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Connections between binary patterns and paperfolding.

par PATRICK MORTON

1. Introduction. This talk will be an overview of some recent work on binary patterns and their relationship to paperfolding sequences. Much of this work was motivated by the Rudin-Shapiro sequence $\{a_{11}(n)\}$ defined by

$$a_{11}(n) = (-1)^{e_{11}(n)},$$

where $e_{11}(n)$ is the number of occurrences of the pattern 11 in the binary representation of n. On the one hand, this definition is easy to generalize. One can consider the analogous sequence

$$(1) \qquad a_P(n) = (-1)^{e_P(n)},$$

where P is any pattern of 0's and 1's and $e_P(n)$ counts occurrences of P in n.

On the other hand, there is the beautiful fact discovered by Mendès France that a_{11} is exactly the direction sequence of the paperfolding sequence obtained by folding a rectangular piece of paper alternately under and over the left edge (which is held fixed). (See [2] or [4].)

This raises the question: is this result about a_{11} isolated or does it point to some deeper connection between binary patterns and paperfolding?

2. Properties of $a_P(n)$. The results of this section represent joint work with David Boyd and Janice Cook. (See [1].)

Let me begin by recalling some of the properties of the sequence $a_P(n)$ defined by (1), where P is any pattern of 0's and 1's. First define

$$(2) \qquad s_P(x) = \sum_{k \leq x} a_P(k).$$

Then the following theorem holds concerning the behavior of $s_P(x)$.

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THEOREM 1. *(See [1].) The partial sum function $s_P(x)$ has the representation*

$$s_P(x) = t(x) + s_0(x),$$

where $t(x)$ is continuous and $s_0(x)$ is bounded. Further,

$$t(x) = \sum_{|\xi_k|>1} \lambda_k(x) x^{\log \xi_k / \log 2},$$

where the ξ_k are the roots outside the unit circle of the polynomial

$$P(x) = (x-2)(x^{d-1} + 2\pi_1 x^{d-2} + \cdots + 2\pi_{d-1}) + 2, \tag{3}$$

and

$$\pi_k = \begin{cases} 1 \text{ if } k \text{ is a period of } P, \\ 0 \text{ if not.} \end{cases}$$

(By definition k is a period of $P = p_{d-1} \cdots p_0$ if and only if $p_i = p_{i+k}$ for $0 \leq i \leq d-1-k$.)

Finally, the $\lambda_k(x)$ are continuous functions of $x > 0$ satisfying $\lambda_k(2x) = \lambda_k(x)$.

As a corollary we have

$$s_P(x) = O(x^\tau),$$

where $\tau = \log\xi / \log 2$ and ξ is the maximum modulus of the roots of $P(x)$. For all but 14 patterns (in particular, if $d \geq 5$) $\xi = \xi_1$ is a root of $P(x)$.

From this follows the formula

$$\lim_{k\to\infty} \frac{s_P(2^k x)}{(2^k x)^\tau} = \lambda_1(x),$$

so in these cases $\lambda(x) = \lambda_1(x)$ (for $1 \leq x \leq 2$) represents the limiting behavior of $s_P(x)/x^\tau$ on the intervals $[2^k, 2^{k+1}]$, as $k \to \infty$. This limit function possesses two rather nice properties:

1) $\lambda(x)$ is nondifferentiable almost everywhere,

2) $x \in Q \Rightarrow x^\tau \lambda(x) \in Q(\xi)$.

Property 1) also implies that $s_P(x) = \Omega(x^\tau)$, so that x^τ is the correct order of magnitude of $s_P(x)$.

The polynomials $P(x)$ in theorem 1 also satisfy a remarkable set of recursions. In order to state these recursions, let P and P' be patterns of length d and $d+1$ recpectively, and let Π and Π' denote their sets of periods, e.g.

$$k \in \Pi \text{ if and only if } k \text{ is a period of } P.$$

If $P(x)$ and $P'(x)$ denote the corresponding polynomials, as in (3), then $P(x)$ and $P'(x)$ depend only on the period sets Π, Π', and for every period set Π' there is a unique period set Π for which

$$P'(x) = \begin{cases} xP(x) - 2, & \text{if } \Pi' = \Pi \cup \{d\} \\ xP(x) - 2x + 2 & \text{if } \Pi' = \Pi. \end{cases}$$

We called these formulae the red and blue rules, respectively. They show that the polynomials $P(x)$ form a tree, with red or blue branches depending on the rule which connects $P(x)$ to $P'(x)$. In [1] we show that this tree has a fractal-like structure. It has a sequence of periodic subtrees which are isomorphic to larger and larger initial pieces of the whole tree; thus the tree reproduces itself in its subtrees.

The sequences a_P are very fundamental. They can be used to study arbitrary sequences of ± 1's. If $a = \{a(n)\}_{n \geq 0}$ is any sequence of ± 1's, then there is a unique sequence $\{P_k\}$ of binary patterns with increasing values (the value of a pattern is the integer it represents) and no leading zeros, for which

$$a = a(0) \prod_{k=1}^{\infty} a_{P_k}.$$

This infinite product is defined using the topology in which two sequences are close if a large number of their initial terms agree. Let us call the set $\{P_k\}$ the binary spectrum of the sequence a.

Now consider the question: is it possible to characterize the sequences a which have a *finite* spectrum?

3. The arithmetic fractal groups $\Gamma_k(G)$.

The answer to the last question is in fact yes, and depends on the following definition. (See Morton and Mourant [4] for the results of this section.)

DEFINITION. *Let G be an abelian group, $a = \{a(n)\}$ an infinite sequence of elements of G, and*

$$X_n^q = (a(k^q n), a(k^q n + 1), \dots, a(k^q n + k^q - 1))$$

the associated sequence of k^q-segments of a. These vectors partition the sequence a into vectors of length k^q. We define

$$\text{(4)} \quad \Gamma_k(G) = \{a : a^{-1}(n)X_n^q \textit{ is periodic with period } M \textit{ for all } q \geq 0\}$$

Here $a^{-1}(n)X_n^q$ is defined using scalar multiplication (or scalar addition if the operation in G is addition). The least period M of a sequence a in $\Gamma_k(G)$ is called its conductor.

The set $\Gamma_k(G)$ is an abelian group under componentwise multiplication.

As an example, let P be a pattern of digits base k and consider the sequence $e_p(n)$ which counts the number of occurrences of P in the base-k representation of n. Then $e_P \in \Gamma_k(Z)$ with conductor dividing k^{d-1}, where d is the number of digits of P. (This holds as long as P is not a pattern of all 0's). This fact follows from the equation

$$X_n^q - e_P(n) = X_m^q - e_P(m), \text{ if } n \equiv m \ (mod\ k^{d-1}).$$

If $k = 2$, reducing e_1 modulo 2 gives the Thue-Morse sequence e_1^*, which satisfies

$$X_n^q = X_0^q + e_1^*(n) \ (\text{mod } 2);$$

hence $e_1^* \in \Gamma_k(Z_2)$. This last formula is a restatement of the familiar fact that e_1^* is invariant under the substitution

$$0 \to 01, \ 1 \to 10.$$

The corresponding equation for the sequence of *integers* e_1 also holds in characteristic 0:

$$X_n^q = X_0^q + e_1(n).$$

The related sequences $a_P = \zeta^{e_P}$, where ζ is a root of unity, generate a special subgroup of $\Gamma_k(<\zeta>)$, namely the subgroup

$$\Lambda_k(<\zeta>) = \{a \in \Gamma_k(<\zeta>) \text{ with conductor dividing a power of } k\},$$

which provides an answer to the question we raised in section 2.

THEOREM 2. *(See [4].) The sequences in $\Lambda_2(\pm 1)$ are exactly the sequences which have a finite binary spectrum.*

Of course an analogous result characterizes sequences in $\Lambda_k(<\zeta>)$ in terms of their base k spectra.

The sequences in $\Gamma_k(G)$ can be thought of as arithmetic analogues of fractals. As an example consider the Rudin-Shapiro sequence a_{11}, whose first four terms are

$$1, 1, 1, -1.$$

We have

$$(5) \qquad X_2^0 = X_0^0 \text{ and } X_3^0 = -X_1^0.$$

Since $a_{11} \in \Gamma_2(\pm 1)$ with $M = 2$, the definition (4) implies that the relations (5) persist for 2^q-segments at all levels:

$$(6) \qquad X_2^q = X_0^q \text{ and } X_3^q = -X_1^q \text{ for } q \geq 0.$$

This can be used to generate a_{11} by considering the first four terms as the elements of X_0^1 and X_1^1. Using (6) with $q = 1$ gives the next four terms of the sequence:

$$11, 1-1, 11, -11 = X_0^1, X_1^1, X_2^1, X_3^1.$$

For $q = 2$ we then get 8 more terms :

$$111-1, 11-11, 111-1, -1-11-1,$$

etc. The same rule connects 2^q segments at ever increasing levels, in analogy to the familiar constructions used to generate certain types of geometric fractals. The same remarks hold for any sequence in $\Gamma_k(G)$ for all k and G.

4. Properties of $\Gamma_k(G)$. Among the properties of these groups I want to particularly mention three. For proofs of 1. and 2. see [4].

1. In the definition of $\Gamma_k(G)$ we only need to require that $a^{-1}(n)X_n^1$ be periodic of period M. The fact that $a^{-1}(n)X_n^q$ has period M for all $q \geq 0$ follows.

2. If $G = U$ is the group of complex roots of unity, $\Gamma_k(U)$ contains an isomorphic copy of every finite abelian group. Every finite abelian group is a homomorphic image of $\Gamma_k(Z)$, for any k.

3. If G is finite, the sequences in $\Gamma_k(G)$ are all k-automatic.

I will prove this last property using the following maps T_i on sequences:

$$(T_i a)(n) = a(kn + i), \; 0 \leq i \leq k - 1.$$

From property 1. above we have the following characterization of sequences in $\Gamma_k(G)$.

LEMMA. *The sequence a lies in $\Gamma_k(G)$ if and only if $a^{-1}T_ia$ is periodic for all $i = 0, 1, \cdots, k-1$.*

Proof. The sequence $a^{-1}(n)X_n^1$ is periodic if and only if all its component sequences are periodic, and its i-th component is just $a^{-1}T_ia$.

If $T_{i_1}, ..., T_{i_r}$ are any r of these maps (with possible repetitions), then obviously

$$(T_{i_1} \cdots T_{i_r}a)(n) = a(k^r n + k^{r-1}i_r + k^{r-2}i_{r-1} + \cdots + i_1).$$

Hence we can use these maps to characterize sequences in $\Gamma_{k^q}(G)$: these are sequences a for which $a^{-1}Ta$ is periodic for all monomials T in $T_0, \cdots, T_{k-1}$ of degree q.

As is well-known, a sequence a is k-automatic if and only if the set of sequences $\{a(k^q n + i)\}$ is a finite set. Using the maps T we can state

a is k-automatic if and only if there are only finitely many images T_a, as T runs over all monomials in $T_0, \cdots, T_{k-1}$.

Using this we prove

THEOREM 3. *If G is a finite abelian group, the sequences in $\Gamma_k(G)$ are all k-automatic.*

Proof. By definition, $a \in \Gamma_k(G)$ has the property that $a^{-1}T_ia$ is periodic for $0 \leq i \leq k-1$. Hence

$$T_ia = ap_i,$$

where p_i is a periodic sequence, of period M, say. If S is any monomial in $T_0, \cdots, T_{k-1}$, and $S = S'T_i$, then

$$Sa = S'T_i(a) = S'(ap_i) = S'(a)S'(p_i) = S'(a).p',$$

where $p' = S'(p_i)$ is also periodic with period M. Peeling off one term T_i at a time gives finally that

$$Sa = ap'p'' \cdots = a\tilde{p},$$

where $\tilde{p}$ is periodic with period M. But there are only finitely many periodic sequences of a given period with terms that are taken from the given finite set G. Hence there are only finitely many distinct sequences Sa, implying that a is k-automatic.

A different proof of this theorem appears in [5], where we show that the sequences in $\Gamma_k(G)$ are essentially fixed points of k^r-substitutions for suitable r. Shallit (private communication) has found yet a third proof.

If $A_k(G)$ is the set of all k-automatic sequences taken from the alphabet G, then it is a consequence of theorem 3 that

$$\Gamma_{k^q}(G) \subset A_k(G), \text{ for all } q \geq 0,$$

and so

$$\varinjlim \Gamma_{k^q} \subset A_k(G).$$

It is not hard to exhibit automatic sequences which are not in Γ_{k^q} for any q. We do this for $q = 2$ in the following

Example. Let $G = Z_2$ and let u be the Baum-Sweet sequence, defined by

$$\begin{aligned} u_0 &= 1, \\ u_{2n+1} &= u_n, \\ u_{4n} &= u_n, \\ u_{4n+2} &= 0. \end{aligned}$$

These formulae may be written as follows:

$$T_1 u = u, \ T_0^2 u = u, \ T_1 T_0 u = 0.$$

It is easy to check that

$$\{Su; S \text{ a monomial in } T_0, T_1\} = \{u, T_0 u, 0\}.$$

To show that $u \notin \Gamma_{2^q}(Z_2)$ for any q, we show that $-u + Su = u + Su$ is not periodic, for a suitable monomial S of degree q. It suffices to take $q \geq 2$ and $S = T_1^{q-1} T_0$. Then

$$u + Su = u + T_1^{q-1} T_0 u = u + T_1^{q-2}(T_1 T_0 u) = u,$$

which is not periodic, since the series $\sum_{n=0}^{\infty} u_n x^n$ satisfies an irreducible cubic equation over $Z_2[x]$. Thus

$$u \in A_2(Z_2) - \varinjlim \Gamma_{2^q}(Z_2).$$

These considerations also raise the following question :

Question. Since $\Gamma_k(G)$ and $A_k(G)$ are abelian groups, they have associated rings of endomorphisms. What are

$$E_1 = \text{ End } (\Gamma_k(G)) \text{ and } E_2 = \text{ End } (A_k(G))?$$

It is obvious that $T_i \in E_2$ and not hard to check that $T_i \in E_1$ for all $i = 0, 1, \cdots, k-1$. Since the T_i do not commute this shows that E_1 and E_2 are non-commutative rings.

5. Paperfolding sequences. I turn now to the connection between binary patterns and paperfolding sequences. First a little notation. I want to consider paperfolding sequences corresponding to an infinite sequence of folding instructions

$$w = \phi_1\phi_2\cdots\phi_n\cdots,$$

where $\phi_n = o$ or u and o, u denote the operations of folding a rectangular piece of paper respectively over or under the left edge, (see [2],[3]). If w_m is the finite initial word of w of length m, then there are two sequences of ± 1's associated to w_m.

The first, f_{w_m}, encodes the sequence of up or down folds obtained by unfolding the paper after applying w_m, according to the rules

$$\vee \to +1$$

$$\wedge \to -1.$$

The second, denoted d_{w_m}, encodes horizontal and vertical directions in the plane curve obtained by making all the folds equal to right angles and looking at the paper edge-on. In d_{w_m}, horizontal and vertical directions are coded by

$$\begin{array}{ll} \rightarrow +1 & \leftarrow -1, \\ \uparrow +1 & \downarrow -1. \end{array}$$

For example,

$$\begin{aligned} f_{oou} &= -1, 1, 1, 1, -1, -1, 1 \\ d_{oou} &= 1, -1, 1, 1, -1, 1, 1, 1. \end{aligned}$$

The limit points of the sets $\{f_{w_m}, m \geq 1\}, \{d_{w_m}, m \geq 1\}$ in the set of all sequences of ± 1's are respectively called paperfolding sequences and direction sequences corresponding to the word w. The topology with respect to which these limits are to be taken is the topology in which two sequences are close if a large number of their initial terms agree. It is not hard to show that a subsequence of $\{f_{w_m}\}$ or $\{d_{w_m}\}$ converges in this topology if and only if the subsequence of corresponding words w_m is *reverse*-convergent, i.e. if and only if an increasing number of the final instructions of the w_m agree. Using this fact it is easy to see that paperfolding sequences and direction sequences of w are in 1-1 correspondence with the reverse infinite words

$$\omega = \cdots \phi_n \cdots \phi_2 \phi_1, \ \phi_n = o \text{ or } u,$$

for which $\phi_n \cdots \phi_1$ are subwords of w. This sequence is called a sequence of *unfolding* instructions, (see [2] or [3]). Let me denote the paperfolding and direction sequences corresponding to ω by f_ω and d_ω. If one defines

$$s_\omega(n) = \sum_{j=1}^{n} f_\omega(j), \ s_w(0) = 0,$$

then one has

$$d_\omega(n) = i^{s_\omega(n)+n^2+n}, n \geq 0,$$

and the following result holds:

THEOREM 4.. *(See [4], theorem 13.) Let the reverse infinite sequence ω be periodic, with period λ. Then*

$$s_\omega \in \Gamma_{2^\lambda}(Z) \text{ and } d_w \in \Gamma_{2^\lambda}(\pm 1).$$

In fact s_ω and d_ω have conductor 2, so they lie in the respective subgroups $\Lambda_{2^\lambda}(Z)$ and $\Lambda_{2^\lambda}(\pm 1)$. It follows by theorem 2 and analogous results that s_ω and d_ω are respectively a sum and product of pattern counting sequences. For example,

$$s_{(oou)^\infty} = -e_1 + e_3 + 2e_4 + e_5 + e_7 + \sum_{i,j=0}^{7} c_{ij} e_{ij},$$

where

$$c_{ij} = \begin{cases} 1, & i \text{ odd}, 0 \leq j \leq 3, \\ -1, & i \text{ odd}, 4 \leq j \leq 7, \\ 0, & \text{otherwise}, \end{cases}$$

and the digits i, j in this formula are taken base 8. This leads to the representation

$$d_{(oou)^\infty} = a_1 a_4 \prod_{i \text{ odd}, \ 0 \leq j \leq 3} a_{ij},$$

where $a_P = (-1)^{e_P}$.

Theorem 4 raises the question: for which unfolding sequences ω is $d_\omega \in \Gamma_{2^\lambda(\pm 1)}$?

We phrase the answer to this question in terms of the map τ defined as follows. Let D_w denote the set of direction sequences which arise from a given infinite word w, and set

$$\tau(d_{\omega\phi}) = d_\omega, \ \phi = o \text{ or } u.$$

Then $\tau : D_\omega \to D_w$ and in [4] we show that

$$d_{\omega\phi} = d_\omega(0)X, d_\omega(1)Y, d_\omega(2)X, d_\omega(3)Y, \dots ,$$

for two fixed vectors X, Y of length two; hence the sequence $\tau(d_{\omega\phi})$ is a "sign sequence" for $d_{\omega\phi}$ with respect to segments of length two. By iterating the map τ we prove that $\tau^k(d_\omega)$ is a sign sequence for d_ω with respect to segments of length 2^k, and this fact leads to the following result, (see [4], theorems 17-18):

THEOREM 5.

1) The sequence d_ω lies in $\Gamma_{2^\lambda}(\pm 1)$ if and only if $d_\omega \tau^\lambda(d_\omega)$ is a periodic sequence.

2) If $d_\omega \in \Gamma_{2^\lambda}(\pm 1)$, then $d_\omega \in \Lambda_{2^{\lambda'}}(\pm 1)$, for a suitable multiple λ' of λ.

3) $d_\omega \in \Lambda_{2^{\lambda'}}(\pm 1)$ if and only if

$$\tau^{\lambda'+n}(d_w) = \tau^n(d_w),$$

for some $n \geq 0$.

Hence direction sequences in Γ_{2^λ} correspond to pre-periodic points of the map τ, and the groups $\Gamma_{2^\lambda}(\pm 1)$ arise naturally in connection with the discrete dynamical system (D_w, τ).

Using the definition of τ and replacing λ' by λ in theorem 5, part 3) gives.

THÉORÈME 6. *(See [4], theorem 19.) The direction sequence d_ω is a finite product of pattern sequences base 2^λ (and so has a finite spectrum base 2^λ) if and only if*

$$\omega = (w_1)^\infty w_0, \ w_1 \textit{ of length } \lambda,$$

or

$$\omega = (\overline{w_2}w_2)^\infty w_0, \ w_2 \textit{ of length } \lambda,$$

where $\overline{w_2}$ is obtained from w_2 by switching $o's$ and $u's$.

As examples with $\lambda = 1$ let me note

$$d_{o^\infty} = a_{10}, \ d_{u^\infty} = a_1 a_{11}, \ d_{(uo)^\infty} = a_{11}, \ d_{(ou)^\infty} = a_1 a_{10},$$

where the patterns in these formulae are all binary patterns. From these relations it is possible to compute the spectra of all direction sequences which lie in $\Lambda_2(\pm 1)$. For example,

$$d_{(ou)^\infty o^k} = a_{10} \prod_P{}' a_P,$$

where the product is over all binary patterns beginning with 1 and having length $k+1$. This shows that every binary pattern (with a leading 1) occurs in the binary spectrum of some d_ω.

I will finish by recalling a recent theorem of Mendès France and Shallit [3] (see their theorem 4.2 for the special case of paperfolding; note that their sequence of turns is here what we have called a paperfolding sequence):

THEOREM. *The paperfolding sequence f_w is 2-automatic if and only if the sequence ω of unfolding instructions is ultimately periodic.*

Together with theorem 6, this gives

THEOREM 7. *A paperfolding sequence f_ω is 2-automatic if and only if its associated direction sequence d_ω has a finite spectrum base 2^λ, for some λ.*

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