in W_{\mathrm{loc}}^{1,2}(\Omega; \mathbb{R}^N)$. This fact, together with the equivalence given in (1.31) and/or using directly Theorem 1.7, leads to the following improvement of (1.29):

$$(6.84) \qquad \dim_{\mathscr{H}}(\Omega \smallsetminus \Omega_u) \leqslant n - 2.$$

The proof is similar to the argument used to deduce Theorem 1.6.

References

[1] E. Acerbi & N. Fusco – "A regularity theorem for minimizers of quasiconvex integrals", *Arch. Rational Mech. Anal.* **99** (1987), no. 3, p. 261–281.

[2] D. R. Adams & L. I. Hedberg – *Function spaces and potential theory*, Grundlehren Math. Wiss., vol. 314, Springer-Verlag, Berlin, 1996.

[3] A. Alberico, A. Cianchi & C. Sbordone – "Continuity properties of solutions to the *p*-Laplace system", *Adv. Calc. Var.* (2015), online.

[4] P. Baroni – "Riesz potential estimates for a general class of quasilinear equations", *Calc. Var. Partial Differential Equations* **53** (2015), no. 3-4, p. 803–846.

[5] L. Brasco & F. Santambrogio – "A sharp estimate à la Calderón-Zygmund for the *p*-Laplacian", `arXiv:1607.06648`, 2016.

[6] A. Cianchi – "Nonlinear potentials, local solutions to elliptic equations and rearrangements", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **10** (2011), no. 2, p. 335–361.

[7] A. Cianchi & V. G. Maz'ya – "Global Lipschitz regularity for a class of quasilinear elliptic equations", *Comm. Partial Differential Equations* **36** (2011), no. 1, p. 100–133.

[8] _____, "Global boundedness of the gradient for a class of nonlinear elliptic systems", *Arch. Rational Mech. Anal.* **212** (2014), no. 1, p. 129–177.

[9] P. Daskalopoulos, T. Kuusi & G. Mingione – "Borderline estimates for fully nonlinear elliptic equations", *Comm. Partial Differential Equations* **39** (2014), no. 3, p. 574–590.

[10] E. De Giorgi – "Frontiere orientate di misura minima", *Sem. di Mat. de Scuola Norm. Sup. Pis.* (1960-61), p. 1–56.

[11] F. Duzaar & G. Mingione – "The *p*-harmonic approximation and the regularity of *p*-harmonic maps", *Calc. Var. Partial Differential Equations* **20** (2004), no. 3, p. 235–256.

[12] _____ , "Regularity for degenerate elliptic problems via *p*-harmonic approximation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004), no. 5, p. 735–766.

[13] _____ , "Local Lipschitz regularity for degenerate elliptic systems", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 6, p. 1361–1396.

[14] F. Duzaar, G. Mingione & K. Steffen – *Parabolic systems with polynomial growth and regularity*, Mem. Amer. Math. Soc., vol. 214, no. 1005, American Mathematical Society, 2011.

[15] M. Foss & G. Mingione – "Partial continuity for elliptic problems", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), no. 3, p. 471–503.

[16] N. Fusco & J. Hutchinson – "Partial regularity for minimisers of certain functionals having non-quadratic growth", *Ann. Mat. Pura Appl. (4)* **155** (1989), p. 1–24.

[17] M. Giaquinta – *Introduction to regularity theory for nonlinear elliptic systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[18] M. Giaquinta & G. Modica – "Remarks on the regularity of the minimizers of certain degenerate functionals", *Manuscripta Math.* **57** (1986), no. 1, p. 55–99.

[19] E. Giusti – *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[20] E. Giusti & M. Miranda – "Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari", *Arch. Rational Mech. Anal.* **31** (1968), no. 3, p. 173–184.

[21] C. Hamburger – "Regularity of differential forms minimizing degenerate elliptic functionals", *J. reine angew. Math.* **431** (1992), p. 7–64.

[22] L. I. Hedberg & T. H. Wolff – "Thin sets in nonlinear potential theory", *Ann. Inst. Fourier (Grenoble)* **33** (1983), no. 4, p. 161–187.

[23] J. Heinonen, T. Kilpeläinen & O. Martio – *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993.

[24] T. Kilpeläinen & J. Malý – "Degenerate elliptic equations with measure data and nonlinear potentials", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **19** (1992), no. 4, p. 591–613.

[25] _____ , "The Wiener test and potential estimates for quasilinear elliptic equations", *Acta Math.* **172** (1994), no. 1, p. 137–161.

[26] R. Korte & T. Kuusi – "A note on the Wolff potential estimate for solutions to elliptic equations involving measures", *Adv. Calc. Var.* **3** (2010), no. 1, p. 99–113.

[27] J. Kristensen & G. Mingione – "The singular set of minima of integral functionals", *Arch. Rational Mech. Anal.* **180** (2006), no. 3, p. 331–398.

[28] J. Kristensen & A. Taheri – "Partial regularity of strong local minimizers in the multi-dimensional calculus of variations", *Arch. Rational Mech. Anal.* **170** (2003), no. 1, p. 63–89.

[29] M. Kronz – "Partial regularity results for minimizers of quasiconvex functionals of higher order", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **19** (2002), no. 1, p. 81–112.

[30] T. Kuusi & G. Mingione – "Nonlinear vectorial potential theory", to appear in *J. Eur. Math. Soc. (JEMS)*.

[31] _____ , "Universal potential estimates", *J. Funct. Anal.* **262** (2012), no. 10, p. 4205–4269.

[32] _____ , "Linear potentials in nonlinear potential theory", *Arch. Rational Mech. Anal.* **207** (2013), no. 1, p. 215–246.

[33] _____ , "Borderline gradient continuity for nonlinear parabolic systems", *Math. Ann.* **360** (2014), no. 3-4, p. 937–993.

[34] _____ , "Guide to nonlinear potential estimates", *Bull. Math. Sci.* **4** (2014), no. 1, p. 1–82.

[35] _____ , "A nonlinear Stein theorem", *Calc. Var. Partial Differential Equations* **51** (2014), no. 1-2, p. 45–86.

[36] V. G. Maz'ja – "The continuity at a boundary point of the solutions of quasi-linear elliptic equations", *Vestnik Leningrad Univ. Math.* **25** (1970), no. 13, p. 42–55.

[37] V. G. Maz'ja & V. P. Havin – "A nonlinear potential theory", *Uspehi Mat. Nauk* **27** (1972), no. 6, p. 67–138.

[38] G. Mingione – "The singular set of solutions to non-differentiable elliptic systems", *Arch. Rational Mech. Anal.* **166** (2003), no. 4, p. 287–301.

[39] _____, "Regularity of minima: an invitation to the dark side of the calculus of variations", *Appl. Math.* **51** (2006), no. 4, p. 355–426.

[40] _____, "Gradient potential estimates", *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 2, p. 459–486.

[41] C. B. Morrey, Jr. – "Partial regularity results for non-linear elliptic systems", *J. Math. Mech.* **17** (1967/1968), p. 649–670.

[42] N. C. Phuc & I. E. Verbitsky – "Quasilinear and Hessian equations of Lane-Emden type", *Ann. of Math. (2)* **168** (2008), no. 3, p. 859–914.

[43] _____, "Singular quasilinear and Hessian equations and inequalities", *J. Funct. Anal.* **256** (2009), no. 6, p. 1875–1906.

[44] T. Schmidt – "Regularity theorems for degenerate quasiconvex energies with (p, q)-growth", *Adv. Calc. Var.* **1** (2008), no. 3, p. 241–270.

[45] J. Simon – "Régularité de solutions de problèmes nonlinéaires", *C. R. Acad. Sci. Paris Sér. A-B* **282** (1976), no. 23, p. A1351–A1354.

[46] N. S. Trudinger & X.-J. Wang – "On the weak continuity of elliptic operators and applications to potential theory", *Amer. J. Math.* **124** (2002), no. 2, p. 369–410.

[47] K. Uhlenbeck – "Regularity for a class of non-linear elliptic systems", *Acta Math.* **138** (1977), no. 3-4, p. 219–240.

[48] N. N. Ural'ceva – "Degenerate quasilinear elliptic systems", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **7** (1968), p. 184–222.

Tuomo Kuusi, Aalto University, Institute of Mathematics
P.O. Box 11100, FI-00076 Aalto, Finland
E-mail : `tuomo.kuusi@aalto.fi`
Url : `https://sites.google.com/site/tuomokuusimath/home`

Giuseppe Mingione, Dipartimento di Matematica e Informatica, Università di Parma
Parco Area delle Scienze 53/a, Campus, 43100 Parma, Italy
E-mail : `giuseppe.mingione@unipr.it`
Url : `https://sites.google.com/site/giuseppemingionemath/`