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On the order of vanishing of modular L -functions at the critical point

par HENRYK IWANIEC

1. Introduction

The nonvanishing of L -functions at special points is an attractive area of research in contemporary number theory, see [7]-[11]. One example is the Rankin-Selberg zeta-function $L(f \otimes g_j, s)$ associated with a holomorphic cusp form f of weight 2 and Maass cusp forms g_j of eigenvalue $\lambda_j = s_j(1 - s_j)$. In this case the nonvanishing of $L(f \otimes g_j, s)$ at $s = s_j$ plays a rôle in the work of R. Phillips and P. Sarnak [6] on deformations of groups and was proved to be true for infinitely many cusp forms g_j by J.-M. Deshouillers and H. Iwaniec [3]. Another example is the Birch-Swinnerton-Dyer conjecture which asserts that the rank of the group of rational points on an elliptic curve E defined over $\mathbb{Q}$ is equal to the order of vanishing of the associated Hasse-Weil L -function $L(s, E)$ at $s = 1$ (the center of the critical strip).

Recently V.A. Kolyvagin [4] has proved that the group of rational points on a modular elliptic curve E is finite if $L(1, E) \neq 0$ and that the L -function $L(s, E, \chi_d)$ twisted by a suitable real character χ_d has simple zero at $s = 1$. The latter condition was subsequently proved to hold true for infinitely many discriminants d by D. Bump, S. Friedberg and J. Hoffstein [2] and independently by K. Murty and R. Murty [5]. In these notes we establish (from scratch) quantitative results on Kolyvagin's condition.

2 - Statement of results

Let E be a modular elliptic curve defined over $\mathbb{Q}$ and

$$L(s, E) = \sum_1^{\infty} a_n n^{-s}$$

be the Hasse-Weil L -function associated with E . Thus

$$f(z) = \sum_1^{\infty} a_n e(nz)$$

is a cusp form of weight 2 which is a newform of level N , where N is the conductor of E . The L -function is entire and it satisfies the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E),$$

where $w = \pm 1$. We are interested in curves E for which $L(1, E) \neq 0$, so the functional equation holds with the sign $w = 1$. The twisted L -function

$$L(s, E, \chi_d) = \sum_1^\infty a_n \chi_d(n) n^{-s},$$

where χ_d is a real primitive character to modulus d prime to N is also entire and it satisfies the functional equation

$$(1) \quad \left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E, \chi_d) = w_d \left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E, \chi_d)$$

with the sign $w_d = w\chi_d(-N)$. In the sequel we let d range over the set

$$\mathcal{D} = \{d : 0 < d \equiv -\nu^2 \pmod{4N} \text{ for some } \nu \text{ prime to } 4N\}$$

and we let $\chi_d(n) = \left(\frac{-d}{n}\right)$ be the Kronecker symbol. Thus if d is squarefree χ_d is the primitive character to the modulus d which is associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Every prime dividing N splits in $\mathbb{Q}(\sqrt{-d})$. Moreover we have $w_d = -1$, so by (1) it follows that

$$(2) \quad L(1, E, \chi_d) = 0.$$

Our aim is to prove that $L(s, E, \chi_d)$ has a simple zero at $s = 1$, i.e. $L'(1, E, \chi_d) \neq 0$ for infinitely many d in $\mathcal{D}$. To this end we shall evaluate two sums of type

$$(3) \quad S_4(Y) = \sum_{d \in \mathcal{D}, d \leq Y}^b |L'(1, E, \chi_d)|^4$$

and

$$(4) \quad S_1(Y) = \sum_{d \in \mathcal{D}}^b L'(1, E, \chi_d) F(d/Y),$$

where $\sum^b$ means that the summation is restricted to squarefree numbers and F is a smooth function, compactly supported in $\mathbb{R}^+$ with positive mean value.

THEOREM. For any $\epsilon > 0$ and $Y \geq 1$ we have

$$(5) \quad S_4(Y) \ll Y^{2+\epsilon}$$

and

$$(6) \quad S_1(Y) = \alpha Y \log Y + \beta Y + O(Y^{13/14+\epsilon})$$

with some constants $\alpha \neq 0$ and β which depend on the curve E and the test function F .

COROLLARY. Suppose $\epsilon > 0$ and $Y > c(\epsilon)$. Then $L'(1, E, \chi_d) \neq 0$ for at least $Y^{2/3-\epsilon}$ real primitive characters χ_d to modulus $d \in \mathcal{D}$, $d \leq Y$.

3. Estimates for the coefficients of f

The Fourier coefficients a_n of the cusp form f are multiplicative. More exactly, for $\operatorname{Re} s > 3/2$ we have the Euler product

$$(7) \quad L(s, E) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

with $\alpha_p = 0, \pm 1, \beta_p = 0$ if $p|N$ and $|\alpha_p| = |\beta_p| = p^{1/2}$ if $p \nmid N$. In the latter case the result was proved by M. Eichler and P. Deligne. It yields the following bound for the coefficient a_n (known as the Ramanujan conjecture)

$$(8) \quad |a_n| \leq n^{1/2} \tau(n),$$

where $\tau(n)$ denotes the divisor function, $\tau(n) \ll n^\epsilon$. This bound can be slightly improved on average. Indeed, arguing as G. Hardy and E. Hecke with Parseval's formula and using the boundedness of $yf(z)$ we get

$$(9) \quad \sum_{m \leq M} |a_m|^2 \ll M^2.$$

Similarly we get

$$(10) \quad \sum_{m \leq M} a_m e(\alpha m) \ll M \log M$$

for any real α and $M \geq 2$, the implied constant depending on f only. In this section we derive three variations on (10).

LEMMA I. Let α be real and ψ be a periodic function of period r . We then have

$$(11) \quad \sum_{m \leq M} a_m \psi(m) e(\alpha m) \ll \Psi M \log M,$$

where

$$\Psi = \frac{1}{r} \sum_{a \pmod{r}} \left| \sum_{b \pmod{r}} \psi(b) e\left(\frac{ab}{r}\right) \right|.$$

Moreover, if $|\psi| \leq 1$ and s is a positive integer then we have

$$(12) \quad \sum_{m \leq M, (m, s)=1} a_m \psi(m) e(\alpha m) \ll \tau(s) r^{\frac{1}{2}} M \log M$$

and

$$(13) \quad \sum_{m \leq M, (m, s)=1}^b a_m \psi(m) e(\alpha m) \ll \tau(s) r^{\frac{1}{2}} M (\log M)^7$$

PROOF: The sum on the left-hand side of (11) is equal to

$$\frac{1}{r} \sum_{a \pmod{r}} \left(\sum_{b \pmod{r}} \psi(b) e\left(\frac{ab}{r}\right) \right) \sum_{m \leq M} a_m e\left(\left(\alpha - \frac{a}{r}\right)m\right),$$

whence the inequality (11) follows by (10). If $|\psi| \leq 1$ we obtain $\Psi \leq r^{1/2}$ by Cauchy's inequality. For the proof of (12) we can assume that $(r, s) = 1$ by changing ψ suitably. Then we apply (11) for $\psi\chi_0$ in place of ψ , where χ_0 is the principal character to the modulus s . We obtain

$$\begin{aligned} \Psi &= \frac{1}{rs} \sum_{a \pmod{r}} \left| \sum_{b \pmod{r}} \psi(b) e\left(\frac{ab}{r}\right) \right| \\ &\quad \sum_{c \pmod{s}} \left| \sum_{d \pmod{s}} \chi_0(d) e\left(\frac{cd}{s}\right) \right| \ll \frac{r^{\frac{1}{2}}}{s} \sum_{c \pmod{s}} \sum_{d|(c, s)} d = \tau(s) r^{\frac{1}{2}}, \end{aligned}$$

which gives (12). Finally we derive (13) from (12). The sum on the left-hand side of (13) is equal to

$$\begin{aligned} &\sum_{\nu^2 m \leq M, (\nu m, s)=1} \sum \mu(\nu) a_{\nu^2 m} \psi(\nu^2 m) e(\alpha \nu^2 m) \\ &= \sum_{\substack{(\nu, s)=1 \\ \nu^2 \lambda \leq M}} \sum_{\lambda | \nu^\infty} \mu(\nu) a_{\nu^2 \lambda} \sum_{\substack{m \leq M/\lambda \nu^2 \\ (m, \nu s)=1}} a_m \psi(\nu^2 \lambda m) e(\alpha \nu^2 \lambda m) \\ &\ll \tau(s) r^{\frac{1}{2}} M (\log M) \sum_{\substack{(\nu, s)=1 \\ \nu^2 \lambda \leq M}} \sum_{\lambda | \nu^\infty} |a_{\nu^2 \lambda}| \frac{\tau(\nu)}{\nu^2 \lambda}. \end{aligned}$$

Hence (13) follows by (8).

4. Approximate formulas for $L'(1, E, \chi_d)$

We shall express $L'(1, E, \chi_d)$ in terms of the rapidly convergent sums

$$\mathcal{A}(X, \chi) = \sum_1^{\infty} a_n \chi(n) n^{-1} V\left(\frac{2\pi n}{X}\right),$$

where V is the incomplete gamma function defined by

$$V(X) = \int_X^{\infty} e^{-t} t^{-1} dt = \frac{1}{2\pi i} \int_{(3/4)} \frac{\Gamma(s)}{s} X^{-s} ds.$$

We have

$$\mathcal{A}(X, \chi_d) = \frac{1}{2\pi i} \int_{(3/4)} L(1+s, E, \chi_d) \frac{\Gamma(s)}{s} \left(\frac{2\pi}{X}\right)^s ds.$$

Moving the integration to the line $\operatorname{Re} s = -3/4$ we pass a simple pole at $s = 0$ with residuum $L'(1, E, \chi_d)$ by virtue of (2). On the other hand the integral over the line $\operatorname{Re} s = -3/4$ is equal to $-\mathcal{A}(d^2 N X^{-1}, \chi_d)$ by the functional equation (1). This gives

$$(14) \quad L'(1, E, \chi_d) = \mathcal{A}(X, \chi_d) + \mathcal{A}(d^2 N X^{-1}, \chi_d)$$

for any $X > 0$ and d in $\mathcal{D}$ which is squarefree. In particular we have

$$(15) \quad L'(1, E, \chi_d) = 2\mathcal{A}(d\sqrt{N}, \chi_d).$$

By (9) we infer trivially that $\mathcal{A}(X, \chi_d) \ll X^{1/2}$ for any $X > 0$ and inserting this to (14) we obtain

$$(16) \quad L'(1, E, \chi_d) = \mathcal{A}(X, \chi_d) + O(dX^{-1/2}).$$

5. Estimation of the fourth moment of $L'(1, E, \chi_d)$

By the large sieve inequality (see [1]) together with (8) we get

$$\sum_{d \leq Y} \sum_{\chi \pmod{d}}^* |\mathcal{A}(X, \chi)|^4 \ll (X + Y)^{2+\epsilon}.$$

On the other hand by (14) we have for any $d \in \mathcal{D}, d \leq Y, d$ squarefree that

$$|L'(1, E, \chi_d)|^4 \ll \int_1^{NY} |\mathcal{A}(X, \chi_d)|^4 X^{-1} dX.$$

Combining both results we infer the upper bound (5) for $S_4(Y)$.

6. An approximate formula for the first moment of $L'(1, E, \chi_d)$

By (15) we obtain

$$S_1(Y) = 2 \sum_{d \in \mathcal{D}} {}^b \mathcal{A}(d\sqrt{N}, \chi_d) F\left(\frac{d}{Y}\right).$$

Now we relax the condition that d is squarefree by introducing the factor $\sum_{a^2|d} \mu(a)$, then we split the sum according to whether $a \leq A$ or $a > A$ and in the latter case we return to squarefree numbers by extracting square divisors of $a^{-2}d$. We obtain $S_1(Y) = S + R$, say, where

$$S = 2 \sum_{a \leq A, (a, 4N)=1} \mu(a) \sum_{d \in \mathcal{D}} \mathcal{A}(a^2 d \sqrt{N}, \chi_{a^2 d}) F\left(\frac{a^2 d}{Y}\right)$$

and

$$R = 2 \sum_{(b, 4N)=1} \left(\sum_{a|b, a > A} \mu(a) \right) \sum_{d \in \mathcal{D}} {}^b \mathcal{A}(b^2 d \sqrt{N}, \chi_{b^2 d}) F\left(\frac{b^2 d}{Y}\right).$$

Here A is a large number to be chosen later. In the term $\mathcal{A}(X, \chi_{b^2 d})$ with $X = b^2 d \sqrt{N}$ we return to $L'(1, E, \chi_d)$ by reversing the arguments as follows

$$\begin{aligned} \mathcal{A}(X, \chi_{b^2 d}) &= \sum_{(n, b)=1} a_n \chi_d(n) n^{-1} V\left(\frac{2\pi n}{X}\right) \\ &= \sum_{k|b} \sum_{\ell|b} \alpha_k \beta_\ell \chi_d(k\ell) \frac{\mu(k)\mu(\ell)}{k\ell} \mathcal{A}\left(\frac{X}{k\ell}, \chi_d\right) \\ &= L'(1, E, \chi_d) \prod_{p|b} \left(1 - \chi_d(p) \frac{\alpha_p}{p}\right) \left(1 - \chi_d(p) \frac{\beta_p}{p}\right) + O(\tau(b) d X^{-\frac{1}{2}}) \end{aligned}$$

the second line being obtained by (7) and the third line by (16). Finally applying (5) and the Hölder inequality we conclude that

$$(17) \quad R \ll \sum_b \left(\sum_{a|b, a > A} 1 \right) (b^{-\frac{5}{2}} Y^{\frac{5}{4}} + b^{-4} Y^{\frac{3}{2}}) Y^\epsilon \ll (A^{-\frac{3}{2}} Y^{\frac{5}{4}} + A^{-3} Y^{\frac{3}{2}}) Y^\epsilon.$$

7. A transformation of S

It remains to evaluate S . For $(a, 4N) = 1$ and $d \in \mathcal{D}$ we have

$$\mathcal{A}(a^2 d \sqrt{N}, \chi_{a^2 d}) = \sum_{(n, a)=1} a_n n^{-1} \chi_d(n) V(2\pi n / a^2 d \sqrt{N}).$$

Every n can be written uniquely as the product $n = k\ell^2 m$, where k has prime factors in $4N$, ℓm is prime to $4N$ and m is squarefree. For n written this way and d in $\mathcal{D}$ we have $\chi_d(n) = \chi_d(m)$ subject to $(d, \ell) = 1$. The last condition is detected by the familiar formula of Möbius giving

$$S = 2 \sum_{\substack{a \leq A \\ (a, 4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 m \\ (n, a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_{dq \in \mathcal{D}} \chi_{dq}(m) F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right).$$

Next, by means of Gauss sums we write

$$\chi_d(m) = \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{2|r| < m} \chi_{Nr}(m) e\left(\frac{\overline{4N}rd}{m}\right),$$

where $\epsilon_m = 1$ if $m \equiv 1 \pmod{4}$, $\epsilon_m = i$ if $m \equiv -1 \pmod{4}$ and $4N\overline{4N} \equiv 1 \pmod{m}$. This gives

$$S = 2 \sum_{\substack{a \leq A \\ (a, 4N)=1}} \sum_{n=k\ell^2 m} \mu(a) a_n n^{-1} \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{q|\ell} \mu(q) \sum_{2|r| < m} \chi_{Nr q}(m) \sum_d,$$

where

$$\sum_d = \sum_{dq \in \mathcal{D}} F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right) e\left(\frac{\overline{4N}rd}{m}\right).$$

We put $\Delta = \min(1/2, a^2 q Y^{\epsilon-1})$ and split $S = S_0 + S_1 + S_2$, where S_0, S_1, S_2 denote the partial sums restricted by the conditions $r = 0, 0 < |r| < \Delta m, \Delta m \leq |r| < m/2$ respectively.

8. Estimates for S_2 and S_1

LEMMA 2. Suppose $g(x)$ is a smooth and integrable function on $\mathbb{R}$ with derivatives $g^{(j)}(x) \ll (|x| + X)^{-j}$ for all $j \geq 1$ the implied constant depending on j only. Suppose α is real and q is a positive integer such that αq is not an integer. We then have

$$(18) \quad \sum_{n \equiv v \pmod{q}} g(n) e(\alpha n) \ll \frac{X}{q} \left(\frac{q}{X \|\alpha q\|} \right)^j$$

for any $j \geq 2$, the implied constant depending on j only.

PROOF: By Poisson's formula the sum is equal to

$$\frac{1}{q} \sum_{u=-\infty}^{\infty} e\left(\frac{uv}{q}\right) \hat{g}\left(\alpha - \frac{u}{q}\right),$$

where $\hat{g}(y)$ denotes the Fourier transform of $g(x)$. We have $\hat{g}(y) \ll X(Xy)^{-j}$ by the partial integration j times, whence (18) follows by trivial summation over u .

To estimate S_2 we sum over d first by an appeal to (18). For any $j \geq 2$ we get $\sum_d \ll (n + Y)^{-j}$, whence $S_2 \ll 1$.

To estimate S_1 we sum over m first using (13) and partial summation together with the relation

$$e\left(\frac{4\overline{N}rd}{m}\right) = e\left(\frac{rd}{4Nm} - \frac{\overline{m}rd}{4N}\right)$$

and then we sum over r trivially getting

$$\begin{aligned} \sum_{0 < |r| < \Delta m} \sum a_n n^{-1} \overline{\epsilon}_m m^{-\frac{1}{2}} \chi_{Nr q}(m) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right) e\left(\frac{4\overline{N}rd}{m}\right) \\ \ll k^{-\frac{3}{2}} \ell^{-3} a^3 q^2 Y^{\epsilon - \frac{1}{2}}. \end{aligned}$$

Hence we conclude that

$$S_1 \ll \sum_{a \leq A} \sum_{k \ell^2} \sum_{q | \ell} \sum_d F\left(\frac{a^2 dq}{Y}\right) k^{-\frac{3}{2}} \ell^{-3} a^3 q^2 Y^{\epsilon - \frac{1}{2}} \ll A^2 Y^{\epsilon + \frac{1}{2}}.$$

9. Evaluation of S_0

Since $r = 0$ we have $\chi_{Nr\ell}(m) = 0$ for all $m > 1$ and the terms with $m = 1$ yield

$$S_0 = 2 \sum_{\substack{a \leq A \\ (a, 4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 \\ (n,a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_d,$$

where

$$\sum_d = \sum_{dq \in \mathcal{D}} F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq \sqrt{N}}\right).$$

We split the summation over d into residue classes modulo $4N$. Each class contributes

$$\frac{Y}{4Na^2q} \int F(t) V\left(\frac{2\pi n}{t\sqrt{N}Y}\right) dt + O\left(\left(1 + \frac{n}{Y}\right)^{-j}\right)$$

for any $j \geq 2$, and the number of relevant classes is

$$\gamma(4N) = \#\{d \pmod{4N} : d \equiv -\nu^2 \pmod{4N}, (\nu, 4N) = 1\}.$$

Hence

$$\begin{aligned} S_0 &= \gamma(4N)Y \sum_{n=k\ell^2} \frac{a_n \varphi(\ell)}{2Nn\ell} \left(\sum_{\substack{a \leq A, \\ (a, 4N\ell)=1}} \mu(a) a^{-2} \right) \int F(t) V\left(\frac{2\pi n}{t\sqrt{N}Y}\right) dt + O\left(AY^{\epsilon+\frac{1}{2}}\right) \\ &= c_N Y \int F(t) \mathcal{B}(t\sqrt{N}Y) dt + O((AY^{\frac{1}{2}} + A^{-1}Y)Y^{\epsilon}), \end{aligned}$$

where

$$c_N = \frac{3\gamma(4N)}{\pi^2 N} \prod_{p|4N} \left(1 - \frac{1}{p^2}\right)$$

and

$$\mathcal{B}(X) = \sum_{n=k\ell^2} \frac{b_n}{n} V\left(\frac{2\pi n}{X}\right)$$

with

$$b_n = a_n \prod_{p|n, p \nmid 4N} \left(1 + \frac{1}{p}\right).$$

To evaluate the series $\mathcal{B}(X)$ we appeal to analytic properties of the zeta-function

$$L(s) = \sum_{n=k\ell^2} b_n n^{-s}.$$

The required properties are inherited from the properties of the Rankin-Selberg zeta-function

$$H(s) = \sum_1^{\infty} a_n^2 n^{-s} .$$

The Rankin-Selberg zeta-function is meromorphic on $\mathbb{C}$, holomorphic on $\operatorname{Re} s \geq 1$ except for a simple pole at $s = 2$ with residuum

$$H = \operatorname{res}_{s=2} H(s) > 0 ,$$

and it satisfies a functional equation which connects $H(s)$ with $H(2-s)$. Moreover, as shown by G. Shimura [12] the function

$$L(s, \operatorname{sym}^2) = \frac{\zeta(2s)}{\zeta(s)} H(s+1)$$

is entire. By the Phragmén-Lindelöf principle, using the functional equation, it follows that

$$L(s, \operatorname{sym}^2) \ll |s| \quad \text{if } \operatorname{Re} s \geq 1/2 .$$

Since $L(s)$ agrees with $L(2s-1, \operatorname{sym}^2)/\zeta(4s-2) = H(2s)/\zeta(2s-1)$ up to an Euler product $P(s)$, say, which converges absolutely in $\operatorname{Re} s \geq 3/4$ we conclude that $L(s)$ is holomorphic in $\operatorname{Re} s \geq 3/4$, it satisfies

$$L(s) \ll |s|^2 \quad \text{if } \operatorname{Re} s \geq 3/4$$

and that

$$(19) \quad L(1) = HP(1) \neq 0 .$$

Now by the contour integration we get

$$\begin{aligned} \mathcal{B}(X) &= \frac{1}{2\pi i} \int_{(3/4)} L(s+1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s ds \\ &= \operatorname{res}_{s=0} L(s+1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s + \frac{1}{2\pi i} \int_{(-1/4)} \\ &= L(1) \left(\log \frac{X}{2\pi} - \gamma\right) + L'(1) + O(X^{-1/4}) \end{aligned}$$

by the expansion $\Gamma(s) = s^{-1} - \gamma + \dots$, where γ is the Euler constant. Integrating against $F(t)$ we conclude that

$$S_0 = \alpha Y \log Y + \beta Y + O((AY^{\frac{1}{2}} + A^{-1}Y)Y^{\epsilon})$$

with

$$(20) \quad \alpha = c_N L(1) \int F(t) dt \neq 0$$

and

$$(21) \quad \beta = c_N \int F(t) \left[L(1) \left(\log \frac{t\sqrt{N}}{2\pi} - \gamma \right) + L'(1) \right] dt.$$

10. Evaluation of the first moment of $L'(1, E, \chi_d)$. Conclusion

Collecting the established evaluations we infer that

$$\begin{aligned} S_1(Y) &= S_0 + S_1 + S_2 + R = \alpha Y \log Y + \beta Y \\ &\quad + O((AY^{\frac{1}{2}} + A^{-1}Y + A^2Y^{\frac{1}{2}} + A^{-\frac{3}{2}}Y^{\frac{5}{4}} + A^{-3}Y^{\frac{3}{2}})Y^{\epsilon}) \end{aligned}$$

which gives (6) on taking $A = Y^{3/14}$.

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