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The $k$-dimensional Duffin and Schaeffer conjecture.

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Résumé — Nous montrons que la conjecture de Duffin et Schaeffer est vraie en toute dimension supérieure à 1.

Abstract — We show that the Duffin and Schaeffer conjecture holds in all dimensions greater than one.

In 1941 Duffin and Schaeffer [1] made the following conjecture:

**Conjecture.** Let $\{\alpha_n\}$ denote a sequence of real numbers with

$$0 \leq \alpha_n < \frac{1}{2}$$

then the inequalities

$$|nx - a| < \alpha_n, \quad (a, n) = 1,$$

have infinitely many solutions for almost all $x$ if and only if

$$\sum_{n=1}^{\infty} \frac{\alpha_n \varphi(n)}{n}$$

diverges.

If (2) converges then it easily follows from the Borel-Cantelli lemma that the set of $x$'s satisfying infinitely many of the inequalities (1) has Lebesgue measure zero. Duffin and Schaeffer gave conditions on the $\alpha_n$ for which the conjecture is true and showed that the condition $(a, n) = 1$ is necessary. In 1970 Erdős [2] showed that the conjecture holds if $\alpha_n = \frac{\varphi(n)}{n}$ or 0. This was later extended by Vaaler [6] who showed that $\alpha_n = O\left(\frac{1}{n}\right)$ is sufficient. In his book on metric number theory [5] Sprindzuk considers a $k$-dimensional analogue of the conjecture in which (1) is replaced by

$$\max(|x_1 n - a_1|, ..., |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, ..., k$$

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and (2) by

\[ \sum_{n=1}^{\infty} \left( \frac{\alpha_n \phi(n)}{n} \right)^k \]

where the measure is now \( k \)-dimensional Lebesgues measure. He states that the study of such approximations subject to the conditions \((a_1, n) = \cdots = (a_k, n) = 1\) is probably a problem of the same degree of complexity as the case \( n = 1 \). This appears not to be the case. For we can now prove the \( k \)-dimensional analogue of the Duffin and Schaeffer conjecture.

We prove the following result:

**Theorem.** Let \( k > 1 \) and let \( \{\alpha_n\} \) denote a sequence of real numbers with

\[ 0 \leq \alpha_n < \frac{1}{2} \]

and suppose that

\[ \sum_{n=1}^{\infty} \left( \frac{\alpha_n \phi(n)}{n} \right)^k \]

diverges. Then the inequalities

\[ \max(|x_1 n - a_1|, \ldots, |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, \ldots, k \]

have infinitely many solutions for almost all \( x \in \mathbb{R}^k \).

Unfortunately our method does not readily extend to the case \( k = 1 \). We are able find some more sequences \( \{\alpha_n\} \) for which the conjecture holds, for example if \( \alpha_n = 0 \) or \( 1 \ll \alpha_n \), but not to settle the dimension one case. In the \( k \)-dimensional case, \( k \geq 2 \), Vilchinski has previously shown that we may take \( \alpha_n = O(n^{-\gamma}) \) for any \( \gamma > 0 \).

Put

\[ E_n = E_n^{(1)} \times \cdots \times E_n^{(k)} \]

where

\[ E_n^{(i)} = \bigcup_{1 \leq a_i \leq n, (a_i, n) = 1} \left( \frac{a_i - \alpha_n}{n}, \frac{a_i + \alpha_n}{n} \right) \cdot \]

Then \( E_n \) is the set counted in (3) and
Thus (4) becomes

\[ \lambda_k(E_n) = \left( \frac{2\alpha_n \varphi(n)}{n} \right)^k. \]

Thus (4) becomes

\[ \sum_{k=1}^{\infty} \lambda_k(E_n) \]

diverges.

We are interested in

\[ \lambda_k \left( \bigcup_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n \right). \]

By an ergodic theorem of Gallagher [3], see Sprindzuk [5], this is either zero or one. Gallagher proves his result for dimension one. The corresponding k-dimensional result is proved by Vilchinski [7]. In order to prove the theorem it therefore suffices to show that (8) is not zero.

Since (7) diverges, and \( \lambda_k(E_n) \to 0 \) as \( n \to \infty \), given any \( 1 > \eta > 0 \), for every \( N \) we can find a finite set \( Z \) so that if \( z \in Z \) then \( z > N \), and

\[ \eta^2 < \Lambda(Z) = \sum_{n \in Z} \lambda_k(E_n) < \eta. \]

By the Cauchy-Schwarz inequality

\[ \lambda_k \left( \bigcup_{n \in Z} E_n \right) \geq \frac{(\sum_n \lambda_k(E_n))^2}{(\sum_n \sum_{m} \lambda_k(E_n \cap E_m))}, \]

this is Lemma 5 of Sprindzuk [5]. So provided there is some absolute constant \( c \) for which

\[ \sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_{n \in Z} \lambda_k(E_n) \]
the theorem is proved.

Thus we need to bound

$$\sum_{n \neq m, m, n \in \mathbb{Z}} \lambda_k(E_n \cap E_m).$$

In the subsequent discussion constants in the Vinogradov $\ll$ symbol depend at most on the dimension $k$.

Note that

$$(12) \quad \lambda_k(E_n) = \prod_{i=1}^{k} \lambda(E_n^{(i)})$$

and

$$(13) \quad \lambda_k(E_n \cap E_m) = \prod_{i=1}^{k} \lambda(E_n^{(i)} \cap E_m^{(i)}).$$

We now concentrate on $k = 1$ and use (12) and (13) to obtain a bound for $k > 1$.

We wish to obtain an estimate for

$$\lambda(E_n \cap E_m) \quad \text{with } n > m.$$

Put

$$(14) \quad \delta = \min\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right) \quad \text{and} \quad \Delta = \max\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right),$$

and

$$d = (m, n) \quad m' = \frac{m}{d} \quad n' = \frac{n}{d}.$$Then

$$(15) \quad \lambda(E_n \cap E_m) \leq 2\delta \sum_{\substack{|a\alpha_n-b\alpha_m| < 2\Delta \\
(a,n)=1 \\
(b,m)=1}} 1.$$By estimating

$$(16) \quad \sum_{\substack{|a\alpha_n-b\alpha_m| < 2\Delta \\
(a,n)=1 \\
(b,m)=1}} 1$$we obtain
LEMMA 1. With $E_n$ as above and $d = (m, n)$ there is an absolute constant $c$ so that

\begin{equation}
\lambda(E_n \cap E_m) \leq c\lambda(E_n)\lambda(E_m) \prod_{\substack{p|m', n' \\
p > \frac{2mn\Delta}{d}}}(1 - \frac{1}{p})^{-1}.
\end{equation}

Let $g(n)$ be defined as the least number so that

\[ \sum_{\substack{p|n \\
p > g(n)}} \frac{1}{p} < 2 \]

Then by Lemma 1

\begin{equation}
\lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m) \prod_{\substack{p|m', n' \\
p > \frac{2mn\Delta}{d} \wedge p < t}}(1 - \frac{1}{p})^{-1}.
\end{equation}

In particular if $\frac{2mn\Delta}{d} \geq t$ then

\begin{equation}
\lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m).
\end{equation}

Now

\begin{equation}
\prod_{\substack{p|\frac{2mn\Delta}{d} \\
p < t}}(1 - \frac{1}{p})^{-1} \leq \prod_{p < t}(1 - \frac{1}{p})^{-1} \ll \log t.
\end{equation}

We now return to the $k$-dimensional case. By (12) and (13)

\begin{equation}
\sum_{\substack{n \neq m \\
\log t \leq \Lambda(Z)^{-1/k}}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)^{-1} \sum_{n \neq m} \lambda_k(E_n)\lambda_k(E_m)
\end{equation}

\[ \ll \Lambda(Z). \]

From now on we shall assume that

\[ \log t > \Lambda(Z)^{-1/k}. \]
We distinguish four cases:

(a) \[ \Delta = \frac{\alpha_m}{m}, \quad \alpha_n < \alpha_m \]

(b) \[ \Delta = \frac{\alpha_m}{m}, \quad \alpha_n \geq \alpha_m \]

(c) \[ \Delta = \frac{\alpha_n}{n}, \quad \alpha_n < \alpha_m \]

(d) \[ \Delta = \frac{\alpha_n}{n}, \quad \alpha_n \geq \alpha_m. \]

Recall that \( m < n \) so (c) is impossible. We will consider the other three cases separately. The first case corresponds to the situation considered by Erdős.

Case (a).

We have

\[ \frac{mn\Delta}{d} = n'\alpha_m. \]

We need to consider pairs \( m, n \) with

\[ 1 < n'\alpha_m < t \]

Let \( A_{u,v} \) denote that part of the sum

\[ \sum_{m \neq n \text{ case (a)}} \lambda_k(E_n \cap E_m) \]

for which \( g(m') = u \) and \( g(n') = v \). Then, by Lemma 1 and (22)

\[ A_{u,v} \ll \log^k t \sum_{m} \lambda_k(E_m) \sum_{(u)} \sum_{(v)} \lambda_k(E_n). \]

Where \( \sum_{(u)} \) means the sum of those \( m' \) with \( g(m') = u \) and \( m = m'd \).

Since

\[ \lambda_k(E_n) = \left(2\alpha_n \frac{\varphi(n)}{n}\right)^k \ll \alpha_m^k \]
then
\[ A_{u,v} \ll \log^k t \sum_{m} \lambda_k(E_m) \alpha_m^k \sum_{(u)} \sum_{(v)} 1. \]
\[ \text{where } m' | m, \alpha_m^{-1} \leq n' < t\alpha_m^{-1}, \]
\[ 1 \leq m' < t\alpha_m^{-1}, n = n'd. \]

To estimate the inner sum we will use the following result due to Erdős which also appears in Vaaler [6]. Let \( N(\xi, v, x) \) be the number of integers \( n \leq x \) which satisfy
\[ \sum_{\substack{p | n \\text{ or } p \geq v}} \frac{1}{p} \geq \xi. \]

Then

**Lemma 2.** For any \( \epsilon > 0 \) and any \( \xi > 0 \) there exists a positive integer \( v_0 = v_0(\xi, \epsilon) \) such that for all \( x > 0 \) and all \( v > v_0 \),
\[ N(\xi, v, x) \leq x \exp\{-v^{\beta(1-\epsilon)}\} \]
where \( \log \beta = \xi \).

Applying this result with \( \xi = 1 \) and \( \beta(1-\epsilon) = \frac{3}{2} \) we have

**Corollary.** Given \( x > 0 \) and \( v \geq 1 \)
\[ N(1, v, x) \ll x \exp\{-v^{\frac{3}{2}}\}. \]

Using (23) and partial summation
\[ \sum_{1 \leq n' \leq t\alpha_m^{-1}} \ll t\alpha_m^{-1} \exp\{-v^{\frac{3}{2}}\}. \]

Applying (23) again
\[ A_{u,v} \ll t^2 \log^k t \exp(-u^{\frac{3}{2}}) \exp(-v^{\frac{3}{2}}) \sum_m \lambda_k(E_m), \]
since \( k \geq 2 \). Then summing over \( u \) and \( v \) we have
\[ \sum_{m \neq n \ \text{case}(a)} \lambda_k(E_n \cap E_m) \ll \Lambda(Z) \]
if \( \eta \) is sufficiently small.

The other cases are similar and we have

\[
\sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_n \lambda_k(E_n)
\]

where \( c \) is a constant depending only on \( k \). This completes the proof of the Theorem.

References


Mots clefs: Diophantine approximation, k-dimensional, Lebesgues measure, Duffin and Schaeffer conjecture.

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