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The $k$-dimensional Duffin and Schaeffer conjecture.

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Résumé — Nous montrons que la conjecture de Duffin et Schaeffer est vraie en toute dimension supérieure à 1.

Abstract — We show that the Duffin and Schaeffer conjecture holds in all dimensions greater than one.

In 1941 Duffin and Schaeffer [1] made the following conjecture:

**Conjecture.** Let $\{\alpha_n\}$ denote a sequence of real numbers with

$$0 \leq \alpha_n < \frac{1}{2}$$

then the inequalities

$$|nx - a| < \alpha_n, \quad (a, n) = 1,$$

have infinitely many solutions for almost all $x$ if and only if

$$\sum_{n=1}^{\infty} \frac{\alpha_n \varphi(n)}{n}$$

diverges.

If (2) converges then it easily follows from the Borel-Cantelli lemma that the set of $x$'s satisfying infinitely many of the inequalities (1) has Lebesgue measure zero. Duffin and Schaeffer gave conditions on the $\alpha_n$ for which the conjecture is true and showed that the condition $(a, n) = 1$ is necessary. In 1970 Erdős [2] showed that the conjecture holds if $\alpha_n = \frac{\varphi(n)}{n}$ or 0. This was later extended by Vaaler [6] who showed that $\alpha_n = O(\frac{1}{n})$ is sufficient. In his book on metric number theory [5] Sprindzuk considers a $k$-dimensional analogue of the conjecture in which (1) is replaced by

$$\max(|x_1n - a_1|, ..., |x_kn - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, ..., k$$

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and (2) by

$$\sum_{n=1}^{\infty} \left( \frac{\alpha_n \varphi(n)}{n} \right)^k$$

where the measure is now $k$-dimensional Lebesgue measure. He states that the study of such approximations subject to the conditions $(a_1, n) = \cdots = (a_k, n) = 1$ is probably a problem of the same degree of complexity as the case $n = 1$. This appears not to be the case. For we can now prove the $k$-dimensional analogue of the Duffin and Schaeffer conjecture.

We prove the following result:

**Theorem.** Let $k > 1$ and let $\{\alpha_n\}$ denote a sequence of real numbers with

$$0 \leq \alpha_n < \frac{1}{2}$$

and suppose that

$$\sum_{n=1}^{\infty} \left( \frac{\alpha_n \varphi(n)}{n} \right)^k$$

diverges. Then the inequalities

$$\max(|x_1 n - a_1|, \ldots, |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, \ldots, k$$

have infinitely many solutions for almost all $x \in R^k$.

Unfortunately our method does not readily extend to the case $k = 1$. We are able find some more sequences $\{\alpha_n\}$ for which the conjecture holds, for example if $\alpha_n = 0$ or $1 \ll \alpha_n$, but not to settle the dimension one case. In the $k$-dimensional case, $k \geq 2$, Vilchinski has previously shown that we may take $\alpha_n = O(n^{-\gamma})$ for any $\gamma > 0$.

Put

$$E_n = E_n^{(1)} \times \cdots \times E_n^{(k)}$$

where

$$E_n^{(i)} = \bigcup_{1 \leq a_i \leq n, (a_i, n) = 1} \left( \frac{a_i - \alpha_n}{n}, \frac{a_i + \alpha_n}{n} \right).$$

Then $E_n$ is the set counted in (3) and
(6) \[ \lambda_k(E_n) = \left(\frac{2\alpha_n \varphi(n)}{n}\right)^k. \]

Thus (4) becomes

(7) \[ \sum_{k=1}^{\infty} \lambda_k(E_n) \]

diverges.

We are interested in

(8) \[ \lambda_k \left( \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n \right) \]

By an ergodic theorem of Gallagher [3], see Sprindzuk [5], this is either zero or one. Gallagher proves his result for dimension one. The corresponding k-dimensional result is proved by Vilchinski [7]. In order to prove the theorem it therefore suffices to show that (8) is not zero.

Since (7) diverges, and \( \lambda_k(E_n) \to 0 \) as \( n \to \infty \), given any \( 1 > \eta > 0 \), for every \( N \) we can find a finite set \( Z \) so that if \( z \in Z \) then \( z > N \), and

(9) \[ \eta^2 < \Lambda(Z) = \sum_{n \in Z} \lambda_k(E_n) < \eta. \]

By the Cauchy-Schwarz inequality

(10) \[ \lambda_k \left( \bigcup_{n \in Z} E_n \right) \geq \frac{(\sum_{n} \lambda_k(E_n))^2}{(\sum_{n} \sum_{m} \lambda_k(E_n \cap E_m))}, \]

this is Lemma 5 of Sprindzuk [5]. So provided there is some absolute constant \( c \) for which

(11) \[ \sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_{n \in Z} \lambda_k(E_n) \]
the theorem is proved.
Thus we need to bound
\[ \sum_{\substack{n \neq m \\ m, n \in \mathbb{Z}}} \lambda_k(E_n \cap E_m). \]
In the subsequent discussion constants in the Vinogradov \( \ll \) symbol depend at most on the dimension \( k \).
Note that
\begin{equation}
\lambda_k(E_n) = \prod_{i=1}^{k} \lambda(E_n^{(i)})
\end{equation}
and
\begin{equation}
\lambda_k(E_n \cap E_m) = \prod_{i=1}^{k} \lambda(E_n^{(i)} \cap E_m^{(i)}).
\end{equation}
We now concentrate on \( k = 1 \) and use (12) and (13) to obtain a bound for \( k > 1 \).
We wish to obtain an estimate for
\[ \lambda(E_n \cap E_m) \text{ with } n > m. \]
Put
\begin{equation}
\delta = \min\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right) \text{ and } \Delta = \max\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right),
\end{equation}
and
\[ d = (m, n) \quad m' = \frac{m}{d} \quad n' = \frac{n}{d}. \]
Then
\begin{equation}
\lambda(E_n \cap E_m) \leq 2\delta \sum_{|\frac{a}{n} - \frac{b}{m}| < 2\Delta \atop (a, n) = 1 \atop (b, m) = 1} 1.
\end{equation}
By estimating
\begin{equation}
\sum_{|\frac{a}{n} - \frac{b}{m}| < 2\Delta \atop (a, n) = 1 \atop (b, m) = 1} 1
\end{equation}
we obtain
**Lemma 1.** With $E_n$ as above and $d = (m, n)$ there is an absolute constant $c$ so that

\[(17) \quad \lambda(E_n \cap E_m) \leq c \lambda(E_n) \lambda(E_m) \prod_{p|m',n'} (1 - \frac{1}{p})^{-1}.\]

Let $g(n)$ be defined as the least number so that

\[\sum_{\substack{p|n \\ p > g(n)}} \frac{1}{p} < 2\]

Put

\[t = \max(g(n'), g(m')).\]

Then by Lemma 1

\[(18) \quad \lambda(E_n \cap E_m) \ll \lambda(E_n) \lambda(E_m) \prod_{p|m',n'} (1 - \frac{1}{p})^{-1}.\]

In particular if $\frac{2mn\Delta}{d} \geq t$ then

\[(19) \quad \lambda(E_n \cap E_m) \ll \lambda(E_n) \lambda(E_m).\]

Now

\[(20) \quad \prod_{\substack{p|m' \\ p < t}} (1 - \frac{1}{p})^{-1} \leq \prod_{p < t} (1 - \frac{1}{p})^{-1} \ll \log t.\]

We now return to the $k$-dimensional case. By (12) and (13)

\[(21) \quad \sum_{\substack{n \neq m \\ \log t \leq \Lambda(Z)^{-1/k}}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)^{-1} \sum_{n \neq m} \lambda_k(E_n) \lambda_k(E_m) \ll \Lambda(Z).\]

From now on we shall assume that

\[\log t > \Lambda(Z)^{-1/k}.\]
We distinguish four cases:

(a) \[ \Delta = \frac{\alpha_m}{m}, \quad \alpha_n < \alpha_m \]

(b) \[ \Delta = \frac{\alpha_m}{m}, \quad \alpha_n \geq \alpha_m \]

(c) \[ \Delta = \frac{\alpha_n}{n}, \quad \alpha_n < \alpha_m \]

(d) \[ \Delta = \frac{\alpha_n}{n}, \quad \alpha_n \geq \alpha_m. \]

Recall that \( m < n \) so (c) is impossible. We will consider the other three cases separately. The first case corresponds to the situation considered by Erdős.

Case (a).

We have

\[ \frac{mn\Delta}{d} = n'\alpha_m. \]

We need to consider pairs \( m, n \) with

\[ 1 < n'\alpha_m < t \]

Let \( A_{u,v} \) denote that part of the sum

\[ \sum_{m \neq n} \lambda_k(E_n \cap E_m) \]

for which \( g(m') = u \) and \( g(n') = v \). Then, by Lemma 1 and (22)

\[ A_{u,v} \ll \log^k t \sum_m \lambda_k(E_m) \sum (u) \sum (v) \lambda_k(E_n). \]

Where \( \sum (u) \) means the sum of those \( m' \) with \( g(m') = u \) and \( m = m'd \). Since

\[ \lambda_k(E_n) = \left(2\alpha_n \frac{\varphi(n)}{n}\right)^k \ll \alpha_m^k \]
then

$$A_{u,v} \ll \log^k t \sum_{m} \lambda_k(E_m) \alpha_m^k \sum_{(u)} \sum_{(v)} \frac{1}{m' | m} \sum_{\alpha_m^{-1} \leq n' < t \alpha_m^{-1}} \sum_{1 \leq m' < t \alpha_m^{-1}}$$

To estimate the inner sum we will use the following result due to Erdős which also appears in Vaaler [6]. Let $N(\xi, v, x)$ be the number of integers $n \leq x$ which satisfy

$$\sum_{\frac{1}{p} \geq \xi} \frac{1}{p} \geq \xi.$$ 

Then

**Lemma 2.** For any $\epsilon > 0$ and any $\xi > 0$ there exists a positive integer $v_0 = v_0(\xi, \epsilon)$ such that for all $x > 0$ and all $v > v_0$,

$$N(\xi, v, x) \leq x \exp\{-v^{\beta(1-\epsilon)}\}$$

where $\log \beta = \xi$.

Applying this result with $\xi = 1$ and $\beta(1 - \epsilon) = \frac{3}{2}$ we have

**Corollary.** Given $x > 0$ and $v \geq 1$

$$N(1, v, x) \ll x \exp(-v^{\frac{3}{2}}).$$

Using (23) and partial summation

$$\sum_{(v)} \ll t \alpha_m^{-1} \exp (-v^{\frac{3}{2}}).$$

Applying (23) again

$$A_{u,v} \ll t^2 \log^k t \exp(-u^{\frac{3}{2}}) \exp(-v^{\frac{3}{2}}) \sum_{m} \lambda_k(E_m),$$

since $k \geq 2$. Then summing over $u$ and $v$ we have

$$\sum_{m \neq n \text{ case}(a)} \lambda_k(E_n \cap E_m) \ll \Lambda(Z).$$
if \( \eta \) is sufficiently small.

The other cases are similar and we have

\[
(25) \quad \sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_n \lambda_k(E_n)
\]

where \( c \) is a constant depending only on \( k \). This completes the proof of the Theorem.

References


Mots clefs: Diophantine approximation, \( k \)-dimensional, Lebesgues measure, Duffin and Schaeffer conjecture.