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On Taylor's conjecture for Kummer orders.*

by PHILIPPE CASSOU-NOGUÈS AND ANUPAM SRIVASTAV

1. Introduction

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and let $\overline{O}$ be the ring of algebraic integers of $\overline{\mathbb{Q}}$. For a number field $F \subseteq \overline{\mathbb{Q}}$ we denote by O_F its ring of algebraic integers and we set $\Omega_F = Gal(\overline{\mathbb{Q}}/F)$.

Let K be a quadratic imaginary number field, L a finite extension of K and (E/L) be an elliptic curve, defined over L, with everywhere good reduction and admitting complex multiplication by O_K.

Let $\mathfrak{A} = (a)$ denote a non-zero integral O_K-ideal. Let us write $G = G(\mathfrak{A})$ for the subgroup of points in $E(\overline{\mathbb{Q}})$ that are killed by all elements of $\mathfrak{A}$. For $P \in E(L)$, we set

$$(1\text{-}1) \qquad G_P = G_P(\mathfrak{A}) = \{R \in E(\overline{\mathbb{Q}}) : [a]R = P\}$$

the corresponding G-space of points on E. We define the corresponding Kummer algebra by

$$(1\text{-}2) \qquad L_P = L_P(\mathfrak{A}) = Map(G_P, \overline{\mathbb{Q}})^{\Omega_L}$$

where the addition and multiplication are given value-wise on Ω_L maps from G_P to $\overline{\mathbb{Q}}$. In [T] M.-J. Taylor considered the O_L-algebra $\mathcal{B}$ which represents the O_L-group scheme of $\mathfrak{A}$ points of E. In fact $\mathcal{B}$ is an O_L Hopf order in the L-algebra $L_O = Map(G, \overline{\mathbb{Q}})^{\Omega_L}$ where O is the origin of E. The O_L-Cartier dual of $\mathcal{B}$ is an O_L-order in the dual algebra $\mathcal{A} = (\overline{\mathbb{Q}}[G])^{\Omega_L}$ that we denote by Λ. Taylor [T] defined the Kummer order $\tilde{O}_P$ as the largest Λ-module contained in O_P the integral closure of O_L in L_P. He showed that $\tilde{O}_P$ is a locally free Λ-module. We write $(\tilde{O}_P)$ for its class in $C\ell(\Lambda)$, the class group of locally free Λ-modules.

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In [T] the map $\psi : E(L) \to C\ell(\Lambda)$, given by $\psi(P) = (\tilde{O}_P)$ is shown to be a group homomorphism. Moreover it follows from the definition of $\tilde{O}_P$ that $[a]E(L) \subset Ker\psi$. Taylor conjectured in [T] :

(1-3) CONJECTURE. For any non-zero principal O_K-ideal,

$$E(L)_{\text{torsion}} \subset Ker\psi.$$

We remark that in [S-T] the above framework was generalised to include the case of non principal O_K-ideals.

Let w_K denote the number of roots of unity of K. The above conjecture was proved in [S-T] under the hypothesis that the ideal $\mathfrak{A}$ be coprime to w_K. In this article we consider the conjecture for the case where $|G| = 2$. We now assume that there is a principal prime ideal $\mathfrak{p} = (\pi)$ dividing 2. Moreover we assume that $\mathfrak{p}$ is either ramified or split in $(K/\mathbb{Q})$ and that $K \neq \mathbb{Q}(\sqrt{-1})$. We set $\mathfrak{A} = \mathfrak{p}$, so that $G = E[\pi]$ and $|G| = 2$. By the theory of complex multiplication we can also deduce that $G \subset E[2] \subset E(L)$.

Therefore $\mathcal{A} = L[G]$ and $\mathfrak{B} = Map(G, L)$. From [T], Proposition 1, we conclude that the order Λ, in the present case, is given by

$$\Lambda = 1_G.O_L + (\pi^{-1}\sigma_G)O_L \tag{1-4}$$

where $\sigma_G = \sum_{g \in G} g$.

Let $\mathfrak{M}$ denote the unique maximal O_L-order of $L[G]$. As usual, we denote by $D(\Lambda)$ the kernel of the extension map $e : C\ell(\Lambda) \to C\ell(\mathfrak{M})$. We define the homomorphism $\psi' : E(L) \to C\ell(\mathfrak{M})$ to be the composite map $e \circ \psi$. For $P \in E(L)$, it is shown in [T] that $|G|$ annihilates $\psi(P)$. Thus, in the present case, $\psi(P)^2 = 1$ in $C\ell(\Lambda)$ and $\psi'(P)^2 = 1$ in $C\ell(\mathfrak{M})$. In the second section we shall prove :

THEOREM 1. *Let $\mathfrak{p} = (\pi)$ be a ramified or split principal prime ideal dividing $2O_K$. Moreover, assume that $E[4] \subset E(L)$. Then for $G = E[\pi]$,*

$$E(L)_{\text{torsion}} \subseteq Ker(\psi').$$

Let Φ denote the quotient *map* : $O_L \to O_L/\overline{\pi}O_L$ where $\overline{\pi}$ is the complex conjugate of π. We denote the image of O_L^* under Φ by Im O_L^*. In section 2 we also calculate $D(\Lambda)$,

THEOREM 2. *The group kernel is given by*

$$D(\Lambda) = (O_L/\overline{\pi}O_L)^*/ImO_L^*.$$

The main aim of section 3 is to treat cases where $E[4]$ is not contained in $E(L)$.

We first assume that 2 is split in $(K/\mathbb{Q})$; we denote by $\mathfrak{p} = (\pi)$ a prime ideal of K above 2. We now fix a fractional ideal Ω of K, viewed as a $\mathbb{C}$ lattice, and a 4-division point ν of $\mathbb{C}/\Omega$ such that 2ν has annihilator $2O_K$. Corresponding to the pair (Ω, ν) we define the "minimal Fueter model" as the elliptic curve E given by :

$$y^2 + \sqrt{t}\ xy = x^3 + x \tag{1-5}$$

where $t = t_{\Omega,\nu} = 12\wp_\Omega(2\nu)/(\wp_\Omega(\nu) - \wp_\Omega(2\nu))$. We let $L = K(\sqrt{t})$. Our model is then defined over L. From $[CN - T_2], IX, (5-4)$, we know that $K(t) = K(4)$, the ray class field mod $4O_K$. Moreover, since 2 is split in $(K/\mathbb{Q})$, we know that $t^2 - 2^6$ is a unit, $[CN - T_2], IX, (5-10)$. Therefore E has good reduction everywhere. One can check, using classfield theory, that $E[\pi] \subset E(L)$. We let Q be the primitive π-division point of E. We now assume that $E[\pi^2] \not\subset E(L)$. We consider the map $h : G_Q \to \bar{O}$ défined by $h(R) = y(R)$, for $R \in G_Q$. It will be proved that h lies in $\tilde{O}_Q$.

Next we consider the Swan module $(\sqrt{t}, \pi^{-1}\sigma_G)\Lambda$. Since $t^2 - 2^6$ is a unit, $\sqrt{t}$ is relatively prime to $|G| = 2$. Then this module is a locally free ideal of Λ (cf. [U],[S]).

THEOREM 3. *Let Q be the primitive π-division point of the minimal Fueter curve E. Then*

$$\sqrt{t}\tilde{O}_Q = h(\sqrt{t}, \pi^{-1}\sigma_G)\Lambda.$$

One can observe that the Swan module is the obstruction to the Λ-freeness of $\tilde{O}_Q$. As a consequence of Theorem 2 and Theorem 3 we obtain :

COROLLARY 1. *Under the hypothesis of Theorem 3, $E(L)_{\text{torsion}} \subseteq Ker\psi$ if and only if there exists a unit u of L such that $\sqrt{t} \equiv u \ mod \bar{\pi} O_L$.*

Proof. Since $E[\pi^2] \not\subset E(L)$ the inclusion $E(L)_{\text{torsion}} \subseteq Ker\psi$ is equivalent with $\psi(Q) = 1$, (see section 2). By Theorem 3 we know that $\psi(Q) = 1$ if and only if $(\sqrt{t}, \pi^{-1}\sigma_G)\ \Lambda$ is a free Λ-module. Since we know that the element of $C\ell(\Lambda)$ défined by $(\sqrt{t}, \pi^{-1}\sigma_G)\Lambda$ belongs to $D(\Lambda)$ and is represented by $\sqrt{t}$, the conclusion follows Theorem 2. □

It will be obviously very interesting to know wether the condition of the corollary is always satisfied. In section 4 we checked that the condition is fulfilled when $K = \mathbb{Q}(\sqrt{-7})$.

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2. Proof of Theorems 1 and 2.

We keep the notations of section 1. Let m be the largest positive integer such that $E[\pi^m] \subset E(L)$. We know that $[\pi]E(L) \subset Ker\psi \subset Ker\psi'$. Therefore, in order to prove Theorem 1, it suffices to show that

$$E[\pi^m] - E[\pi^{m-1}] \subset Ker\psi'.$$

Let us now fix $Q \in E(L)$ such that $G_Q \not\subset E(L)$. In this case L_Q can be identified with $L(Q)$, the field generated over L by the coordinates of all points of G_Q. Of course, now $[L(Q) : L] = 2$. Let $R \in E(\bar{\mathbb{Q}})$ be such that

$$\pi R = Q.$$

Then the map :

$$Gal(L(Q)/L) \to G$$
$$\omega \to R^\omega - R$$

induces a group isomorphism which is independent of the particular choice of R. We may identify these two groups. Let γ be the non trivial element of G.

Proof of Theorem 1.

The proof splits in two steps.

(I) Preliminary step

Let $\hat{G}$ denote the group of characters of G. We have an isomorphism

$$\theta : C\ell(\mathfrak{M}) \simeq \prod_{\chi \in \hat{G}} C\ell(O_L). \tag{2-1}$$

For $y \in C\ell(\mathfrak{M})$ we write $\theta_\chi(y)$ to denote its projection on the χ-component $C\ell(O_L)$. Now G acts as automorphisms on $L(Q)$. We write this action exponentially. For $\chi \in \hat{G}$ and $b \in Map(G_Q, \bar{\mathbb{Q}})$, the Lagrange resolvent of b is defined by

$$(b|\chi) = \sum_{g \in G} b^g \chi(g^{-1}) \tag{2-2}$$

Proposition 1. *Let $\chi \in \hat{G}$ and $y \in L(Q)$ be such that $y^g = y.\chi(g)$, $\forall g \in G$. Then there exists a fractional ideal $I(\chi)$ of L whose class in $C\ell(O_L)$ is independent of the choice of y, such that $y^2 O_L = I(\chi)^2$. Moreover, $\theta_\chi(\psi'(Q)) = [I(\chi)]^{-1}$.*

Proof. Clearly the class of $I(\chi)$ does not depend on the choice of y. We may, therefore, take $y = \pi^{-1}(d|\chi)$ where d generates a normal basis of $L(Q)$ over L. From [T], Proposition 6 and Theorem 3, we deduce that there exists a fractional ideal $I(\chi)$ of L such that $\theta_\chi(\psi'(Q)) = [I(\chi)]^{-1}$ and $I(\chi)O_{L(Q)} = \pi^{-1}(d|\chi)O_{L(Q)}$.

□

Corollary 2. *The following statements are equivalent*

i) $\psi'(Q) = 1$

ii) *There exists $y \in L(Q) \setminus L$ such that $y^2 \in L$ and $y^2 O_L$ is a square of a principal O_L-ideal.*

iii) *There exists a unit $u \in L$ such that $L(Q) = L(\sqrt{u})$.*

(II) Construction of a unit.

Let us now assume that $E[4] \subset E(L)$ and fix $Q \in E[\pi^m]$. Therefore, in this case $m > 1$. We consider a general Weierstrass model of E defined over L. Let us fix $R \in G_Q$. Let S be the primitive π-division point and V a primitive 4-division point of $E(L)$. As $G_Q \not\subset E(L)$, the points $[2]R$ and $[2](R+V)$ are both distinct from S. Thus $x(R)^\gamma = x(R+S) \neq x(R)$ and $x(R+V)^\gamma = x(R+V+S) \neq x(R+V)$.

We then have

$$L(Q) = L(x(R)) = L(x(R+V)).$$

Thus, by the theorem of Fueter-Hasse, [CN-T 2, IX]

$$L(Q) = \begin{cases} L.K(\mathfrak{p}^{m+1}) & \text{if 2 is ramified in } (K/\mathbb{Q}) \\ L.K(4\mathfrak{p}^{m-1}) & \text{if 2 is split in } (K/\mathbb{Q}) \end{cases} \tag{2-3}$$

where $K(f)$ denotes the K-ray class field mod f for any O_K- ideal f.

Next we fix an analytic parametrisation

$$\mathbb{C}/\Omega \xrightarrow{\sim} E(\mathbb{C})$$

for a certain lattice Ω of $\mathbb{C}$.

We now set :

(2-4) $$A_Q = \begin{cases} \frac{h_\Omega(R)-h_\Omega(R+S)}{h_\Omega(Q)-h_\Omega(Q+S)}, & \text{if 2 is ramified in } (K/\mathbb{Q}) \\ \frac{h_\Omega(R+V)-h_\Omega(R+V+S)}{h_\Omega(Q+V)-h_\Omega(Q+V+S)}, & \text{if 2 is split in } (K/\mathbb{Q}) \end{cases}$$

where h_Ω is the first Weber's function. Once again from the theory of complex multiplication we know that $A_Q \in K(\mathfrak{p}^{m+1})$ (resp. $K(4\mathfrak{p}^{m-1})$) if 2 is ramified (resp. split) in $(K/\mathbb{Q})$. Moreover we obtain that

(2-5) $$\begin{cases} K(\mathfrak{p}^{m+1}) = K(\mathfrak{p}^m)(A_Q), & \text{if 2 is ramified in } (K/\mathbb{Q}) \\ K(4\mathfrak{p}^{m-1}) = K(4\mathfrak{p}^{m-2})(A_Q), & \text{if 2 is split in } (K/\mathbb{Q}) \end{cases}$$

From (2.3) we then deduce that

(2-6) $$L(Q) = L(A_Q) \text{ and } A_Q^2 \in L.$$

Let $\wp_\Omega$ be the Weierstrass $\wp$ function for Ω. From the definition of h_Ω we deduce that

(2-7) $$A_Q = \begin{cases} \frac{\wp_\Omega(R)-\wp_\Omega(R+S)}{\wp_\Omega(Q)-\wp_\Omega(Q+S)}, & \text{if 2 is ramified in } (K/\mathbb{Q}) \\ \frac{\wp_\Omega(R+V)-\wp_\Omega(R+V+S)}{\wp_\Omega(Q+V)-\wp_\Omega(Q+V+S)}, & \text{if 2 is split in } (K/\mathbb{Q}) \end{cases}$$

Let $\mathcal{H}$ denote the upper half plane. Let $\tau \in \mathcal{H}$ be such that $\Omega = \lambda(\mathbb{Z}\tau+\mathbb{Z})$ for some $\lambda \in \mathbb{C}^*$. For $z \in \mathcal{H}$ we write $\Omega_z = \mathbb{Z}z + \mathbb{Z}$. For $a \in (\mathbb{Q}/\mathbb{Z})^2$ we choose the unique representative $(a_1, a_2) \in \mathbb{Q}^2$ with $a_1, a_2 \in [0, 1[$. We write $az = a_1 z + a_2$. We define $r(resp.s, resp.v, resp.q)$ in $(\mathbb{Q}/\mathbb{Z})^2$ such that $\lambda(r\tau)(resp.\ \lambda(s\tau),\ resp.\lambda(v\tau),\ resp.\lambda(q\tau))$ represents $R(resp.S,$ $resp.\ V, resp.\ Q)$ in $\mathbb{C}\ mod.\Omega$. We now consider functions $F(r, q, s)$ and $G(r, q, s, v)$ defined by

(2-8.a) $$F(r,q,s)(z) = \frac{\wp_{\Omega_z}(rz) - \wp_{\Omega_z}(rz+sz)}{\wp_{\Omega_z}(qz) - \wp_{\Omega_z}(qz+sz)}$$

and

(2-8.b) $$G(r,q,s,v)(z) = \frac{\wp_{\Omega_z}(rz+vz) - \wp_{\Omega_z}(rz+vz+sz)}{\wp_{\Omega_z}(qz+vz) - \wp_{\Omega_z}(qz+vz+sz)}$$

(2-9) $$A_Q = \begin{cases} F(r,q,s)(\tau) & \text{if 2 ramified,} \\ G(r,q,s,v)(\tau) & \text{if 2 splits.} \end{cases}$$

Functions F and G are modular Weierstrass units of a level which is an appropriate power of 2.

When f and g are functions defined on $\mathcal{H}$ we write

$$f \approx g$$

if there exist integers n and m such that f^n/g^m is a modular function, which is a unit over $\mathbb{Z}$.

For $a \in (\mathbb{Q}/\mathbb{Z})^2$ we introduced in $[CN-T_1]$, (2-7), a function $\tilde{\Psi}(a)$ defined on $\mathcal{H}$. In fact an appropriate power of $\tilde{\Psi}(a)$ is a ratio of Deuring modular units. From $[CN - T_1]$, Proposition 2-8, we obtain

LEMMA 1. *There are equivalences*

$$F(r,q,s) \approx \frac{\tilde{\Psi}^2(q)\tilde{\Psi}^2(q+s)\tilde{\Psi}(2r+s)}{\tilde{\Psi}^2(r)\tilde{\Psi}^2(r+s)\tilde{\Psi}(2q+s)}$$

and

$$G(r,q,s,v) \approx \frac{\tilde{\Psi}^2(q+v)\tilde{\Psi}^2(q+s+v)\tilde{\Psi}(2r+2v+s)}{\tilde{\Psi}^2(r+v)\tilde{\Psi}^2(r+v+s)\tilde{\Psi}(2q+2v+s)}$$

We now show :

LEMMA 2. *(i) If 2 is ramified in $(K/\mathbb{Q})$, then $F(r,q,s)(\tau)$ is a unit.*

(ii) If 2 is split in $(K/\mathbb{Q})$ and $m > 2$, then $G(r,q,s,v)(\tau)$ is a unit.

Proof (i) Let 2 be ramified in $(K/\mathbb{Q})$ and suppose $m = 2t,\ t > 1$ (if m is odd the proof is similar). Then, $q\tau(resp.(q+s)\tau,\ resp.(2q+s)\tau,\ resp.r\tau,\ resp.(r+s)\tau,\ resp.(2r+s)\tau)$ defines a primitive $\mathfrak{p}^{2t}(resp.\mathfrak{p}^{2t},\ resp.\mathfrak{p}^{2(t-1)},\ resp.\mathfrak{p}^{2t+1}, resp.\mathfrak{p}^{2t+1},\ resp.\mathfrak{p}^{2t-1})$-division point of $\mathbb{C}/\Omega_\tau$.

For two algebraic numbers a, b we write $a \sim b$ if ab^{-1} is a unit. From $[CN - T_1]$, Proposition 3-5, we deduce that

$$\text{(2-10)} \qquad \begin{cases} \tilde{\Psi}(q)(\tau) \sim \tilde{\Psi}(q+s)(\tau) \sim 2^{(2^{-2t})} \\ \tilde{\Psi}(r)(\tau) \sim \tilde{\Psi}(r+s)(\tau) \sim 2^{(2^{-2t-1})} \\ \tilde{\Psi}(2q+s)(\tau) \sim 2^{(2^{2-2t})} \\ \tilde{\Psi}(2r+s)(\tau) \sim 2^{(2^{1-2t})}. \end{cases}$$

Thus from Lemma 1 and (2-10) we conclude that $F(q,r,s)(\tau)$ is a unit.

(ii) Now suppose that 2 is split in $(K/\mathbb{Q})$ and $m>2$. Then $(q+v)\tau$ and $(q+v+s)\tau$ are primitive $\mathfrak{p}^m\bar{\mathfrak{p}}^2$ division points ; $(r+v)\tau$ and $(r+v+s)\tau$ are primitive $\mathfrak{p}^{m+1}\bar{\mathfrak{p}}^2$-division points. Moreover $(2r+2v+s)\tau$ $(resp.(2q+2v+s)\tau)$ is a primitive $\mathfrak{p}^m\bar{\mathfrak{p}}$ $(resp.\mathfrak{p}^{m-1}\bar{\mathfrak{p}})$-division point. Since these points are primitive of composite order, it follows from $[CN-T_1]$, Proposition 3-5, that each factor in the right hand side of the equivalence in Lemma 1 gives a unit when evaluated at τ. From (2-9) and Lemma 2 we now conclude that A_Q is a unit. Therefore Theorem 1 is proved, via Corollary 2, except in the case where 2 is split in $(K/\mathbb{Q})$ and $m=2$. We can, nevertheless, treat this case in a similar fashion by replacing A_Q by A_Q^1 given by

$$A_Q^1=\pi^{-1}(P_\Omega(R+V)-P_\Omega(R+V+S)) \tag{2-11}$$

where P_Ω is the function considered by Schertz [Sh]. We know that

$$A_Q^1=\kappa(h_\Omega(R+V)-h_\Omega(R+V+S))$$

where $\kappa\in K(1)$. We thus have $A_Q^1\in L_Q\setminus L$ and $(A_Q^1)^2\in L$. We now deduce from [Sch], (12) and Satz 3, that A_Q^1 is a unit. This now completes the proof of Theorem 1.

□

Proof of Theorem 2. We recall that the order Λ is explicitly given by (1-4). Let us consider the fiber product of orders

$$\begin{array}{ccc}\Lambda & \xrightarrow{\epsilon} & O_L\\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\phi}\\ \Lambda/(\pi^{-1}\sigma_G) & \xrightarrow{\bar\epsilon} & O_L/\bar\pi\, O_L\end{array}$$

where η and ϕ are the quotient maps, ϵ is the augmentation map and $\bar\epsilon$ is induced by ϵ. Using the Mayer-Vietoris sequence of Reiner-Ullom, [S], [U], we obtain an exact sequence of groups and homomorphisms.

$$O_L^*\times(\Lambda/(\pi^{-1}\sigma_G))^*\xrightarrow{\phi\bar\epsilon^{-1}}(O_L/\bar\pi\, O_L)^*\xrightarrow{\delta}D(\Lambda)\to\{1\} \tag{2-13}$$

where δ is the connecting homomorphism.

We also need to observe that

$$D(O_L) = D(\Lambda/(\pi^{-1}\sigma_G)) = \{1\}.$$

Moreover, for s coprime with $\bar{\pi}$, $\delta(s \; mod \; \bar{\pi}O_L)$ is given by the class of the corresponding Swan module $(s, \pi^{-1}\sigma_G)\Lambda$. Since O_L and $\Lambda/(\pi^{-1}\sigma_G)$ can be naturally identified as rings, we conclude that

$$D(\Lambda) = (O_L/\bar{\pi}O_L)^*/Im \; O_L^*.$$

□

3. Minimal Fueter model

We recall in this section that $\mathfrak{p} = (\pi)$ is a principal, prime ideal of K, above 2, which is split in $(K/\mathbb{Q})$. Moreover we suppose that $E[\pi] \subset E(L)$ and $E[\pi^2] \not\subset E(L)$. We let Ω be a fractional ideal of K and ν a primitive $4O_K$-division point of $\mathbb{C}/\Omega$.

In $[CN - T_2]$ a Fueter elliptic curve was considered, corresponding to the pair (Ω, ν), given by

$$y^2 = 4x^3 + tx^2 + 4x \tag{3-1}$$

with

$$t = 12\wp_\Omega((2\nu)/(\wp_\Omega(\nu) - \wp_\Omega(2\nu)).$$

In fact one defines a complex analytic isomorphism between $\mathbb{C}/\Omega$ and the complex points of this curve by considering

$$z \rightarrow \begin{cases} (T(z), T_1(z), 1) & \text{if } z \neq 2\nu \\ (0, 1, 0) & \text{if } z = 2\nu \end{cases} \tag{3-2}$$

where T and T_1 are Fueter's elliptic functions, $[CN - T_2], IV$. The minimal Fueter model E is obtained from (3.1) by the change of coordinates

$$(x, y) \rightarrow (x, \sqrt{t}x + 2y). \tag{3-3}$$

From (3-2) and (3-3) we deduce an isomorphism between $\mathbb{C}/\Omega$ and the $\mathbb{C}$-points of E given by

$$z \rightarrow \begin{cases} (T(z), U(z), 1) & \text{if } z \neq 2\nu \\ (0, 1, 0) & \text{if } z = 2\nu \end{cases} \tag{3-4}$$

where $U(z) = (1/2)(T_1(z) - \sqrt{t}\; T(z))$.

We remark that $0 = (0, 0, 1)$ is taken to be the identity of the group law. It is also worth remarking that $i \in K(t)$. We set $A = (i, 0, 1)$. It is worth to notice that, using the theory of complex multiplication, one can show that $A \in E(L)$ and has infinite order. Let α be the parameter of A in $\mathbb{C}/\Omega$ under the isomorphism (3-4).

The divisor of T is given by

$$(T) = 2(0) - 2(2\nu). \tag{3-5}$$

From $[CN - T_2]$,IV we know that

$$T(z).T(z + 2\nu) = 1. \tag{3-6.a}$$

$$T_1(z + 2\nu) = -T_1(z)/T^2(z). \tag{3-6.b}$$

Therefore, since T is an even function and T_1 is an odd function, we deduce that

$$U(2\nu - z) = U(z)/T^2(z). \tag{3-7}$$

Moreover, the elliptic function U has divisor

$$(U) = (0) + (\alpha) + (2\nu - \alpha) - 3(2\nu). \tag{3-8}$$

We denote by N the point of $E(\bar{\mathbb{Q}})_{\text{torsion}}$ defined by ν. Let Q be the primitive π-division point of E. We fix a point $R \in G_Q$ and denote by ρ its parameter in $\mathbb{C}/\Omega$.

Now $R + Q = -R$, therefore $G_Q = \{R, -R\}$.

Thus, $x(R)^\gamma = x(R + Q) = x(-R) = x(R)$. Then $L(Q) = L(y(R)) = L(T_1(\rho)) = L(D(\rho))$ where $D(\rho) = T_1(\rho)/T(\rho)$.

From $[CN - T_2]$, IX, (6-7) we know that $D^4(\rho) = t^2 - 2^6$, which is a unit. Since $D^2(\rho) \in L$ we conclude from Corollary 2 that $\psi'(Q) = 1$.

Until the end of this section the x and y coordinates are those of model (1-5).

We now want to study $\psi(Q)$. First, we have

LEMMA 3. *Let $\mathfrak{P}$ be a prime ideal of O_K. Let $P \in E(\bar{\mathbb{Q}})_{\text{torsion}}$ be such that $\{P, [2]N - P\} \bigcap_{n>0} E[\mathfrak{P}^n]) = \phi$. Then $x(P)$ is a $\mathfrak{P}$-unit (i.e. unit at all primes dividing $\mathfrak{P}$).*

Proof. We first observe that for any $P \in E(\bar{\mathbb{Q}})$, $P \neq [2]N$, $x(P)$ is a $\mathfrak{P}$-integer if and only if $y(P)$ is a $\mathfrak{P}$-integer. Under the given hypothesis both

$x(P)$ and $y(P)$ are well defined and are non zero. Since $x(P).x([2]N-P) = 1$, it suffices to show that $x(P)$ is a $\mathfrak{P}$-integer.

Let M be a finite extension of L such that $\{P, [2]N-P\} \subset E(M)$. Suppose $x(P)$ is not a $\mathfrak{P}$-integer. Then there exists $\mathfrak{P}_M$, a maximal O_M-ideal, with $\mathfrak{P}_M \cap O_K = \mathfrak{P}$ and $v(x(P)) < 0$ where v denote the standard valuation on the completion of M at $\mathfrak{P}_M$. From the equation of the minimal Fueter model E we see that $2v(y(P)) = 3v(x(P))$. Thus, under the reduction mod $\mathfrak{P}_M$, P is mapped onto $(0,1,0)$. This means that $[2]N-P$ is in the kernel of reduction mod$\mathfrak{P}_M$ which is impossible since the set of torsion points in the kernel of reduction is precisely $\bigcup_{n>0} E[\mathfrak{P}^n]$.

□

LEMMA 4. $x(R) \sim \pi$

Proof. Since R is a primitive π^2-division point of E, $[2]N-R$ is a torsion point of composite order. From Lemma 3 we conclude that $x(R)$ is a unit outside the prime divisors of $\mathfrak{p} = (\pi)$. For a prime $\mathfrak{P}$ of $L(Q)$ that divides $\mathfrak{p}$, using that R is a primitive π^2-division point in the kernel of reduction mod $\mathfrak{P}_{L(Q)}$ and that $x(R)/y(R)$ is the parameter of R in the associated formal group we can find the valuation $v_{\mathfrak{P}_{L(Q)}}(x(R))$.

□

Remark : Lemma 3 and 4 can both be proved using the technique of modular functions as developed in section 2, Lemma 1 and 2.

It follows from the equation of E that $y(R)^2/\pi$ is an algebraic integer and a $\mathfrak{p}$-unit.

We now consider the map

$$\begin{aligned} h : &G_Q \to \bar{\mathbb{Q}} \\ &M \to y(M). \end{aligned} \tag{3-9}$$

PROPOSITION 2.

i) The map h lies in $\tilde{O}_Q$

ii) Let $\chi \in \hat{G}$ and $M \in G_Q$, then

$$(h|\chi)(M) \sim \begin{cases} \sqrt{t}x(M), & \text{if } \chi \text{ is trivial} \\ x(M) & \text{otherwise.} \end{cases}$$

Proof. We first prove (ii). Since x is an even function and T_1 an odd function, we obtain from the definition of h and (3-4)

$$(h|\chi)(M) = \begin{cases} -\sqrt{t}x(m), & \text{if } \chi \text{ is the identity character} \\ T_1(m) & \text{otherwise} \end{cases}$$

where m is the parameter of M in $\mathbb{C}/\Omega$. Since $m = \pm\rho$ we have $T_1(m) = \pm D(\rho)x(M)$ and then, since $D(\rho)$ is a unit, $T_1(m) \sim x(M)$. We now prove i). By lemma 4 it is evident that $h \in O_Q$. Since

$$\Lambda = 1_G O_L + (\pi^{-1}\sigma_G)O_L,$$

we need only check that $h.(\pi^{-1}\sigma_G) \in O_Q$. For $M \in G_Q$ we obtain

$$h(\pi^{-1}\sigma_G)(M) = \pi^{-1}(h|\epsilon)(M) = -\pi^{-1}\sqrt{t}\cdot x(M)$$

where ϵ is the identity character. Using Lemma 4 we conclude that $h(\pi^{-1}\sigma_G)(M) \in \bar{O}$. Hence h lies in $\tilde{O}_Q$.

□

Proof of Theorem 3. The proof is similar to that of Theorem 5 in [S-T]. We must show the equality locally. For each prime $\mathfrak{P}$ of L we write

$$\tilde{O}_{Q,\mathfrak{P}} = \theta_{\mathfrak{P}}\Lambda_{\mathfrak{P}}$$

(3-10) $$(\sqrt{t}, \pi^{-1}\sigma_G)\Lambda_{\mathfrak{P}} = a_{\mathfrak{P}}\Lambda_{\mathfrak{P}}$$

where $\theta_{\mathfrak{P}}(resp. a_{\mathfrak{P}})$ belongs to $\tilde{O}_{Q,\mathfrak{P}}(resp. \Lambda_{\mathfrak{P}})$. From Theorem 3 of [T] we know that for $M \in G_Q$ and $\chi \in \hat{G}$ we have

(3-11) $$(\theta_{\mathfrak{P}}|\chi)(M) \sim \pi.$$

We let χ act on $L_{\mathfrak{P}}[G]$ by $L_{\mathfrak{P}}$-linearity. We first observe that $\chi(\Lambda_{\mathfrak{P}}) = O_{L,\mathfrak{P}}$. Then, by looking at $\chi(a_{\mathfrak{P}}\Lambda_{\mathfrak{P}})$, we obtain

(3-12) $$\chi(a_{\mathfrak{P}}) \sim \begin{cases} 1, & \text{if } \chi \text{ is the identity character} \\ \sqrt{t} & \text{otherwise.} \end{cases}$$

We now can write

(3-13) $$\theta_{\mathfrak{P}}.(\sqrt{t}b_{\mathfrak{P}}) = ha_{\mathfrak{P}}$$

with $b_{\mathfrak{P}} \in L_{\mathfrak{P}}[G]$. In order to prove the theorem we must show that $b_{\mathfrak{P}} \in \Lambda^*_{\mathfrak{P}}$. Since $h \in \tilde{O}_{Q,\mathfrak{P}}$, $h\sqrt{t}$ and $h(\pi^{-1}\sigma_G)$ lie in $\tilde{O}_{Q,\mathfrak{P}}$. We conclude from (3-10) that $ha_{\mathfrak{P}} \in \tilde{O}_{Q,\mathfrak{P}}$ and, from (3-13), that $\sqrt{t}b_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}$.

For $\chi \in \hat{G}$ we consider the Lagrange resolvent of both sides of (3-13). We obtain

$$\sqrt{t}(\theta_{\mathfrak{P}}|\chi)\chi(b_{\mathfrak{P}}) = (h|\chi)\chi(a_{\mathfrak{P}}). \tag{3-14}$$

Using Lemma 4, Proposition 2, (3-11) and (3-12), we deduce from (3-14) that $\chi(b_{\mathfrak{P}}) \sim 1$.

We now consider two cases.

Case 1. $\mathfrak{P} \nmid \sqrt{t}$. In this case $\sqrt{t}b_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}$ implies that $b_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}$; so $b_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^*$, since $\chi(b_{\mathfrak{P}})$ is a unit for all $\chi \in \hat{G}$.

Case 2. $\mathfrak{P} \mid \sqrt{t}$. Since $\sqrt{t}$ is coprime with 2, $\mathfrak{P} \nmid 2$. Then $\Lambda_{\mathfrak{P}}$ is the unique maximal order and $b_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^*$ since $\chi(b_{\mathfrak{P}})$ is a unit for all $\chi \in \hat{G}$.

□

Remark : If 2 splits in $(K/\mathbb{Q}), (2) = (\pi)(\overline{\pi})$ and E denotes the Fueter minimal model

$$y^2 + \sqrt{t}xy = x^3 + x$$

then for any number field $L \supset K(\sqrt{t})$ and $G = E[\pi]$ we have that $E(L)_{torsion} \subset Ker\psi'$.

One can easily check that if $E[\pi^2] \subset E(L)$ then $E[4] \subset E(L)$ and we can use the results of section 2.

4. Examples

In this section we consider the set up of section 3 for the particular case of $K = \mathbb{Q}(\sqrt{-7})$.

We set $\pi = (1 + \sqrt{-7})/2$ and $2 = \pi\bar{\pi}$, where $\bar{\pi}$ is the complex conjugate of π. We note that the class number of K is 1, $K(2) = K$ and $[K(4) : K] = 2$. Since $i \in K(t) = K(4)$ we must have $K(t) = K(i)$. Moreover, since $t^2 - 2^6$ is a unit in $K(2)$, we know that $t^2 - 2^6 = \pm 1$. The possibility $t^2 - 2^6 = 1$ contradicts the fact that $K(t) = K(i)$. Hence $t^2 = 63$ and $L = K(\sqrt[4]{63})$; therefore L is the splitting field of $X^4 - 63$.

We first determine the group kernel $D(\Lambda)$ considered in Theorem 2.

PROPOSITION 3. $D(\Lambda) = \{1\}$.

Proof. By Theorem 2 we know that

$$D(\Lambda) = (O_L/\bar{\pi}O_L)^*/Im\ O_L^*.$$

It is easily checked that the ramification index of (2) in L is 4. Hence the group $(O_L/\bar{\pi}O_L)^*$ is of order 8. We have to show that $\mathrm{Im}(O_L)^*$ also has order 8. Let $\alpha = \sqrt[4]{63}$ and $\beta = (1+i)\alpha$. We set $u = (1-i)(1+\pi)+\alpha$, $v = 1 - 3\alpha + \alpha^3/3$ and $w = 5 - 2\beta - 12\pi - 2\pi\beta$. We verify that

$$u^2 = iw, \ w.(5 + 2\beta - 12\pi + 2\pi\beta) = 1$$

$$v(127 + 45\alpha + 12\alpha^2 + 17\alpha^3/3) = 1 \tag{4-1}$$

Therefore u, v and w are all units of L. We also have

$$u^2 \equiv i \bmod \bar{\pi}O_L, \ v^2 \equiv 1 \bmod \bar{\pi}O_L$$

$$i^2 \equiv 1 \bmod \bar{\pi}O_L \tag{4-2}$$

and

$$i \not\equiv 1 \bmod \bar{\pi}O_L, \ v \not\equiv 1 \bmod \bar{\pi}0_L,$$

$$v \not\equiv i \bmod \bar{\pi}O_L \tag{4-3}$$

Let Φ be the quotient map

$$\Phi : O_L \to (O_L/\bar{\pi}O_L)$$

It follows from (4-2) and (4-3) that $\Phi(u)$ is of order 4 and that $\Phi(v)$ doesn't lie in the subgroup generated by $\Phi(u)$. Hence we must have that the order of $\mathrm{Im}(O_L^*)$ is 8.

□

We know from section 3 that

$$L(E[\pi^2]) = L((t^2 - 2^6)^{1/4}) = L(\sqrt{i}).$$

Therefore :

$$E(L)_{torsion} \subset Ker\psi' .$$

Hence, from Proposition 3, we conclude

COROLLARY 3.

$$E(L)_{\text{torsion}} \subset Ker\psi.$$

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