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LONG MEMORY IN ECONOMICS DISCUSSION AND COMMENTS * **

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The article of Lardic and Mignon (1999) provides a useful survey on applications of long memory in Economics. It begins with a description of the ARFIMA models, describes a number of estimation methods and then considers several extensions such as seasonal ARFIMA, the GARMA where the poles of the spectral density are not at zero frequency, the FIGARCH which allows a random variance and then surveys applications in Finance and Economics.

The ARFIMA models are based on a "story". One has the ARIMA(p, d, q) which involves the difference operator Δ to the power d, where d is a non-negative integer. When $d \geq 1$, the ARIMA is not stationary and differencing it produces a stationary series. This is why Δ^d is used. What then compels us to choose d fractional? Mathematical aesthetics, in part : one can make sense of the resulting process. As a bonus, the process is stationary and has long memory if $0 < d < 1/2$.

Using Δ^d, however, is only one way to generate long memory. One can start, more generally, with a linear process

$$X_t = \sum_{k=-\infty}^{\infty} c_{t-k} \epsilon_k, \quad t \in \mathbb{Z}, \tag{1}$$

with coefficients c_k behaving asymptotically like k^{d-1}, $0 < d < 1/2$, as $k \to \infty$. The resulting process $X(t)$, $t \in \mathbb{Z}$, has long memory because its spectral density explodes at the origin. The ARFIMA models are then special cases, but with the following two features : (i) they involve a finite number of unknown parameters and (ii) once d has been estimated, one falls back on the well-known ARMA case, thus allowing a two-stage estimation procedure : first d, then the parameters of the ARMA.

Note that the first feature in itself, is relatively weak because it is possible to have a non-ARFIMA linear process (1) with long memory which also depends on a finite number of parameters. Fractional Gaussian noise, which

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is a stationary process with long memory derived from fractional Brownian motion, can be regarded as an example, although strictly speaking, one has to replace the sum in (1) by an integral.

Incidently, fractional Brownian motion does not have the representation (13) stated in the paper. Its representation for $t \geq 0$ is (up to a multiplicative constant)

$$B_H(t) = \int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right) dB(s) + \int_{0}^{t} (t-s)^{H-1/2} dB(s).$$

The first integral, which was omitted in Relation (13) of the paper guarantees the stationarity of the increment, and hence that the covariance is given by Relation (15) of the paper (again, up to a multiplicative constant). Mandelbrot and Van Ness (1968) represented fractional Brownian motion as a fractional integral of Brownian motion. For a mathematical review of the main properties of fractional Brownian motion, and for its connection with fractional integration, see Taqqu (2001) and Pipiras and Taqqu (2001). These papers will appear in a book about long memory, with review articles describing different aspects of the subject.

When dealing with the problem of estimating d, one should distinguish between graphical methods and variants of maximum likelihood. Since the ARMA part of the ARFIMA can significantly affect the estimation of d, a graphical method is often very useful in practice because it allows one to see, using a log-log plot, where linearity breaks down. Although it is not easy to derive a precise statistical test, a graphical method has the advantage of offering an estimate which is useful in practice. In the papers Taqqu, Teverovsky and Willinger (1995) and Taqqu and Teverovsky (1998) one can find a systematic presentation of various estimates of d, some graphical, some not. These papers use simulated data (fractional Gaussian noise, ARFIMA, and also innovations that may have finite or infinite variance) and they provide benchmarks to the practitioner. The methods considered include the aggregated variance method, a method based on differencing the variance, the absolute value of the aggregated series method, Higuchi's method, the residual of regression method and R/S. A number of periodogram-based methods are also included, namely the regression on the periodogram (the Geweke and Porter-Hudak (1983) method – for a good theoretical treatment see Robinson (1995b)), the Whittle method (which is a variant of the maximum likelihood), the aggregated Whittle method and the local Whittle method developed by Robinson (1995a).

Theoretical results for the Whittle method, obtained by Fox and Taqqu (1986), were extended to the full maximum likelihood by Dahlhaus (1989). In practice, the Whittle is almost as effective as the maximum likelihood and much more computationally efficient. However, the Whittle method is parametric : one needs to know the exact model. The method is excellent if one has this information but can yield poor results otherwise.

The other methods are essentially semi-parametric. They require making the assumption of long-memory (a spectral density which explodes at the origin) without specifying the spectrum at higher frequencies. One is then faced with a bias versus variance problem. In the spectral setting, one has many data points for estimating far from the zero frequency but d is not relevant there, so there is bias; estimating close to the zero frequency decreases the bias but increases the variance since one does not have a lot of data points.

The local Whittle method, due to Robinson, is one of the best semi-parametric methods. The idea is simply to use the periodogram-based Whittle estimation but apply it only to the m Fourier frequencies closest to the origin. The practical choice of m is analyzed through simulations in Taqqu and Teverovsky (1996), (1997). It is analyzed theoretically in Giraitis, Robinson and Samarov (1997).

A method which is as good as the local Whittle but which has additional advantages is the wavelet-based method (see Abry and Veitch (1998), Abry, Flandrin, Taqqu and Veitch (2000, 2001), Misiti, Misiti, Oppenheim, Poggi (1998)).

The idea is as follows. Start with a (mother) wavelet, that is, a function $\psi(t)$ satisfying

$$\int_{-\infty}^{\infty} t^k \psi(t)dt = 0, \quad k = 0, \dots, N-1$$

for some $N \geqslant 1$. One typically uses the so-called Daubechies wavelets which have bounded support and are localized in frequency. Starting from $\psi(t)$, one defines the functions

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k), \quad j \in \mathbb{Z}, \ k \in \mathbb{Z},$$

which are dilations and translations of ψ. If the long-memory stationary process is $\{X(t), \ t \in \mathbb{R}\}$, then one computes the coefficients

$$d_{j,k} = \int_{\mathbb{R}} X(t)\psi_{j,k}(t)dt, \quad j, k \in \mathbb{Z}$$

In practice, the $d_{j,k}$ are obtained by using the so-called multiresolution algorithm which does not require integration, except perhaps at the initialization stage. (The case of t discrete and the question of initialization are discussed in detail in Veitch, Taqqu and Abry (2000)). The coefficients $d_{j,k}$ are stationary and have short memory in k, and satisfy $Ed_{j,k}^2 \approx 2^{2jd}$ for large j. Because of stationarity, $Ed_{j,k}^2$ does not depend on k. One estimates d by first plotting the sample version of log $Ed_{j,k}^2$ versus log j and then estimating d through a weighted regression. The wavelet method works well. It works even if $X(t)$ is fractional Brownian motion and it also eliminates polynomial trends. Its effectiveness on ARFIMA sequences is analyzed in Abry, Flandrin, Taqqu and Veitch (2001).

Finally, some short remarks about seasonal ARFIMA and the FIGARCH models. For theoretical properties of (multiplicative) seasonal ARFIMA, see also Giraitis and Leipus (1995). For change points problems, see Giraitis and Leipus (1990, 1992). Interesting results related to GARCH models have recently been obtained by Giraitis, Kokoszka and Leipus (1999).

Lardic and Mignon also discuss Lo's (1991) technique for "correcting" the R/S method by taking into account short memory. Such a correction must be done with care because it may eliminate long memory when present, as demonstrated in Teverovsky, Taqqu and Willinger (1999) and Willinger, Taqqu and Teverovsky (1999).

Lardic and Mignon conclude their paper with a useful review of the literature about the empirical evidence on whether financial time series, such as exchange rates, interest rates, and price indices display long memory. As they indicate, the presence of long memory is incompatible with efficient markets. As is well known, the fractional Brownian motion process $B_H(t),\ t \geqslant 0$, $1/2 < H < 1$ (which is the long memory version of Brownian motion – here $H = d + 1/2$) is not a semimartingale (Liptser and Shiryayev (1989)), and hence is incompatible with the "no-arbitrage" assumption, which is made in the context of efficient markets[1]. In fact, Rogers (1997) and Kallianpur and Karandikar (2000) show that if the model for the stock is geometric fractional Brownian motion, that is,

$$S(t) = e^{\mu t + \sigma B_H(t)},\ t \geqslant 0,$$

with $1/2 < H < 1$, then there are arbitrage opportunities.

The paper Lardic and Mignon (1999) contains an extensive bibliography. For additional references, particularly in regard to theoretical issues and other areas of application, see Taqqu (1986) and Willinger, Taqqu and Erramilli (1996).

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